1. Let + and \cdot be defined on \{♠, †, ⊙, A\} through:

\[\begin{array}{c|cccc}
+ & ⊙ & † & ⊙ & A \\
\hline
♠ & ⊙ & † & ⊙ & A \\
† & † & ⊙ & A & ♠ \\
⊙ & ⊙ & A & † & ⊙ \\
A & A & ⊙ & † & ⊙ \\
\end{array}\]

\[\begin{array}{c|cccc}
\cdot & ⊙ & † & ⊙ & A \\
\hline
♠ & ⊙ & † & ⊙ & A \\
† & † & ⊙ & A & ♠ \\
⊙ & ⊙ & A & † & ⊙ \\
A & A & ⊙ & † & ⊙ \\
\end{array}\]

Do these turn \{♠, †, ⊙, A\} into a field?

**Solution:** The neutral element of \{♠, †, ⊙, A\} with respect to +, i.e., the zero, is ♠. According to the second table, ⊙ · ⊙ = ♠ holds, which is impossible in a field.

2. Show that \( \mathbb{Q}[i] := \{ p + iq : p, q \in \mathbb{Q} \} \subset \mathbb{C} \)

with + and \cdot inherited from \mathbb{C}, is a field. Is there a way to turn \( \mathbb{Q}[i] \) into an ordered field?

**Hint:** Many of the field axioms are true for \( \mathbb{Q}[i] \) simply because they are true for \mathbb{C}; in this case, just point it out and don’t verify the axiom in detail.

**Solution:** Let \( p, q, r, s \in \mathbb{Q} \). Then

\[ (p + iq) + (r + is) = (p + r) + (q + s) \in \mathbb{Q}[i] \]

and

\[ (p + iq)(r + is) = (pr - qs) + i(qr + ps) \in \mathbb{Q}[i] \]

hold, so that (F 1) is satisfied.

Since (F 2), (F 3), and (F 4) hold for \mathbb{C}, they also hold for \( \mathbb{Q}[i] \).

Since 0 = 0 + i 0, 1 = 1 + i 0 ∈ \( \mathbb{Q}[i] \), (F 5) is satisfied as well.

Let \( p, q \in \mathbb{Q} \), and let \( x = p + iq \). Then \( -x = -p + i(-q) \in \mathbb{Q}[i] \) as well. Suppose that \( x \neq 0 \), so that \( p^2 + q^2 \neq 0 \). Set

\[ y := \frac{p}{p^2 + q^2} - i \frac{q}{p^2 + q^2} \in \mathbb{Q}[i]. \]

It is immediate that \( xy = 1 \). Hence, (F 6) is also satisfied.
Assume that there is \( P \subset \mathbb{Q}[i] \) as in the definition of an ordered field. Then either \( i \in P \) or \( -i \in P \) holds, so that in either case \(-1 = i^2 = (-i)^2 \in P\), which contradicts the fact that \( 1 \in P \).

3. Let \( \emptyset \neq S \subset \mathbb{R} \) be bounded below, and let \(-S := \{ -x : x \in S \}\). Show that:

(a) \(-S\) is bounded above;
(b) \( S \) has an infimum, namely \( \inf S = - \sup(-S) \).

Solution:

(a) Let \( L \) be a lower bound for \( S \), i.e., \( L \leq x \) for all \( x \in S \). It follows that \(-x \leq -L \) for each \( x \in S \) and thus \( x \leq -L \) for each \( x \in -S \). Hence, \(-L\) is an upper bound for \(-S\).

(b) Let \( C := \sup(-S) \), so that \( x \leq C \) for all \( x \in -S \). It follows that \(-x \geq -C \) for all \( x \in -S \), i.e., \( x \geq -C \) for all \( x \in S \). Hence, \(-C\) is a lower bound for \( S \). Let \( C' \) be another other lower bound for \( S \). In the solution to (a), we have seen that \(-C'\) is an upper bound for \(-S\), and thus \(-C' \geq C \) by the definition of a supremum. It follows that \( C' \leq -C \). Hence, \(-C = \inf S\) holds.

4. Find \( \sup S \) and \( \inf S \) in \( \mathbb{R} \) for

\[
S := \left\{ (-1)^n \left( 1 - \frac{1}{n} \right) : n \in \mathbb{N} \right\}.
\]

Justify, i.e., prove, your findings.

Solution: For odd \( n \in \mathbb{N} \), \((-1)^n \left( 1 - \frac{1}{n} \right)\) is negative, and for even \( n \), we have

\[
(-1)^n \left( 1 - \frac{1}{n} \right) = 1 - \frac{1}{n} \leq 1.
\]

Hence, \( S \) is bounded above by 1. Assume that \( \sup S < 1 \), and let \( \epsilon := 1 - \sup S \). In class, we saw that there is \( n \in \mathbb{N} \) with \( 0 < \frac{1}{n} < \epsilon \), so that

\[
1 - \frac{1}{2\underbrace{n}} > 1 - \frac{1}{n} > 1 - \epsilon = \sup S,
\]

which is impossible.

Similarly, one sees that \( \inf S = -1 \).

5. Let \( S, T \subset \mathbb{R} \) be non-empty and bounded above. Show that

\[
S + T := \{ x + y : x \in S, y \in T \}
\]
is also bounded above with
\[
\sup(S + T) = \sup S + \sup T.
\]

**Solution:** Let \( x \in S \) and \( y \in T \). Then \( x \leq \sup S \) and \( y \leq \sup T \). It follows that
\[
x + y \leq \sup S + \sup T,
\]
so that \( \sup S + \sup T \) is an upper bound for \( S + T \). Consequently,
\[
\sup(S + T) \leq \sup S + \sup T
\]
holds.

Assume that \( \sup(S + T) < \sup S + \sup T \). Let \( \epsilon := \frac{1}{2}(\sup S + \sup T - \sup(S + T)) \).
Choose \( x \in S \) and \( y \in T \) such that
\[
x > \sup S - \epsilon \quad \text{and} \quad y > \sup T - \epsilon.
\]
It follows that
\[
x + y > \sup S + \sup T - 2\epsilon = \sup(S + T),
\]
which is a contradiction.

6*. An ordered field \( \mathbb{O} \) is said to have the **nested interval property** if \( \bigcap_{n=1}^{\infty} I_n \neq \emptyset \) for each decreasing sequence \( I_1 \supset I_2 \supset I_3 \supset \cdots \) of closed intervals in \( \mathbb{O} \).

Show that an Archimedean ordered field with the nested interval property is complete.

**Solution:** Let \( \emptyset \neq S \subset \mathbb{O} \) be bounded above. Choose \( a_1 \in S \) and let \( b_1 > a_1 \) be an upper bound for \( S \). Let \( I_1 := [a_1, b_1] \), and let \( c_1 := \frac{1}{2}(b_1 - a_1) \). There are two possibilities:

**Case 1:** \( c_1 \) is an upper bound for \( S \). In this case, let \( a_2 := a_1, \ b_2 := c_1, \) and \( I_2 := [a_2, b_2] \).

**Case 2:** \( c_1 \) is not an upper bound for \( S \). In this case, there is \( a_2 \in S \) with \( a_2 > c_1 \).
Let \( b_2 := b_1 \), and define \( I_2 := [a_2, b_2] \).
Let \( c_2 := \frac{1}{2}(b_2 - a_2) \). Depending on whether \( c_2 \) is an upper bound for \( S \) or not, we find \( a_3 \) and \( b_3 \) as we found \( a_2 \) and \( b_2 \) and define \( I_3 := [a_3, b_3] \).
Continuing in this fashion, we obtain a decreasing sequence \( I_1 \supset I_2 \supset I_3 \supset \cdots \) of closed intervals in \( \mathbb{O} \) with the following properties for all \( n \in \mathbb{N} \):

- \( I_n = [a_n, b_n] \), where \( a_n \in S \) and \( b_n \in \mathbb{O} \) is an upper bound for \( S \);
- \( (b_{n+1} - a_{n+1}) \leq \frac{1}{2}(b_n - a_n) \).
This second fact yields that

\[(b_{n+1} - a_{n+1}) \leq \frac{1}{2n}(b_1 - a_1) \leq \frac{1}{n}(b_1 - a_1)\]

for all \( n \in \mathbb{N} \) by induction on \( n \).

Since \( \mathbb{O} \) has the nested interval property, there is \( x \in \bigcap_{n=1}^{\infty} I_n \). We claim that \( x \) is the supremum of \( S \) in \( \mathbb{O} \).

Assume that \( x \) is not an upper bound for \( S \), i.e., there is \( y \in S \) such that \( y > x \). Use the fact that \( \mathbb{O} \) is Archimedean to find \( n \in \mathbb{N} \) such that

\[(b_{n+1} - a_{n+1}) \leq \frac{1}{n}(b_1 - a_2) < y - x.\]

Since \( x \geq a_{n+1} \), we obtain

\[y - x > b_{n+1} - a_{n+1} \geq b_{n+1} - x,\]

and adding \( x \) on both sides yields \( y > b_{n+1} \), which contradicts \( b_{n+1} \) being an upper bound for \( S \).

Hence, \( x \) is an upper bound for \( S \).

Assume that there is an upper bound \( y \in \mathbb{O} \) with \( y < x \). Again use the fact that \( \mathbb{O} \) is Archimedean to find \( n \in \mathbb{N} \) such that

\[(b_{n+1} - a_{n+1}) \leq \frac{1}{n}(b_1 - a_2) < x - y.\]

Since \( b_{n+1} \geq x \), we obtain

\[x - y > b_{n+1} - a_{n+1} \geq x - a_{n+1},\]

and subtracting \( x \) and multiplying with \(-1\) on both sides yields that \( a_{n+1} > y \) which contradicts \( y \) being an upper bound for \( S \).

Hence, \( x \) is the least upper bound for \( S \), i.e., \( x = \sup S \).