MATH 518—Functional Analysis
(Winter 2021)

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Contents

Chapter 1

Topological Vector Spaces

1.1 A Vector Space That Cannot Be Turned into a Banach Space

Throughout these notes, we use $\mathbb{F}$ to denote a field that can either be the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$.

Recall that a *Banach space* is a vector space E over $\mathbb{F}$ equipped with a norm $\|\cdot\|$, i.e., a *normed space*, such that every Cauchy sequence in $(E, \|\cdot\|)$ converges.

Examples.

1. Consider the vector space

$$\mathcal{C}([0,1]) := \{f\colon [0,1] \to \mathbb{F} : f \text{ is continuous}\}.$$

The *supremum norm* on $\mathcal{C}([0,1])$ is defined to be

$$\|f\|_\infty := \sup_{t\in[0,1]} |f(t)| \qquad (f \in \mathcal{C}([0,1])).$$

It is well known that $(\mathcal{C}([0,1]), \|\cdot\|_\infty)$ is a Banach space.

2. For $n \in \mathbb{N}$, consider the vector space

$$\mathcal{C}^n([0,1]) := \{f\colon [0,1] \to \mathbb{F} : f \text{ is } n\text{-times continuously differentiable}\}.$$

The Weierstraß Approximation Theorem, states that the polynomials are dense in $(\mathcal{C}([0,1]), \|\cdot\|_\infty)$. Therefore $(\mathcal{C}^n([0,1]), \|\cdot\|_\infty)$ cannot be a Banach space. However, if we define

$$\|f\|_{\mathcal{C}^n} := \sum_{j=0}^{n} \left\|f^{(j)}\right\|_\infty \qquad (f \in \mathcal{C}^n([0,1])),$$

then $(\mathcal{C}^n([0,1]), \|\cdot\|_{\mathcal{C}^n})$ is a Banach space.

3. Consider the vector space

$$\mathcal{C}^\infty([0,1]) := \bigcap_{n=1}^{\infty} \mathcal{C}^n([0,1]).$$

Again, the Weierstraß Approximation Theorem yields that $(\mathcal{C}^\infty([0,1]), \|\cdot\|_\infty)$ is not a Banach space. With a little more work one can see that $(\mathcal{C}^\infty([0,1]), \|\cdot\|_{\mathcal{C}^n})$ is not a Banach space either for all $n \in \mathbb{N}$.

We claim that more is true:

Claim 1 There is *no* norm $\|\cdot\|$ on $\mathcal{C}^\infty([0,1])$ such that:

(a) $(\mathcal{C}^\infty([0,1]), \|\cdot\|)$ is a Banach space;

(b) for each $t \in [0,1]$, the linear functional

$$\mathcal{C}^\infty([0,1]) \to \mathbb{F}, \quad f \mapsto f(t)$$

is continuous with respect to $\|\cdot\|$.

In what follows, we will suppress the symbol $\|\cdot\|$ in $(\mathcal{C}^\infty([0,1]), \|\cdot\|)$ and simply write $\mathcal{C}^\infty([0,1])$.

We will prove Claim 1 by contradiction. Assume that there is a norm $\|\cdot\|$ on $\mathcal{C}^\infty([0,1])$ such that (a) and (b) hold. We prove two auxiliary claims that will lead us towards a contradiction.

Claim 2 For each $t \in [0,1]$, the linear functional

$$\mathcal{C}^\infty([0,1]) \to \mathbb{F}, \quad f \mapsto f'(t) \tag{1.1}$$

is continuous with respect to $\|\cdot\|$.

Proof. Fix $t \in [0,1]$, and let the functional (1.1) be denoted by ϕ. Let $(h_n)_{n=1}^\infty$ be a sequence of non-zero reals such that $t + h_n \in [0,1]$ for all $n \in \mathbb{N}$ and $h_n \to 0$. For $n \in \mathbb{N}$, set

$$\phi_n : \mathcal{C}^\infty([0,1]) \to \mathbb{F}, \quad f \mapsto \frac{f(t+h_n) - f(t)}{h_n}.$$

By (b), each ϕ_n is a linear combination of continuous, linear functionals and therefore itself continuous. As $\langle f, \phi \rangle = \lim_{n\to\infty} \langle f, \phi_n \rangle$ for all $f \in \mathcal{C}^\infty([0,1])$ by the very definition of the first derivative (a well known corollary of) the Uniform Boundedness Theorem yields the continuity of ϕ. □

Claim 3 The linear operator

$$\frac{d}{dx} : \mathcal{C}^\infty([0,1]) \to \mathcal{C}^\infty([0,1]), \quad f \mapsto f'$$

is continuous.

Proof. We will apply the Closed Graph Theorem. Let $(f_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{C}^{\infty}([0,1])$, and let $g \in \mathcal{C}^{\infty}([0,1])$ be such that

$$f_n \to 0 \qquad \text{and} \qquad f_n' \to g.$$

We need to show that $g \equiv 0$. To see this, let $t \in [0,1]$. By Claim 2, we have $f_n'(t) \to 0$ whereas (b) yields $f_n'(t) \to g(t)$, so that $g(t) = 0$. As $t \in [0,1]$ is arbitrary, this means $g \equiv 0$. □

To complete the proof of Claim 2, choose $C > \left\|\frac{d}{dx}\right\|$ where $\left\|\frac{d}{dx}\right\|$ is the operator norm of $\frac{d}{dx}$ on the Banach space $\mathcal{C}^{\infty}([0,1])$. Define

$$f\colon [0,1] \to \mathbb{R}, \quad t \mapsto e^{Ct},$$

so that $f \in \mathcal{C}^{\infty}([0,1])$ and

$$f'(t) = C\, e^{Ct} = C\, f(t) \qquad (t \in [0,1]).$$

This yields

$$C\|f\| = \|C\, f\| = \|f'\| \leq \left\|\frac{d}{dx}\right\| \|f\|$$

and, consequently,

$$C \leq \left\|\frac{d}{dx}\right\|,$$

which contradicts the choice of C.

The last example strongly suggests that we may have to look beyond the realm of normed and Banach spaces if we want to investigate certain natural examples of function spaces with functional analytic methods.

1.2 Locally Convex Spaces

Unless specified otherwise, all vector spaces will from now on be over $\mathbb{F}$, i.e., over $\mathbb{R}$ or over $\mathbb{C}$

Definition 1.2.1. A *topological vector space*—short: *TVS*—is a vector space E over $\mathbb{F}$ equipped with a Hausdorff topology $\mathcal{T}$ such that the maps

$$E \times E \to E, \quad (x,y) \mapsto x + y$$

and

$$\mathbb{F} \times E \to E, \quad (\lambda, x) \mapsto \lambda x$$

are continuous, where $E \times E$ and $\mathbb{F} \times E$ are equipped with the respective product topologies.

We refer to a topological vector space E with its given topology $\mathcal{T}$ by the symbol $(E, \mathcal{T})$ or simply by E if no confusion can occur about $\mathcal{T}$.

Of course, every normed space is a topological vector space with $\mathcal{T}$ being the topology induced by the norm.

Definition 1.2.2. Let E be a vector space. Then a map $p : E \to [0, \infty)$ is called a *seminorm* if

$$p(x + y) \le p(x) + p(y) \qquad (x, y \in E)$$

and

$$p(\lambda x) = |\lambda| p(x) \qquad (\lambda \in \mathbb{F},\, x \in E).$$

Remarks. 1. If $p(x) = 0$ implies that $x = 0$ for all $x \in E$, then p is, in fact, a norm.

2. We have

$$|p(x) - p(y)| \le p(x - y) \qquad (x, y \in E). \tag{1.2}$$

This is proved exactly like the corresponding statement for norms.

Proposition 1.2.3. *Let E be a topological vector space, and let p be a seminorm on E. Then the following are equivalent:*

(i) *p is continuous;*

(ii) *$\{x \in E : p(x) < 1\}$ is open;*

(iii) *$0 \in \operatorname{int}\{x \in E : p(x) < 1\}$;*

(iv) *$0 \in \operatorname{int}\{x \in E : p(x) \le 1\}$;*

(v) *p is continuous at 0;*

(vi) *there is a continuous seminorm q on E such that $p \le q$.*

Proof. (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) are straightforward.

(iv) $\Longrightarrow$ (v): Let U be a neighborhood of 0 in $\mathbb{F}$. We need to show that $p^{-1}(U)$ is a neighborhood of 0 in E. Choose $\epsilon > 0$ such that $(-\epsilon, \epsilon) \subset U$. It follows that

$$p^{-1}((-\epsilon, \epsilon)) \supset p^{-1}\left(\left[-\frac{\epsilon}{2}, \frac{\epsilon}{2}\right]\right) = \frac{\epsilon}{2} p^{-1}([-1, 1]) = \frac{\epsilon}{2}\{x \in E : p(x) \le 1\}.$$

By (iv), $0 \in E$ is an interior point of $\{x \in E : p(x) \le 1\}$ and therefore of $p^{-1}((-\epsilon, \epsilon))$. This means that $p^{-1}((-\epsilon, \epsilon))$ is a neighborhood of 0 as is, consequently, $p^{-1}(U)$.

(v) $\Longrightarrow$ (i): Let $x \in E$ and let $(x_\alpha)_\alpha$ be a net in E such that $x_\alpha \to x$. It follows that $x_\alpha - x \to 0$, so that—by (1.2)—

$$|p(x_\alpha) - p(x)| \le p(x_\alpha - x) \to 0.$$

This proves the continuity of p at x.

(i) $\Longrightarrow$ (vi) is obvious

(vi) $\Longrightarrow$ (v): Let $(x_\alpha)_\alpha$ be a net in E such that $x_\alpha \to 0$. It follows that

$$p(x_\alpha) \leq q(x_\alpha) \to 0,$$

so that p is continuous at 0. $\square$

Remarks. 1. If $p_1, \ldots, p_n$ are continuous seminorms on E, then so are $p_1 + \cdots + p_n$ and $\max_{j=1,\ldots,n} p_j$.

2. If $\mathcal{P}$ is a family of seminorms on E such that there is a continuous seminorm q with $p \leq q$ for all $p \in \mathcal{P}$, then

$$E \to [0, \infty), \quad x \mapsto \sup_{p \in \mathcal{P}} p(x)$$

is a continuous seminorm.

Definition 1.2.4. A vector space E equipped with a family $\mathcal{P}$ of seminorms such that $\bigcap \{p^{-1}(\{0\}) : p \in \mathcal{P}\} = \{0\}$ is a called a *locally convex (vector) space*—in short: *LCS.*

We will write $(E, \mathcal{P})$ for a locally convex space with its given family of seminorms and often suppress the symbol $\mathcal{P}$ if no confusion can arise.

Note that we do not *a priori* require that a locally convex space be a topological vector space. Next, we will see that a locally convex space $(E, \mathcal{P})$ is equipped with a canonical topology induced by $\mathcal{P}$.

Theorem 1.2.5. *Let $(E, \mathcal{P})$ be a locally convex space, and consider the collection of all subsets U of E such that, for each $x \in U$, there are $p_1, \ldots, p_n \in \mathcal{P}$ and $\epsilon_1, \ldots, \epsilon_n > 0$ such that*

$$\{y \in E : p_j(x - y) < \epsilon_j \text{ for } j = 1, \ldots, n\} \subset U.$$

Then these subsets of E form Hausdorff topology $\mathcal{T}$ over E turning E into a topological vector space. Furthermore, for any $x \in E$ and any net $(x_\alpha)_\alpha$ in E, we have

$$x_\alpha \xrightarrow{\mathcal{T}} x \quad \Longleftrightarrow \quad p(x_\alpha - x) \to 0 \text{ for all } p \in \mathcal{P}.$$

Proof. We start with proving that $\mathcal{T}$ is indeed a topology over E.

It is obvious that $\varnothing, E \in \mathcal{T}$.

Let $U_1, U_2 \in \mathcal{T}$, and let $x \in U_1 \cap U_2$. For $j = 1, 2$, there are $p_1^{(j)}, \ldots, p_{n_j}^{(j)} \in \mathcal{P}$ and $\epsilon_1^{(j)}, \ldots, \epsilon_{n_j}^{(j)} > 0$ such that

$$\left\{ y \in E : p_\nu^{(j)}(x - y) < \epsilon_\nu^{(j)} \text{ for } \nu = 1, \ldots, n_j \right\} \subset U_j.$$

It follows that

$$\left\{y \in E : p_\nu^{(j)}(x-y) < \epsilon_\nu^{(j)} \text{ for } \nu = 1,\ldots,n_j \text{ and } j = 1,2\right\} \subset U_1 \cap U_2,$$

and therefore $U_1 \cap U_2 \in \mathcal{T}$.

Let $\mathcal{U} \subset \mathcal{T}$, and let $x \in \bigcup\{U : U \in \mathcal{U}\}$. Then there is $U_0 \in \mathcal{U}$ with $x \in U_0$. As $U_0 \in \mathcal{T}$, there are $p_1,\ldots,p_n \in \mathcal{P}$ and $\epsilon_1,\ldots,\epsilon_n > 0$ such that

$$\{y \in E : p_j(x-y) < \epsilon_j \text{ for } j = 1,\ldots,n\} \subset U_0 \subset \bigcup\{U : U \in \mathcal{U}\}.$$

It follows that $\bigcup\{U : U \in \mathcal{U}\} \in \mathcal{T}$.

Next, we show that $\mathcal{T}$ is a Hausdorff topology. Let $x, y \in E$ be such that $x \neq y$. As $\bigcap\{p^{-1}(\{0\}) : p \in \mathcal{P}\} = \{0\}$, there is $p \in \mathcal{P}$ such that $p(x-y) > 0$. Set $\epsilon := \frac{1}{2}p(x-y)$, and define

$$U := \{z \in E : p(x-z) < \epsilon\} \qquad \text{and} \qquad V := \{z \in E : p(y-z) < \epsilon\};$$

it is easy to see that $U, V \in \mathcal{T}$, and it is obvious that $x \in U$ and $y \in V$. Assume that there is $z \in U \cap V$. Then we have

$$p(x-y) \leq p(x-z) + p(z-y) < 2\epsilon = p(x-y),$$

which is a contradiction. It follows that $U \cap V = \varnothing$, so that $\mathcal{T}$ is a Hausdorff topology.

Before we prove that $\mathcal{T}$ turns E into a topological vector space, we show the "Furthermore" part of Theorem 1.2.5. Let $x \in E$, and let $(x_\alpha)_\alpha$ be a net in E. Suppose that $x_\alpha \xrightarrow{\mathcal{T}} x$. It is immediate from the definition of $\mathcal{T}$ that every $p \in \mathcal{P}$ is continuous, so that

$$p(x_\alpha - x) \to 0 \qquad (p \in \mathcal{P}). \tag{1.3}$$

Conversely, suppose that (1.3) holds, and assume towards a contradiction that $x_\alpha \not\to x$ with respect to $\mathcal{T}$. This means that there are $U \in \mathcal{T}$ with $x \in U$ and a subnet $(x_\beta)_\beta$ of $(x_\alpha)_\alpha$ such that $x_\beta \notin U$ for all indices β. By the definition of $\mathcal{T}$, there are $p_1,\ldots,p_n \in \mathcal{P}$ and $\epsilon_1,\ldots,\epsilon_n > 0$ such that

$$\{y \in E : p_j(x-y) < \epsilon_j \text{ for } j = 1,\ldots,n\} \subset U.$$

As $\lim_\beta p_j(x_\beta - x) = 0$ for $j = 1,\ldots,n$, there are $\beta_1,\ldots,\beta_n$ in the index set of $(x_\beta)_\beta$ such that $p_j(x_\beta - x) < \epsilon_j$ for all $j = 1,\ldots,n$ and all indices β with $\beta_j \preceq \beta$. Choose an index β_0 such that $\beta_1,\ldots,\beta_n \preceq \beta_0$, which is possible because the index set of a net is directed. It follows that $p_j(x_\beta - x) < \epsilon_j$ for $j = 1,\ldots,n$ and all β with $\beta_0 \preceq \beta$ and therefore $x_\beta \in U$ for those β. This contradicts the choice of $(x_\beta)_\beta$.

To complete the proof, let $(x_\alpha)_\alpha$ and $(y_\beta)_\beta$ be nets in E, and let $x, y \in E$ be such that $x_\alpha \to x$ and $y_\beta \to y$. By the foregoing this means that

$$p(x_\alpha - x) \to 0 \quad \text{and} \quad p(y_\beta - y) \to 0 \qquad (p \in \mathcal{P}),$$

so that

$$p(x_\alpha + y_\beta - (x+y)) \leq p(x_\alpha - x) + p(y_\beta - y) \to 0 \qquad (p \in \mathcal{P})$$

and therefore $x_\alpha + y_\beta \to x + y$. This proves the continuity of addition in E; the proof of the continuity of scalar multiplication is similarly easy. $\square$

We call the topology described in Theorem 1.2.5, *the topology induced by* $\mathcal{P}$.

Examples. 1. Let $E := \mathcal{C}^\infty([0,1])$, and let $\mathcal{P} := \{\|\cdot\|_{\mathcal{C}^n} : n \in \mathbb{N}\}$. A net $(f_\alpha)_\alpha$ in E the converges to $f \in E$ if and only if $\left(f_\alpha^{(n)}\right)_\alpha$ converges to $f^{(n)}$ uniformly on $[0,1]$ for each $n \in \mathbb{N}_0$.

2. Let $\varnothing \neq S$ be any set, let $E := \mathbb{F}^S$, and let $\mathcal{P} := \{p_s : s \in S\}$ where

$$p_s \colon E \to \mathbb{F}, \quad f \mapsto |f(s)| \qquad (s \in S).$$

Then a net $(f_\alpha)_\alpha$ in E converges to $f \in E$ if and only if $f_\alpha \to f$ pointwise on S.

3. Let X be a topological space, and let

$$E := \mathcal{C}(X) := \{f \colon X \to \mathbb{F} : f \text{ is continuous}\};$$

note that we do not require the functions in E to be bounded. For $\varnothing \neq K \subset X$ compact, define $\|\cdot\|_K \colon E \to [0,\infty)$ by letting

$$\|f\|_K := \sup_{x \in K} |f(x)|.$$

Set $\mathcal{P} := \{\|\cdot\|_K : \varnothing \neq K \subset X \text{ compact}\}$. Let $f \in E \setminus \{0\}$. Then there is $x \in X$ with $f(x) \neq 0$, so that $\|f\|_{\{x\}} > 0$. It follows that $\bigcap\{p^{-1}(\{0\}) : p \in \mathcal{P}\} = \{0\}$, so that $(E,\mathcal{P})$ is a locally convex vector space. It is easy to see that a net $(f_\alpha)_\alpha$ in E the converges to $f \in E$ if and only if $f_\alpha \to f$ uniformly on all compact subsets of X.

4. Let E be any normed space with dual space E^*. For $\phi \in E^*$, define

$$p_\phi \colon E \to [0,\infty), \quad x \mapsto |\langle x, \phi\rangle|.$$

Set $\mathcal{P} := \{p_\phi : \phi \in E^*\}$. By the Hahn–Banach Theorem, there is, for each $x \in E \setminus \{0\}$, a functional $\phi \in E^*$ with $\langle x, \phi \rangle \neq 0$. It follows that $\bigcap\{p^{-1}(\{0\}) : p \in \mathcal{P}\} = \{0\}$, so that $(E,\mathcal{P})$ is a locally convex space. The resulting topology on E is called the *weak topology* on E and denoted by $\sigma(E,E^*)$.

5. Let E be any normed space with dual space E^*. For $x \in E$, define

$$p_x \colon E^* \to [0,\infty), \quad \phi \mapsto |\langle x, \phi\rangle|.$$

Set $\mathcal{P} := \{p_x : x \in E\}$. Then $(E^*,\mathcal{P})$ is a locally convex space. The resulting topology on E^* is called the *weak*[*] *topology* on E and denoted by $\sigma(E^*,E)$.

Why are locally convex spaces called "locally convex"? This will soon become apparent.

Recall that a subset C of a vector space E is called *convex* if $tx + (1-t)y \in C$ for all $x, y \in C$ and $t \in [0,1]$.

Definition 1.2.6. Let E be a topological vector space, and let $S \subset E$. Then:

(a) the *convex hull* $\operatorname{conv} S$ of S is defined as the intersection of all convex subsets of E containing S;

(b) the *closed convex hull* $\overline{\operatorname{conv}}\, S$ of S is defined as the intersection of all closed convex subsets of E containing S.

Lemma 1.2.7. *Let E be a topological vector space, and let $C \subset E$ be convex. Then $\overline{C}$ is convex.*

Proof. Let $x \in C$, let $y \in \overline{C}$, and let $t \in [0,1]$. We claim that $tx + (1-t)y \in \overline{C}$. As $y \in \overline{C}$, there is a net $(y_\alpha)_\alpha \in C$ with $y_\alpha \to y$. It follows that

$$\underbrace{tx + (1-t)y_\alpha}_{\in C} \to tx + (1-t)y \in \overline{C}.$$

Now, let $x, y \in \overline{C}$, and let $t \in [0,1]$. Let $(x_\alpha)_\alpha$ be a net in C with $x_\alpha \to x$. It follows that

$$\underbrace{tx_\alpha + (1-t)y}_{\in \overline{C}} \to tx + (1-t)y \in \overline{C},$$

so that $\overline{C}$ is convex. $\square$

Proposition 1.2.8. *Let E be a topological vector space, and let $S \subset E$. Then*

$$\operatorname{conv} S = \left\{ \sum_{j=1}^{n} t_j x_j : n \in \mathbb{N},\, x_1, \ldots, x_n \in S,\, t_1, \ldots, t_n \geq 0,\, \sum_{j=1}^{n} t_j = 1 \right\} \tag{1.4}$$

and $\overline{\operatorname{conv}}\, S = \overline{\operatorname{conv} S}$.

Proof. To prove (1.4), let the right hand side of (1.4) be denoted by C. Obviously, $S \subset C$, and it is routinely checked that C is convex, so that $\operatorname{conv} S \subset C$.

For the reverse inclusion, we show that $\sum_{j=1}^{n} t_j x_j \in \operatorname{conv} S$ for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in S$, and $t_1, \ldots, t_n \geq 0$ with $\sum_{j=1}^{n} t_j = 1$. We proceed by induction on n.

The case where $n = 1$ is trivial. So, let $n \geq 2$ be such that $\sum_{j=1}^{n-1} t_j x_j \in \operatorname{conv} S$ for all $x_1, \ldots, x_{n-1} \in S$, and $t_1, \ldots, t_{n-1} \geq 0$ with $\sum_{j=1}^{n-1} t_j = 1$. Let $x_1, \ldots, x_{n-1}, x_n \in S$, and let $t_1, \ldots, t_{n-1}, t_n \geq 0$ be such that $\sum_{j=1}^{n} t_j = 1$. Without loss of generality suppose that $t_n \neq 0$. If $\sum_{j=1}^{n-1} t_j = 0$ we have $t_1 = \cdots = t_{n-1} = 0$ and $t_n = 1$, so nothing needs to be shown. Suppose therefore that $t := \sum_{j=1}^{n-1} t_j \in (0,1]$. It follows that $\sum_{j=1}^{n-1} t^{-1} t_j = 1$, so that $\sum_{j=1}^{n-1} t^{-1} t_j x_j \in \operatorname{conv} S$ by the induction hypothesis. It follows that

$$\sum_{j=1}^{n} t_j x_j = \sum_{j=1}^{n-1} t_j x_j + t_n x_n = t \underbrace{\left(\sum_{j=1}^{n-1} t^{-1} t_j x_j \right)}_{\in \operatorname{conv} S} + (1-t) \underbrace{x_n}_{\in \operatorname{conv} S} \in \operatorname{conv} S.$$

(It should be noted that nothing in the proof of (1.4) requires the presence of a topology, i.e., (1.4) holds in any vector space.)

By Lemma 1.2.7, $\overline{\operatorname{conv} S}$ is convex, so that $\overline{\operatorname{conv}}\, S \subset \overline{\operatorname{conv} S}$. On the other hand, $\operatorname{conv} S \subset \overline{\operatorname{conv}}\, S$ by definition, so that $\overline{\operatorname{conv} S} \subset \overline{\operatorname{conv}}\, S$ as well. $\square$

Definition 1.2.9. Let E be a vector space. A set $S \subset E$ is called:

(a) *balanced* if $\lambda x \in S$ for all $x \in S$ and all $\lambda \in \mathbb{F}$ with $|\lambda| \leq 1$;

(b) *absorbing* if, for each $x \in E$, there is $\epsilon > 0$ such that $tx \in S$ for all $t \in (0, \epsilon)$;

(c) *absorbing at* $x \in S$ if $S - x$ is absorbing.

Remark. It is clear that a balanced set must contain 0, and that a set is absorbing if and only if it is absorbing at 0.

Example. Let E be a vector space, and let p be a seminorm on E. Then

$$\{x \in E : p(x) < 1\}$$

is convex, balanced, and absorbing at each of its points.

As it turns out, *all* convex and balanced sets that are absorbing at each of their points arise in this fashion.

Proposition 1.2.10. *Let E be a vector space, and let C be a convex, balanced set that is absorbing at each of its points. Then there is a unique seminorm p_C on E such that $C = \{x \in E : p_C(x) < 1\}$*

Proof. The uniqueness assertion is clear.

To prove the existence of p_C, define

$$p_C(x) := \inf\{t \geq 0 : x \in tC\} \qquad (x \in E).$$

As C is absorbing, this is well defined. It is obvious that $p(0) = 0$.

Let $x \in E$, and let $\lambda \in \mathbb{F} \setminus \{0\}$. We obtain

$$\begin{aligned}
p_C(\lambda x) &= \inf\{t \geq 0 : \lambda x \in tC\} \\
&= \inf\left\{t \geq 0 : x \in t\left(\frac{1}{\lambda}C\right)\right\} \\
&= \inf\left\{t \geq 0 : x \in t\left(\frac{1}{|\lambda|}C\right)\right\}, \qquad \text{because } C \text{ is balanced,} \\
&= |\lambda| \inf\left\{\frac{t}{|\lambda|} : t \geq 0,\, x \in t\left(\frac{1}{|\lambda|}C\right)\right\} \\
&= |\lambda| \inf\{t \geq 0 : x \in tC\} \\
&= |\lambda| p_C(x).
\end{aligned}$$

Let $x, y \in E$, and let $\epsilon > 0$. Choose $s, t > 0$ such that $x \in sC$, $y \in tC$, $s < p_C(x) + \frac{\epsilon}{2}$, and $t < p_C(y) + \frac{\epsilon}{2}$. It follows that

$$\frac{x}{s+t} \in \frac{s}{s+t}C \qquad \text{and} \qquad \frac{y}{s+t} \in \frac{t}{s+t}C$$

and

$$\frac{x+y}{s+t} \in \frac{s}{s+t}C + \frac{t}{s+t}C \subset C,$$

so that $x + y \in (s+t)C$. It follows that

$$p_C(x+y) \leq s + t \leq p_C(x) + p_C(y) + \epsilon.$$

All in all, p_C is a seminorm.

Let $x \in E$ be such that $p_C(x) < 1$, and let $t \in [0, 1)$ be such that $x \in tC$. As C is balanced, $tC \subset C$, so that $x \in C$. On the other hand, let $x \in C$. It is then immediate that $p_C(x) \leq 1$. As C is absorbing at x, there is $\epsilon > 0$ such that $x + tx \in C$ for all $t \in (0, \epsilon)$. It follows that

$$p_C(x) = \frac{1}{1+t} p_C((1+t)x) \leq \frac{1}{1+t} < 1,$$

which completes the proof. □

Lemma 1.2.11. *Let E be a topological vector space, and the $U \subset E$ be open. Then U is absorbing at each of its points.*

Proof. Let $x \in U$. Then $U - x$ is open and contains zero. We can therefore suppose that $0 \in U$, and it is enough to show that U is absorbing.

Let $x \in E$ be arbitrary and consider the map

$$f \colon \mathbb{R} \to E, \quad t \mapsto tx.$$

Then f is continuous, so that $f^{-1}(U)$ is open in $\mathbb{R}$ and contains 0. Therefore, there is $\epsilon > 0$ such that $(-\epsilon, \epsilon) \subset f^{-1}(U)$, i.e., $tx \in U$ for all $t \in \mathbb{R}$ with $|t| < \epsilon$. In particular, this means that $tx \in U$ for all $t \in (0, \epsilon)$, i.e., U is absorbing. □

We can now characterize the locally convex vector spaces among the topological ones. This also reveals why those spaces were name “locally convex” in the first place.

Theorem 1.2.12. *The following are equivalent for a topological vector space* $(E, \mathcal{T})$*:*

(i) *there is a family* $\mathcal{P}$ *of seminorms on* E *with* $\bigcap\{p^{-1}(\{0\}) : p \in \mathcal{P}\} = \{0\}$ *such that* $\mathcal{T}$ *is the topology induced by* $\mathcal{P}$*;*

(ii) *the open, convex, balanced subsets of* E *form a base of neighborhoods for* 0*.*

Proof. (i) $\Longrightarrow$ (ii): The sets of the form

$$\{x \in E : p_j(x) < \epsilon_j \text{ for } j = 1, \dots, n\}$$

where $n \in \mathbb{N}$, $p_1, \dots, p_n \in \mathcal{P}$, and $\epsilon_1, \dots, \epsilon_n > 0$ are open, balanced, and convex, and form a base of neighborhoods of 0.

(ii) $\Longrightarrow$ (i): Let $\mathcal{C}$ be the collection of all open, convex, balanced subsets of $(E, \mathcal{T})$, and let $\mathcal{P} := \{p_C : C \in \mathcal{C}\}$ with p_C as in Proposition 1.2.10 for each $C \in \mathcal{C}$. As $\{x \in E : p_C(x) < 1\} = C$, it follows that p_C is continuous by Proposition 1.2.3 with respect to $\mathcal{T}$ for each $C \in \mathcal{C}$.

Let $x \in E \setminus \{0\}$. Then there is $C \in \mathcal{C}$ such that $x \notin C$ and therefore $p_C(x) \geq 1$. It follows that $\bigcap\{p^{-1}(\{0\}) : p \in \mathcal{P}\} = \{0\}$.

Let $(x_\alpha)_\alpha$ be a net in E such that $x_\alpha \xrightarrow{\mathcal{T}} 0$. As all $p \in \mathcal{P}$ are continuous, this means that $p(x_\alpha) \to 0$ for all $p \in \mathcal{P}$. Consequently, the topology on E induced by $\mathcal{P}$ is coarser than $\mathcal{T}$.

On the other hand, let U be a neighborhood of 0 with respect to $\mathcal{T}$. As $\mathcal{C}$ is a base of neighborhoods for 0, there is $C \in \mathcal{C}$ such that $C \subset U$. Now, C is open with respect to the topology induced by $\mathcal{P}$, which means that U is a neighborhood of 0 in that topology, too. $\square$

We conclude this section with a topological vector space that is *not* locally convex.

Example. Let $p \in (0, 1)$, and let $L^p([0, 1])$ denote the (equivalence classes) of all measurable functions $f : [0, 1] \to \mathbb{F}$ such that

$$((f))_p := \int_0^1 |f(t)|^p \, dt < \infty.$$

Claim 1 Let $x, y \in [0, \infty)$. Then $(x + y)^p \leq x^p + y^p$.

Proof. Fix $x \in [0, \infty)$, and define

$$f : [0, \infty) \to \mathbb{R}, \quad y \mapsto x^p + y^p - (x + y)^p.$$

Then f is differentiable on $(0, \infty)$ such that

$$f'(y) = py^{p-1} - p(x+y)^{p-1} \qquad (y \in (0, \infty)).$$

As $p - 1 < 0$ and $x + y \geq y$, it is clear that $f'(y) \geq 0$ for all $y > 0$, so that f is increasing and, in particular,

$$0 = f(0) \leq f(y) = x^p + y^p - (x+y)^p \qquad (y \in [0, \infty)).$$

This proves the claim. $\square$

Claim 2 $L^p([0,1])$ is a vector space.

Proof. Let $f \in L^p([0,1])$, and let $\lambda \in \mathbb{F}$. Then it is immediate that $\lambda f \in L^p([0,1])$ as well.

Let $f, g \in L^p([0,1])$. By Claim 1, this means that

$$((f+g))_p = \int_0^1 |f(t) + g(t)|^p \, dt \leq \int_0^1 |f(t)|^p \, dt + \int_0^1 |g(t)|^p \, dt < \infty,$$

so that $f + g \in L^p([0,1])$ as well. $\square$

Claim 3 Setting

$$d(f,g) := ((f-g))_p \qquad (f, g \in L^p([0,1]))$$

defines a *translation invariant* metric on $L^p([0,1])$, i.e.,

$$d(f+h, g+h) = d(f,g) \qquad (f, g, h \in L^p([0,1])).$$

Proof. Let $f, g, h \in L^p([0,1])$, and observe that

$$\begin{aligned} d(f,h) &= \int_0^1 |f(t) - h(t)|^p \, dt \\ &= \int_0^1 (|f(t) - g(t)| + |g(t) - h(t)|)^p \, dt \\ &\leq \int_0^1 |f(t) - g(t)|^p + |g(t) - h(t)|^p \, dt, \qquad \text{by Claim 1,} \\ &\leq \int_0^1 |f(t) - g(t)|^p \, dt + \int_0^1 |g(t) - h(t)|^p \, dt \\ &= d(f,g) + d(g,h). \end{aligned}$$

This proves the triangle inequality. The other axioms of a metric are obvious, as is translation invariance. $\square$

Claim 4 $L^p([0,1])$ equipped with the topology induced by d is a topological vector space.

Proof. Let $(f_n)_{n=1}^\infty$ and $(g_n)_{n=1}^\infty$ be sequences in $L^p([0,1])$ such that $f_n \to f \in L^p([0,1])$ and $g_n \to g \in L^p([0,1])$, and let $(\lambda_n)_{n=1}^\infty$ be a sequence in $\mathbb{F}$ such that $\lambda_n \to \lambda$. We obtain

$$\begin{aligned} d(f_n + g_n, f + g) &\leq d(f_n + g_n, f_n + g) + d(f_n + g, f + g)), \\ &= d(g_n, g) + (f_n, f), \qquad \text{by translation invariance,} \\ &\to 0 \end{aligned}$$

as well as

$$\begin{aligned} d(\lambda_n f_n, \lambda f) &\leq d(\lambda_n f_n, \lambda_n f) + d(\lambda_n f, \lambda f) \\ &= ((\lambda_n f_n - \lambda_n f))_p + ((\lambda_n f - \lambda f))_p \\ &= \underbrace{|\lambda_n|^p}_{\text{bounded}} \underbrace{((f_n - f))_p}_{= d(f_n, f) \to 0} + \underbrace{|\lambda_n - \lambda|^p}_{\to 0} ((f))_p \\ &\to 0, \end{aligned}$$

which proves the claim. ∎

For $R > 0$, we now define

$$B_R := \{f \in L^p([0,1]) : ((f))_p < R\}.$$

Claim 5 $B_{2^{n(1-p)}R} \subset \operatorname{conv} B_R$ for all $R > 0$ and all $n \in \mathbb{N}$.

Proof. We proceed by induction on n.

Let $R > 0$ be arbitrary, and let $f \in B_{2^{(1-p)}R}$, i.e, $r := ((f))_p < 2^{(1-p)}R$. Use the Intermediate Value Theorem to obtain $x \in (0,1)$ such that $\int_0^x |f(t)|^p \, dt = \frac{r}{2}$. Define $g, h \colon [0,1] \to \mathbb{F}$ by letting

$$g(t) := \begin{cases} f(t), & t \leq x, \\ 0, & t > x, \end{cases}$$

and

$$h(t) := \begin{cases} 0, & t \leq x, \\ f(t), & t > x. \end{cases}$$

It follows that

$$f = \frac{1}{2}(2g + 2h) \qquad \text{and} \qquad ((2g))_p = ((2h))_p = 2^p \frac{r}{2} = r 2^{p-1} = \frac{r}{2^{1-p}} < \frac{2^{(1-p)}R}{2^{1-p}} = R,$$

so that $f \in \operatorname{conv} B_R$.

Let $n \geq 2$ be such that $B_{2^{(n-1)(1-p)}R} \subset \operatorname{conv} B_R$ for all $R > 0$. It follows that

$$\begin{aligned} B_{2^{n(1-p)}R} &= B_{2^{(n-1)(1-p)} 2^{1-p} R} \\ &\subset \operatorname{conv} B_{2^{1-p}R}, \qquad \text{by the induction hypothesis with } R \textit{ en lieu } \text{of } 2^{1-p}R, \\ &\subset \operatorname{conv} B_R, \qquad \text{by the base step.} \end{aligned}$$

This completes the proof. ∎

Claim 6 The only open, convex subsets of $L^p([0,1])$ are $\varnothing$ and $L^p([0,1])$.

Proof. Let $\varnothing \neq C \subset L^p([0,1])$ be open and convex, and suppose without loss of generality that $0 \in C$. By the definition of the topology of $L^p([0,1])$, there is $R > 0$ such that $B_R \subset C$. Let $f \in L^p([0,1])$. As $1 - p > 0$, there is $n \in \mathbb{N}$ such that $((f))_p < 2^{n(1-p)}R$, so that

$$f \in B_{2^{n(1-p)}R} \subset \operatorname{conv} B_R \subset C.$$

As $f \in L^p([0,1])$ is arbitrary, this means that $C = L^p([0,1])$. $\square$

In view of Claim 6, it is clear that $L^p([0,1])$ cannot be locally convex.

1.3 Geometric Consequences of the Hahn–Banach Theorem

Definition 1.3.1. Let E be a vector space. Then a map $p \colon E \to \mathbb{R}$ is called a *sublinear functional* if

$$p(x+y) \leq p(x) + p(y) \qquad (x, y \in E)$$

and

$$p(tx) = t\,p(x) \qquad (t \geq 0,\ x \in E).$$

Examples. 1. Every seminorm is a sublinear functional.

2. Let $E := \ell^\infty_{\mathbb{R}}$ denote the space of all bounded sequences in $\mathbb{R}$. Define

$$p \colon E \to \mathbb{R}, \quad (x_n)_{n=1}^\infty \mapsto \limsup_{n\to\infty} x_n.$$

Then p is a sublinear functional on E, but not a seminorm.

We recall:

Theorem 1.3.2 (Hahn–Banach Theorem)**.** *Let E be a vector space over $\mathbb{R}$, let F be a subspace of E, let $p \colon E \to \mathbb{R}$ be a sublinear functional, and let $\phi \colon F \to \mathbb{R}$ be linear such that*

$$\langle x, \phi \rangle \leq p(x) \qquad (x \in F).$$

Then there is a linear functional $\tilde{\phi} \colon E \to \mathbb{R}$ such that $\tilde{\phi}|_F = \phi$ and

$$\left\langle x, \tilde{\phi} \right\rangle \leq p(x) \qquad (x \in E).$$

Corollary 1.3.3. *Let $(E, \mathcal{P})$ be a locally convex vector space. Then, for each $x \in E \setminus \{0\}$, there is a continuous linear functional $\phi \colon E \to \mathbb{F}$ such that $\langle x, \phi \rangle \neq 0$.*

Proof. Let $x \in E \setminus \{0\}$. Then there is $p \in \mathcal{P}$ such that $p(x) > 0$.

We first treat the case where $\mathbb{F} = \mathbb{R}$. Set $F = \mathbb{R}\,x$. Define

$$\psi \colon F \to \mathbb{R}, \quad tx \mapsto t\,p(x)$$

Then ψ is a linear functional on F with $\langle x, \psi \rangle = p(x) \neq 0$ such that

$$\langle tx, \psi \rangle = t\,p(x) = p(tx) \qquad (t \geq 0)$$

and

$$\langle tx, \psi \rangle = t\,p(x) = -|t|p(x) = -p(tx) \leq p(tx) \quad (t \leq 0),$$

so that

$$\langle y, \psi \rangle \leq p(y) \qquad (y \in F).$$

By the Hahn–Banach Theorem, there is therefore a linear functional $\phi : E \to \mathbb{R}$ with $\phi|_F = \psi$—so that, in particular, $\langle x, \phi \rangle \neq 0$—and

$$\langle y, \phi \rangle \leq p(y) \qquad (y \in E).$$

Let $y \in E$. We claim that $|\langle y, \phi \rangle| \leq p(y)$. If $\langle y, \phi \rangle \geq 0$, this is clear, so suppose that $\langle y, \phi \rangle < 0$. In this case, we have

$$|\langle y, \phi \rangle| = -\langle y, \phi \rangle = \langle -y, \phi \rangle \leq p(-y) = p(y),$$

which proves the claim. This entails that ϕ is continuous.

Suppose now that $\mathbb{F} = \mathbb{C}$. As any any vector space over $\mathbb{C}$ is also one over $\mathbb{R}$, we use the case where $\mathbb{F} = \mathbb{R}$ to obtain a continuous $\mathbb{R}$-linear functional $\phi_{\mathbb{R}} : E \to \mathbb{R}$ with $\langle x, \phi_{\mathbb{R}} \rangle \neq 0$. Define

$$\phi \colon E \to \mathbb{C}, \quad y \mapsto \langle y, \phi_{\mathbb{R}} \rangle - i\langle iy, \phi_{\mathbb{R}} \rangle. \tag{1.5}$$

It is clear that ϕ is $\mathbb{R}$-linear and continuous such that $\operatorname{Re}\langle y, \phi \rangle = \langle y, \phi_{\mathbb{R}} \rangle$ for $y \in E$; in particular, $\langle x, \phi \rangle \neq 0$. Observe that

$$\begin{aligned} \langle iy, \phi \rangle &= \langle iy, \phi_{\mathbb{R}} \rangle - i\langle -y, \phi_{\mathbb{R}} \rangle \\ &= \langle iy, \phi_{\mathbb{R}} \rangle + i\langle y, \phi_{\mathbb{R}} \rangle = i(\langle y, \phi_{\mathbb{R}} \rangle - i\langle iy, \phi_{\mathbb{R}} \rangle) = i\langle y, \phi \rangle \qquad (y \in E), \end{aligned}$$

so that ϕ is, in fact, $\mathbb{C}$-linear. $\square$

Example. A *Banach limit* is a linear functional $\mathrm{LIM} \colon \ell^\infty_{\mathbb{R}} \to \mathbb{R}$ such that, for each $(x_n)_{n=1}^\infty \in \ell^\infty_{\mathbb{R}}$,

(a) $\liminf_{n\to\infty} x_n \leq \mathrm{LIM}((x_n)_{n=1}^\infty) \leq \limsup_{n\to\infty} x_n$, and

(b) $\mathrm{LIM}((x_n)_{n=1}^\infty) = \mathrm{LIM}((x_{n+1})_{n=1}^\infty)$.

We claim that Banach limits exist.

Define

$$p\colon \ell^\infty_{\mathbb{R}} \to \mathbb{R}, \quad (x_n)_{n=1}^\infty \mapsto \limsup_{n\to\infty} x_n,$$

so that p is a sublinear functional.

Let

$$F := \left\{ (x_n)_{n=1}^\infty \in \ell^\infty_{\mathbb{R}} : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n x_k \text{ exists} \right\},$$

and define

$$\mathrm{Lim}\colon F \to \mathbb{R}, \quad (x_n)_{n=1}^\infty \mapsto \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n x_k.$$

Let $(x_n)_{n=1}^\infty \in F$. For $n \in \mathbb{N}$, let $c_n := \sup_{k\geq n} x_k$. It follows that $x_n \leq c_n$ for each $n \in \mathbb{N}$ and that $(c_n)_{n=1}^\infty$ converges (to $\limsup_{n\to\infty} x_n$). Hence, we have

$$\begin{aligned}
\mathrm{Lim}((x_n))_{n=1}^\infty) &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n x_k \\
&\leq \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n c_k \\
&= \lim_{n\to\infty} c_n \\
&= \limsup_{n\to\infty} x_n \\
&= p((x_n)_{n=1}^\infty).
\end{aligned}$$

The Hahn–Banach Theorem thus yields a linear functional $\mathrm{LIM} : \ell^\infty_{\mathbb{R}} \to \mathbb{R}$ with $\mathrm{LIM}\,|_F = \mathrm{Lim}$ such that

$$\mathrm{LIM}((x_n)_{n=1}^\infty) \leq p((x_n)_{n=1}^\infty) \qquad ((x_n)_{n=1}^\infty \in \ell^\infty_{\mathbb{R}}),$$

i.e.,

$$\mathrm{LIM}((x_n)_{n=1}^\infty) \leq \limsup_{n\to\infty} x_n \qquad ((x_n)_{n=1}^\infty \in \ell^\infty_{\mathbb{R}}).$$

This, in turn, implies that

$$\begin{aligned}
\liminf_{n\to\infty} x_n &= -\limsup_{n\to\infty} -x_n \\
&\leq -\mathrm{LIM}((-x_n)_{n=1}^\infty) = \mathrm{LIM}((x_n)_{n=1}^\infty) \qquad ((x_n)_{n=1}^\infty \in \ell^\infty_{\mathbb{R}}),
\end{aligned}$$

so that (a) has been proven.

For $(x_n)_{n=1}^\infty \in \ell^\infty_{\mathbb{R}}$, note that

$$\frac{1}{n} \sum_{k=1}^n (x_k - x_{k+1}) = \frac{1}{n}(x_1 - x_{n+1}) \to 0,$$

so that, in particular, $(x_n - x_{n+1})_{n=1}^\infty \in F$. Hence, we obtain

$$\begin{aligned}
\mathrm{LIM}((x_n)_{n=1}^\infty) - \mathrm{LIM}((x_{n+1})_{n=1}^\infty) &= \mathrm{LIM}((x_n - x_{n+1})_{n=1}^\infty) \\
&= \mathrm{Lim}((x_n - x_{n+1})_{n=1}^\infty) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (x_k - x_{k+1}) \\
&= 0.
\end{aligned}$$

This proves (b).

Our first lemma has a proof very similar to that of Proposition 1.2.10, so we omit it.

Lemma 1.3.4. *Let E be a vector space, and let $\varnothing \neq C \subset E$ be convex and absorbing and contain* 0*. Then*

$$p_C(x) := \inf\{t \geq 0 : x \in tC\} \qquad (x \in E)$$

defines a sublinear functional on E. If, moreover, C is absorbing at each of its points, then

$$C = \{x \in E : p_C(x) < 1\}.$$

Our second lemma is yet another variation of the theme that boundedness implies continuity:

Lemma 1.3.5. *Let E be a topological vector space, and let U be a neighborhood of zero. Then any linear functional $\phi \colon E \to \mathbb{F}$ with $\sup\{|\langle x, \phi\rangle| : x \in U\} < \infty$ is continuous.*

Proof. Let $\epsilon > 0$, and set $C := \sup\{|\langle x, \phi\rangle| : x \in U\}$. It follows that $\left\{\frac{\epsilon}{C+1}x : x \in U\right\}$ is a neighborhood of zero contained in $\phi^{-1}((-\epsilon, \epsilon))$, so that $\phi^{-1}((-\epsilon, \epsilon))$ is a neighborhood of 0. □

As for normed spaces, we shall, from now on, use the symbol E^* for the collection of all continuous linear functionals on a locally convex space E.

Proposition 1.3.6. *Let E be a topological vector space, and let $U, C \subset E$ be non-empty and convex with U open such that $U \cap C = \varnothing$. Then there are $\phi \in E^*$ and $c \in \mathbb{R}$ such that*

$$\mathrm{Re}\langle x, \phi\rangle < c \leq \mathrm{Re}\langle y, \phi\rangle \qquad (x \in U,\ y \in C).$$

Proof. Suppose first that $\mathbb{F} = \mathbb{R}$.

Fix $x_0 \in U$, $y_0 \in C$, set $z_0 := y_0 - x_0$, and define

$$V := U - C + z_0 = \{x - y + z_0 : x \in U,\ y \in C\}.$$

Then V is open and convex and contains 0, so that p_V is a sublinear functional on E with $V = \{x \in E : p_V(x) < 1\}$ by Lemma 1.3.4. As $U \cap C = \varnothing$, we have $z_0 \notin V$, so that $p_V(z_0) \geq 1$. Set $F := \mathbb{R}\, z_0$, and define $\psi\colon F \to \mathbb{R}$ by letting $\langle tz_0, \psi \rangle = t$ for $t \in \mathbb{R}$, so that

$$\langle tz_0, \psi \rangle = \begin{cases} t \leq t\, p_V(z_0) = p_V(tz_0), & t \geq 0, \\ t < 0 \leq p_V(tz_0), & t < 0. \end{cases}$$

Use the Hahn–Banach Theorem to find $\phi\colon E \to \mathbb{R}$ with

$$\phi|_F = \psi \qquad \text{and} \qquad \langle x, \phi \rangle \leq p_V(x) \quad (x \in E).$$

Let $x \in U$ and $y \in C$. It follows that

$$\langle x, \phi \rangle - \langle y, \phi \rangle + 1 = \langle x - y + z_0, \phi \rangle \leq p_V(\underbrace{x - y + z_0}_{\in V}) < 1.$$

It follows that

$$\langle x, \phi \rangle < \langle y, \phi \rangle \qquad (x \in U,\, y \in C).$$

As $\langle x, \phi \rangle < 1$ for $x \in V$, we have $\langle x, \phi \rangle > -1$ for $x \in -V$, so that $|\langle x, \phi \rangle| < 1$ for $x \in V \cap (-V)$. By Lemma 1.3.5, this means that $\phi \in E^*$.

The subsets $\phi(U)$ and $\phi(C)$ of $\mathbb{R}$ are convex and disjoint. Set $c := \sup_{x \in U} \langle x, \phi \rangle$, so that

$$\langle x, \phi \rangle \leq c \leq \langle y, \phi \rangle \qquad (x \in U,\, y \in C).$$

Assume towards a contradiction that there is $x \in U$ with $\langle x, \phi \rangle = c$. As U is open, there $\epsilon > 0$ such that $x + tz_0 \in U$ for all $t \in \mathbb{R}$ with $|t| < \epsilon$. It follows that

$$c \geq \left\langle x + \frac{\epsilon}{2} z_0, \phi \right\rangle = \langle x, \phi \rangle + \frac{\epsilon}{2} = c + \frac{\epsilon}{2},$$

which is a contradiction.

We have thus established the claim in the case where $\mathbb{F} = \mathbb{R}$. Suppose now that $\mathbb{F} = \mathbb{C}$. By the $\mathbb{F} = \mathbb{R}$ case, there is a continuous, $\mathbb{R}$-linear $\phi_{\mathbb{R}}\colon E \to \mathbb{R}$ such that there is $c \in \mathbb{R}$ with

$$\langle x, \phi_{\mathbb{R}} \rangle < c \leq \langle y, \phi_{\mathbb{R}} \rangle \qquad (x \in U,\, y \in C).$$

Construct $\phi\colon E \to \mathbb{C}$ from $\phi_{\mathbb{R}}$ as in (1.5). $\square$

Lemma 1.3.7. *Let E be a locally convex vector space, and let $F, K \subset E$ be non-empty and convex with F and K compact such that $K \cap F = \varnothing$. Then there is an open, balanced, convex neighborhood C of 0 such that $(K + C) \cap F = \varnothing$.*

Proof. For each $x \in K$, the set $F - x$ is closed and does not contain 0. Consequently, $E \setminus (F - x)$ is an open neighborhood of 0. As E is locally convex, there is an open, balanced, convex neighborhood C_x of 0 with $C_x \subset E \setminus (F - x)$, i.e., $C_x \cap (F - x) = \varnothing$ and therefore $(x + C_x) \cap F = \varnothing$.

The family $\left\{x + \frac{1}{2}C_x : x \in K\right\}$ is an open cover for K. As K is compact, there are $x_1, \ldots, x_n \in K$ such that $K \subset \bigcup_{j=1}^n \left(x_j + \frac{1}{2}C_{x_j}\right)$. Set $C := \bigcap_{j=1}^n \frac{1}{2}C_{x_j}$, so that C is an open, balanced, convex neighborhood of 0. It follows that

$$\begin{aligned} K + C &\subset \bigcup_{j=1}^n \left(x_j + \frac{1}{2}C_{x_j} + C\right) \\ &\subset \bigcup_{j=1}^n \left(x_j + \frac{1}{2}C_{x_j} + \frac{1}{2}C_{x_j}\right) \\ &\subset \bigcup_{j=1}^n (x_j + C_{x_j}) \\ &\subset E \setminus F, \end{aligned}$$

i.e., $(K + C) \cap F = \varnothing$. □

Theorem 1.3.8 (Hahn–Banach Separation Theorem)**.** *Let E be a locally convex vector space, and let $F, K \subset E$ be non-empty and convex with F and K compact such that $K \cap F = \varnothing$. Then there are $\phi \in E^*$ and $c_1, c_2 \in \mathbb{R}$ such that*

$$\operatorname{Re}\langle x, \phi\rangle \leq c_1 < c_2 \leq \operatorname{Re}\langle y, \phi\rangle \qquad (x \in K,\, y \in F).$$

Proof. By Lemma 1.3.7, there is an open, balanced, convex neighborhood C of 0 such that $(K + C) \cap F = \varnothing$. As $K + C$ is open and convex, Proposition 1.3.6 applies, so that there are $\phi \in E^*$ and $c \in \mathbb{R}$ with

$$\operatorname{Re}\langle x, \phi\rangle < c \leq \operatorname{Re}\langle y, \phi\rangle \qquad (x \in K + C,\, y \in F).$$

Set

$$c_1 := \sup_{x \in K} \operatorname{Re}\langle x, \phi\rangle \qquad \text{and} \qquad c_2 := \inf_{y \in F} \operatorname{Re}\langle y, \phi\rangle.$$

It follows that

$$c \leq c_2 \leq \operatorname{Re}\langle y, \phi\rangle \qquad (y \in E). \tag{1.6}$$

Also, as K is compact, there is $x_0 \in K$ such that $\operatorname{Re}\langle x_0, \phi\rangle = c_1$, so that

$$\operatorname{Re}\langle x, \phi\rangle \leq c_1 = \operatorname{Re}\langle x_0, \phi\rangle < c \qquad (x \in K). \tag{1.7}$$

Together, (1.6) and (1.7) prove the claim. □

Loosely speaking, the Hahn–Banach Separation Theorem assert that two non-empty, closed, disjoint subsets of a locally convex vector space, of which one is compact, can be separated by a hyperplane.

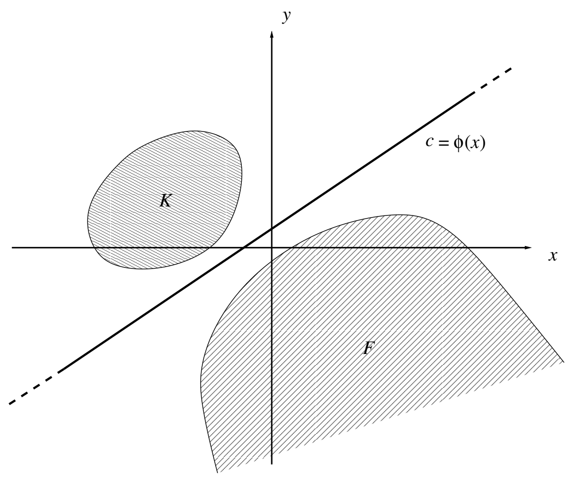

Figure 1.1: Separating convex subsetsets of $\mathbb{R}^2$ by a line

Corollary 1.3.9. *Let E be a locally convex vector space, and let $\varnothing \neq S \subset E$ be arbitrary. Then, for each $x_0 \in E \setminus \overline{\operatorname{conv} S}$, there are $\phi \in E^*$ and $c \in \mathbb{R}$ such that*

$$\operatorname{Re}\langle x_0, \phi \rangle < c \leq \operatorname{Re}\langle x, \phi \rangle \qquad (x \in S).$$

Corollary 1.3.10. *Let E be a locally convex vector space, and let F be a closed subspace of E. Then, for each $x_0 \in E \setminus F$, there is $\phi \in E^*$ such that*

$$\langle x_0, \phi \rangle = 1 \qquad \text{and} \qquad \phi|_F \equiv 0.$$

Proof. By the Corollary 1.3.9, there is $\phi \in E^*$ such that

$$\operatorname{Re}\langle x_0, \phi \rangle < \operatorname{Re}\langle x, \phi \rangle \qquad (x \in F). \tag{1.8}$$

Assume towards a contradiction that there is $x \in F$ with $\operatorname{Re}\langle x, \phi \rangle \neq 0$. Choose $t \in \mathbb{R}$, such that

$$\operatorname{Re}\langle tx, \phi \rangle = t \operatorname{Re}\langle x, \phi \rangle \leq \operatorname{Re}\langle x_0, \phi \rangle.$$

As $tx \in F$ as well, this contradicts (1.8). It follows that $\operatorname{Re} \phi|_F \equiv 0$. If $\mathbb{F} = \mathbb{R}$, this means $\phi|_F \equiv 0$. For $\mathbb{F} = \mathbb{C}$, note that

$$\langle x, \phi \rangle = \operatorname{Re}\langle x, \phi \rangle - i \operatorname{Re}\langle ix, \phi \rangle \qquad (x \in E),$$

so that $\phi|_F \equiv 0$ in this case as well. Finally, since $0 \in F$, we have $\operatorname{Re}\langle x_0, \phi \rangle < 0$ and therefore $\langle x_0, \phi \rangle \neq 0$. Replacing ϕ by $\langle x_0, \phi \rangle^{-1} \phi$, we obtain $\phi \in E^*$ with the desired properties. $\square$

Corollaries 1.3.9 and 1.3.10 can be used to prove approximation theorems. As an application, we will use Corollary 1.3.9 to prove the Bipolar Theorem.

Definition 1.3.11. We call $(E, F, \langle \cdot, \cdot \rangle)$ a *dual pairing* of vector spaces if E and F are vector spaces and $\langle \cdot, \cdot \rangle \colon E \times F \to \mathbb{F}$ is bilinear such that:

(a) for each $x \in E \setminus \{0\}$, there is $y \in F$ with $\langle x, y \rangle \neq 0$;

(b) for each $y \in F \setminus \{0\}$, there is $x \in E$ with $\langle x, y \rangle \neq 0$.

Example. Let E be a locally convex vector space. Then $(E, E^*, \langle \cdot, \cdot \rangle)$ is a dual pairing of vector spaces with $\langle \cdot, \cdot \rangle$ being the usual duality between E and E^*, i.e.,

$$\langle \cdot, \cdot \rangle \colon E \times E^* \to \mathbb{F}, \quad (x, \phi) \mapsto \langle x, \phi \rangle.$$

Given a dual pairing $(E, F, \langle \cdot, \cdot \rangle)$, we define a family $\{p_y : y \in F\}$ of seminorms on E by letting

$$p_y(x) = |\langle x, y \rangle| \qquad (x \in E).$$

These seminorms turn E into a locally convex space; the induced topology is denoted by $\sigma(E, F)$. In the same vein, we define a family $\{p_x : x \in E\}$ of seminorms on F by letting

$$p_x(y) = |\langle x, y \rangle| \qquad (y \in F),$$

which turns F into a locally convex space as well; the induced topology is denoted by $\sigma(F, E)$.

Lemma 1.3.12. *Let $(E, F, \langle \cdot, \cdot \rangle)$ be a dual pairing of vector spaces. Then a linear functional $\phi \colon E \to \mathbb{F}$ is $\sigma(E, F)$ continuous if and only if there is $y \in F$ such that*

$$\langle x, \phi \rangle = \langle x, y \rangle \qquad (x \in E). \tag{1.9}$$

Proof. Only the "only if" part needs proof.

Suppose that ϕ is $\sigma(E, F)$ continuous. By Problem 2 on Assignment #1, there are $C \geq 0$ and $y_1, \ldots, y_n \in F$ such that

$$|\langle x, \phi \rangle| \leq C \max_{j=1,\ldots,n} p_{y_j}(x) = C \max_{j=1,\ldots,n} |\langle x, y_j \rangle| \qquad (x \in E). \tag{1.10}$$

For $j = 1, \ldots, n$, define $\phi_j \colon E \to \mathbb{F}$ by letting

$$\langle x, \phi_j \rangle := \langle x, y_j \rangle \qquad (x \in E).$$

From (1.10), it is evident that $\bigcap_{j=1}^n \ker \phi_j \subset \ker \phi$. By Problem 3 on Assignment #1, this means that there are $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ such that $\phi = \sum_{j=1}^n \lambda_j \phi_j$. Set $y := \sum_{j=1}^n \lambda_j y_j$. Then (1.9) holds. $\square$

Remark. It is clear that $y \in F$ satisfying (1.9) is necessarily unique.

Definition 1.3.13. Let E be a vector space. Then:

(a) a subset of E is called *absolutely convex* if it is both convex and balanced;

(b) the *absolutely convex hull* absconv S of a set $S \subset E$ is the intersection of all absolutely convex subsets of E containing S.

Remarks. 1. A set $C \subset E$ is absolutely convex if and only if $\lambda_1 x_1 + \cdots \lambda_n x_n \in C$ whenever $x_1, \ldots, x_n \in C$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ with $\sum_{j=1}^n |\lambda_j| \leq 1$.

2. Given $S \subset E$, we have

$$\text{absconv } S = \left\{ \sum_{j=1}^n \lambda_j x_j : n \in \mathbb{N},\, x_1, \ldots, x_n \in S,\, \lambda_1, \ldots, \lambda_n \in \mathbb{F},\, \sum_{j=1}^n |\lambda_j| \leq 1 \right\}.$$

3. If E is a topological vector space and $C \subset E$ is absolutely convex, then $\overline{C}$ is absolutely convex as well.

Definition 1.3.14. Let $(E, F, \langle \cdot, \cdot \rangle)$ be a dual pairing of vector spaces, and let $S \subset E$ and $T \subset F$. Then:

(a) the *polar* of S in F is defined as

$$S^\circ := \{ y \in F : |\langle x, y \rangle| \leq 1 \text{ for all } x \in S \};$$

(b) the *polar* of T in E is defined as

$$T_\circ := \{ x \in E : |\langle x, y \rangle| \leq 1 \text{ for all } y \in T \}.$$

The following is immediate:

Proposition 1.3.15. *Let $(E, F, \langle \cdot, \cdot \rangle)$ be a dual pairing of vector spaces, and let $S \subset E$ and $T \subset F$. Then:*

(i) *S° is a $\sigma(E, F)$ closed, absolutely convex subset of F;*

(ii) *$T_\circ$ is a $\sigma(F, E)$ closed, absolutely convex subset of E.*

Theorem 1.3.16 (Bipolar Theorem). *Let $(E, F, \langle \cdot, \cdot \rangle)$ be a dual pair of vector spaces, and let $\varnothing \neq S \subset E$. Then*

$$\overline{\text{absconv } S}^{\sigma(E,F)} = (S^\circ)_\circ.$$

Proof. As $S \subset (S^\circ)_\circ$, and since $(S^\circ)_\circ$ is $\sigma(E, F)$ closed and absolutely convex, we have

$$\overline{\text{absconv } S}^{\sigma(E,F)} \subset (S^\circ)_\circ.$$

Assume towards a contradiction that there is $x_0 \in (S^\circ)_\circ \setminus \overline{\operatorname{absconv} S}^{\sigma(E,F)}$. From the Hahn–Banach Separation Theorem, we conclude that there are a $\sigma(E,F)$ continuous, linear functional $\phi\colon E \to \mathbb{F}$ as well as $c_1, c_2 \in \mathbb{R}$ such that

$$\operatorname{Re}\langle x, \phi\rangle \leq c_1 < c_2 \leq \operatorname{Re}\langle x_0, \phi\rangle \qquad (x \in \overline{\operatorname{absconv} S}).$$

(Apply Theorem 1.3.8 with $K = \{x_0\}$ and $F = \overline{\operatorname{absconv} S}$, and then replace ϕ, c_1, and c_2 with their respective negatives.) Making c_1 a little larger, we can suppose that

$$\operatorname{Re}\langle x, \phi\rangle < c_1 < c_2 \leq \operatorname{Re}\langle x_0, \phi\rangle \qquad (x \in \overline{\operatorname{absconv} S}). \tag{1.11}$$

As $0 \in \operatorname{absconv} S$, it is then clear that $c_1 > 0$. Without loss of generality, suppose that $c_1 = 1$. From (1.11), it follows that there is $\epsilon > 0$ such that

$$\operatorname{Re}\langle x, \phi\rangle \leq 1 < 1 + \epsilon \leq \operatorname{Re}\langle x_0, \phi\rangle \qquad (x \in \overline{\operatorname{absconv} S}).$$

By Lemma 1.3.12, there is $y \in F$ such that

$$\langle x, \phi\rangle = \langle x, y\rangle \qquad (x \in E).$$

Let $x \in S$, and choose $\lambda \in \mathbb{F}$ with $|\lambda| = 1$ such that $\langle x, \phi\rangle = \lambda|\langle x, \phi\rangle|$. It follows that $\lambda^{-1}x \in \operatorname{absconv} S$, so that

$$|\langle x, y\rangle| = |\langle x, \phi\rangle| = \langle \lambda^{-1}x, \phi\rangle = \operatorname{Re}\langle \lambda^{-1}x, \phi\rangle < 1$$

and therefore $y \in S^\circ$.

On the other hand, we have

$$|\langle x_0, y\rangle| = |\langle x_0, \phi\rangle| \geq \operatorname{Re}\langle x_0, \phi\rangle \geq 1 + \epsilon,$$

so that $x_0 \notin (S^\circ)_\circ$, which is a contradiction. □

Given a normed space E and $r > 0$, we denote by $\operatorname{ball}_r(E)$ and $\operatorname{Ball}_r(E)$ the open and the closed ball in E, respectively, with radius r centered at 0; if $r = 1$, we simply write $\operatorname{ball}(E)$ and $\operatorname{Ball}(E)$.

Corollary 1.3.17 (Goldstine's Theorem)**.** *Let E be a normed space. Then* $\operatorname{Ball}(E)$ *is* $\sigma(E^{**}, E^*)$ *dense in* $\operatorname{Ball}(E^{**})$.

Proof. Consider the dual pairing $(E^{**}, E^*, \langle\cdot,\cdot\rangle)$.

As $\operatorname{Ball}(E)$ is absolutely convex, the Bipolar Theorem yields

$$\overline{\operatorname{Ball}(E)}^{\sigma(E^{**},E^*)} = (\operatorname{Ball}(E)^\circ)_\circ.$$

Since

$$\operatorname{Ball}(E)^\circ = \{\phi \in E^* : |\langle x, \phi\rangle| \leq 1 \text{ for all } x \in \operatorname{Ball}(E)\} = \operatorname{Ball}(E^*)$$

and

$$(\operatorname{Ball}(E)^\circ)_\circ = \operatorname{Ball}(E^*)_\circ = \{X \in E^{**} : |\langle X, \phi\rangle| \leq 1 \text{ for all } \phi \in \operatorname{Ball}(E^*)\} = \operatorname{Ball}(E^{**}),$$

this yields the claim. □

1.4 The Krein–Milman Theorem

Definition 1.4.1. Let E be a vector space, and let $C \subset V$ be convex. Then:

(a) a convex subset C_0 of C is called *extremal* if $tx + (1-t)y \in C_0$ with $x, y \in C$ and $t \in (0,1)$ if and only if $x, y \in C_0$;

(b) $x \in C$ is called an *extreme point* of C if $\{x\}$ is an extremal subset of C.

The set of extreme points of C is denoted by ext C.

Examples. 1. Let E be a normed space.

Claim.

$$\text{ext Ball}(E) \subset \{x \in E : \|x\| = 1\}.$$

Proof. Let $x \in E$ be such that $\|x\| < 1$. If $x = 0$, pick any $y \in \text{Ball}(E) \setminus \{0\}$, and note that

$$0 = \frac{1}{2}y + \frac{1}{2}(-y).$$

If $x \neq 0$, set $t := \|x\|$ and $y := t^{-1}x$, so that

$$x = ty + (1-t)0.$$

In either case, $x \notin \text{ext Ball}(E)$. $\square$

2. Let $E = \mathbb{R}^N$ be equipped with the Euclidean norm $\|\cdot\|_2$.

Claim.

$$\text{ext Ball}(E) = \{x \in E : \|x\|_2 = 1\}.$$

Proof. The function

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^2$$

is convex, i.e.,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \qquad (x, y \in \mathbb{R},\ t \in [0,1]).$$

We first show that, if $x, y \in \mathbb{R}$ and $t \in (0,1)$ are such that

$$f(tx + (1-t)y) = tf(x) + (1-t)y,$$

then $x = y$. To see this, let $x < y$ and $t \in (0,1)$, and set $z := tx + (1-t)y$, so that $x < z < y$. By the Mean Value Theorem, there are $\xi \in (x,z)$ and $\eta \in (z,y)$ such that

$$\frac{f(z) - f(x)}{z - x} = 2\xi < 2\eta = \frac{f(y) - f(z)}{y - z}.$$

As $z - x = (1-t)(y-x)$ and $y - z = t(y-x)$, this implies

$$\frac{f(z)-f(x)}{1-t} < \frac{f(y)-f(x)}{t},$$

so that

$$f(z) < tf(x) + (1-t)f(y).$$

Let $x \in E$ be such that $\|x\|_2 = 1$, and let $y, z \in \mathrm{Ball}(E)$ and $t \in (0,1)$ be such that $x = ty + (1-t)y$. Since

$$1 = \|x\|_2 \leq t\|y\|_2 + (1-t)\|z\|_2,$$

it clear that $\|y\|_2 = \|z\|_2 = 1$ as well. Assume that $y \neq z$. Then there is at least one $j_0 \in \{1, \ldots, N\}$ such that $y_{j_0} \neq z_{j_0}$. It follows that

$$\begin{aligned} 1 &= \|x\|_2^2 \\ &= \|ty + (1-t)z\|_2^2 \\ &= \sum_{j=1}^{n} (ty_j + (1-t)z_j)^2 \\ &< \sum_{j=1}^{n} ty_j^2 + (1-t)z_j^2 \\ &= t\|y\|_2^2 + (1-t)\|z\|_2^2 \\ &= 1, \end{aligned}$$

which is a contradiction; consequently, $x \in \mathrm{ext}\ \mathrm{Ball}(E)$. □

3. Let $E = \mathbb{R}^N$ be equipped with the ℓ^1-norm $\|\cdot\|_1$, i.e.,

$$\|x\|_1 := \sum_{j=1}^{N} |x_j| \qquad (x \in \mathbb{R}^N).$$

Claim.

$$\mathrm{ext}\ \mathrm{Ball}(E) = \{\epsilon \boldsymbol{e}_j : \epsilon \in \{-1, 1\},\ j = 1, \ldots, N\}$$

where $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_N$ are the canonical basis vectors of $\mathbb{R}^N$.

Proof. We first show that $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_N \in \mathrm{ext}\ \mathrm{Ball}(E)$. Fix $k \in \{1, \ldots, N\}$. Let $x = (x_1, \ldots, x_N), y = (y_1, \ldots, y_N) \in \mathrm{Ball}(E)$ and $t \in (0,1)$ be such that $\boldsymbol{e}_k = tx + (1-t)y$. Since $x_k, y_k \in [-1, 1]$ and $1 = tx_k + (1-t)y_k$, it follows that $x_k = y_k = 1$, and since $\|x\|_1, \|y\|_1 \leq 1$, this is possible only if $x_j = y_j = 0$ for $j \neq k$, i.e., if $x = y = \boldsymbol{e}_k$.

As $x \in \mathrm{ext}\ \mathrm{Ball}(E)$ implies $-x \in \mathrm{Ball}(E)$, this means that

$$\mathrm{ext}\ \mathrm{Ball}(E) \supset \{\epsilon \boldsymbol{e}_j : \epsilon \in \{-1, 1\},\ j = 1, \ldots, N\}$$

Conversely, let $x \in \text{ext Ball}(E)$, and assume that $x \notin \{\epsilon \boldsymbol{e}_j : \epsilon \in \{-1, 1\}, j = 1, \ldots, N\}$. Set

$$m := \max\{j = 1, \ldots, N - 1 : x_{j+1} \neq 0\}.$$

As $x \notin \{\epsilon \boldsymbol{e}_j : \epsilon \in \{-1, 1\}, j = 1, \ldots, N\}$, there must be $j_0 \in \{1, \ldots, m\}$ with $x_{j_0} \neq 0$, so that

$$0 < \sum_{j=1}^{m} |x_j| < 1 \qquad \text{and} \qquad \sum_{j=1}^{m+1} |x_j| = 1.$$

Set $t := \sum_{j=1}^{m} |x_j|$, and let

$$y := t^{-1}(x_1, \ldots, x_m, 0, \ldots, 0)$$

and

$$z := (1 - t)^{-1}(\underbrace{0, \ldots, 0}_{m \text{ times}}, x_{m+1}, 0, \ldots, 0).$$

It follows that $\|y\|_1 = \|z\|_1 = 1$, $y \neq z$, and $x = ty + (1 - t)z$, which contradicts $x \in \text{ext Ball}(E)$. $\square$

4. Let $E = L^1([0, 1])$ be equipped with the L^1-norm, i.e.,

$$\|f\|_1 := \int_0^1 |f(t)|\, dt \qquad (f \in L^1([0, 1])).$$

Claim.

$$\text{ext Ball}(E) = \varnothing.$$

Proof. Assume that there is $f \in \text{ext Ball}(E)$; then, in particular, $\|f\|_1 = 1$.

By the Intermediate Value Theorem, there is $x \in (0, 1)$ such that $\int_0^x |f(t)|\, dt = \frac{1}{2}$. Define

$$g \colon [0, 1] \to \mathbb{F}, \quad t \mapsto \begin{cases} 2f(t), & t \leq x, \\ 0, & t > x, \end{cases}$$

and

$$h \colon [0, 1] \to \mathbb{F}, \quad t \mapsto \begin{cases} 0, & t \leq x, \\ 2f(t), & t > x, \end{cases}$$

so that $f \in L^1([0, 1])$ with $\|f\|_1 = \|g\|_1 = 1$ and $f = \frac{1}{2}g + \frac{1}{2}h$. This contradicts $f \in \text{ext Ball}(E)$. $\square$

Our last example shows that the closed unit ball of a normed space need not have extremal points. Our next theorem will guarantee the existence of extremal points of non-empty, compact, convex subsets of locally convex spaces.

Lemma 1.4.2. *Let E be a locally convex vector space, let $\varnothing \neq K \subset E$ be compact and convex, let $\phi \in E^*$, and set $M := \sup_{x \in K} \operatorname{Re}\langle x, \phi \rangle$. Then*

$$K_\phi := \{x \in K : \operatorname{Re}\langle x, \phi \rangle = M\}$$

is a non-empty, compact, convex, and extremal subset of K.

Proof. It is straightforward that K_ϕ is non-empty, convex, and closed in K (and therefore compact).

Let $x, y \in K$ and $t \in (0,1)$ be such that $tx + (1-t)y \in K_\phi$. It follows that

$$\begin{aligned} M &= tM + (1-t)M \\ &\geq t \operatorname{Re}\langle x, \phi \rangle + (t - t) \operatorname{Re}\langle y, \phi \rangle \\ &= \operatorname{Re}\langle tx + (1-t)y, \phi \rangle \\ &= M. \end{aligned}$$

This is possible only if $\operatorname{Re}\langle x, \phi \rangle = \operatorname{Re}\langle y, \phi \rangle = M$, i.e., $x, y \in K_\phi$. $\square$

Theorem 1.4.3 (Krein–Milman Theorem)**.** *Let E be a locally convex vector space, and let $\varnothing \neq K \subset E$ be compact and convex. Then $K = \overline{\operatorname{conv}}(\operatorname{ext} K)$; in particular, $\operatorname{ext} K \neq \varnothing$.*

Proof. Let

$$\mathcal{K} := \{\varnothing \neq C \subset K : C \text{ is compact, convex and extremal in } K\}.$$

By Lemma 1.4.2, $\mathcal{K} \neq \varnothing$.

Fix $C_0 \in \mathcal{K}$, and let

$$\mathcal{K}_0 := \{C \in \mathcal{K} : C \subset C_0\}.,$$

Let $\mathcal{K}_0$ be ordered by set inclusion. We will use Zorn's Lemma to show that $\mathcal{K}_0$ has minimal elements. Let $\mathcal{L}_0 \subset \mathcal{K}_0$ be totally ordered. We claim that $\bigcap\{C : C \in \mathcal{L}_0\} \in \mathcal{K}_0$. Assume first that $\bigcap\{C : C \in \mathcal{L}_0\} = \varnothing$. As K is compact, this means that there are $C_1, \ldots, C_n \in \mathcal{L}_0$ such that $C_1 \cap \cdots \cap C_n = \varnothing$. As $\mathcal{L}_0$ is totally ordered, we can suppose without loss of generality that $C_1 \subset \cdots \subset C_n$, so that $C_1 \cap \cdots \cap C_n = C_1 \neq \varnothing$, which is a contradiction. It follows that $\bigcap\{C : C \in \mathcal{L}_0\} \in \mathcal{K}_0 \neq \varnothing$. It is straightforward to check that $\bigcap\{C : C \in \mathcal{L}_0\} \subset C_0$ is compact, convex, and extremal in K and therefore belongs to $\mathcal{K}_0$. Zorn's Lemma therefore yields a minimal element $C_{\min}$ of $\mathcal{K}_0$. Let $x \in C_{\min}$ and assume that there is $y \in C_{\min} \setminus \{x\}$. By the Hahn–Banach Separation Theorem, there is $\phi \in E^*$ with $\operatorname{Re}\langle y, \phi \rangle < \operatorname{Re}\langle x, \phi \rangle$; in particular, $y \notin (C_{\min})_\phi$, so that $(C_{\min})_\phi \subsetneq C_{\min}$. By Lemma 1.4.2, $(C_{\min})_\phi$ is an extremal subset of $C_{\min}$, and since $C_{\min}$ is an extremal subset of K, $(C_{\min})_\phi$ is also an extremal subset of K. This contradicts the minimality of $C_{\min}$.

We conclude that $C_{\min} = \{x\}$, so that $x \in \text{ext } K$. In particular, we have $C_0 \cap \text{ext } K \neq \varnothing$. As $C_0 \in \mathcal{K}$ is arbitrary, this means that

$$C \cap \text{ext } K \neq \varnothing \qquad (C \in \mathcal{K}) \tag{1.12}$$

Set $\tilde{K} := \overline{\text{conv}}(\text{ext } K)$. Then $\varnothing \neq \tilde{K} \subset K$ is compact and convex. Assume that there is $x_0 \in K \setminus \tilde{K}$. By Corollary 1.3.9, there are $\phi \in E^*$ and $c \in \mathbb{R}$ such that

$$\text{Re}\langle x, \phi \rangle \leq c < \text{Re}\langle x_0, \phi \rangle \qquad (x \in \tilde{K}). \tag{1.13}$$

By Lemma 1.4.2, $K_\phi \in \mathcal{K}$, and by (1.13), $K_\phi \cap \text{ext } K = \varnothing$, but this contradicts (1.12). $\square$

Corollary 1.4.4. *Let E be a finite dimensional vector space. Then we have* $\text{Ball}(E) = \overline{\text{conv}}(\text{ext Ball}(E))$*; in particular,* $\text{ext Ball}(E) \neq \varnothing$.

Lemma 1.4.5. *Let E be a topological vector space, and let $K_1, \ldots, K_n \subset E$ be non-empty, compact, and convex. Then*

$$\text{conv}(K_1 \cup \cdots \cup K_n) = \left\{ \sum_{j=1}^n t_j x_j : x_1 \in K_1, \ldots, x_n \in K_n,\, t_1, \ldots, t_n \geq 0,\, \sum_{j=1}^n t_j = 1 \right\} \tag{1.14}$$

is compact; in particular,

$$\text{conv}(K_1 \cup \cdots \cup K_n) = \overline{\text{conv}}(K_1 \cup \cdots \cup K_n).$$

Proof. It is straightforward that $\text{conv}(K_1 \cup \cdots \cup K_n)$ is indeed of the form given in (1.14) (compare the proof of Proposition 1.2.8).

Let

$$I := \{(t_1, \ldots, t_n) \in \mathbb{R}^n : t_1, \ldots, t_n \geq 0,\, t_1 + \cdots + t_n = 1\}$$

Then I is a closed subset of $[0,1]^n$ and therefore compact. Consequently, $I \times K_1 \times \cdots \times K_n \subset \mathbb{R}^n \times E^n$ is compact with respect to the product topology. The map

$$\Phi \colon \mathbb{R}^n \times E^n \to E, \quad ((t_1, \ldots, t_n), x_1, \ldots, x_n) \mapsto \sum_{j=1}^n t_j x_j$$

is continuous, so that

$$\Phi(I \times K_1 \times \cdots \times K_n) = \text{conv}(K_1 \cup \cdots \cup K_n)$$

is compact. As compact subsets of Hausdorff spaces are always closed, this implies $\text{conv}(K_1 \cup \cdots \cup K_n) = \overline{\text{conv}}(K_1 \cup \cdots \cup K_n)$. $\square$

Theorem 1.4.6. *Let E be a locally convex vector space, let $\varnothing \neq K \subset E$ be compact and convex, and let $S \subset K$ be such that $K = \overline{\text{conv}}\, S$. Then* $\text{ext } K \subset \overline{S}$.

Proof. Without loss of generality suppose that S is closed.

Assume that there is $x_0 \in (\text{ext } K) \setminus S$. Then there is an open, balanced, convex neighborhood C of 0 such that $(x_0 + C) \cap S = \varnothing$. Set

$$U := \left\{ x \in E : p_C(x) < \frac{1}{3} \right\};$$

it follows that $(x_0 + U) \cap (S + U) = \varnothing$ and, in particular, $x_0 \notin \overline{S + U}$. Since S is closed in K and therefore compact itself, there are $x_1, \ldots, x_n \in S$ such that $S \subset \bigcup_{j=1}^n (x_j + U)$. For $j = 1, \ldots, n$, set

$$K_j := \overline{\text{conv}}(S \cap (x_j + U)).$$

It follows that $K_1, \ldots, K_n$ are compact with

$$S \subset K_1 \cup \cdots \cup K_n \subset K,$$

so that

$$K = \overline{\text{conv}}\, S = \overline{\text{conv}}(K_1 \cup \cdots \cup K_n) = \text{conv}(K_1 \cup \cdots \cup K_n)$$

where the last equality is due to Lemma 1.4.5. By (1.14) there are therefore $y_1 \in K_1, \ldots, y_n \in K_n$ and $t_1, \ldots, t_n \geq 0$ with $\sum_{j=1}^n t_j = 1$ such that

$$x_0 = t_1 y_1 + \cdots + t_n y_n.$$

As $x_0 \in \text{ext } K$, there must be some $k \in \{1, \ldots, n\}$ such that $x_0 = y_k$. It follows that

$$x_0 \in K_k \subset x_k + \overline{U} \subset \overline{S + U},$$

which is a contradiction. $\square$

Chapter 2

Weak and Weak* Topologies

2.1 The Weak* Topology on the Dual of a Normed Space

Theorem 2.1.1 (Alaoğlu–Bourbaki Theorem). *Let E be a normed space. Then* $\operatorname{Ball}(E^*)$ *is $\sigma(E^*, E)$ compact.*

Proof. For $x \in E$, set

$$K_x := \{\lambda \in \mathbb{F} : |\lambda| \leq \|x\|\},$$

so that K_x is compact. Let

$$K := \prod_{x \in E} K_x$$

be equipped with the product topology; then K is compact by Tychonoff's Theorem. Define

$$\iota\colon \operatorname{Ball}(E^*) \to K, \quad \phi \mapsto (\langle x, \phi \rangle)_{x \in E};$$

it is obvious that ι is injective.

We claim that ι is a homeomorphism onto its range.

Let $(\phi_\alpha)_\alpha$ be a net in $\operatorname{Ball}(E^*)$ such that $\phi_\alpha \overset{\sigma(E^*,E)}{\longrightarrow} \phi \in \operatorname{Ball}(E^*)$, i.e.,

$$\langle x, \phi_\alpha \rangle \to \langle x, \phi \rangle \qquad (x \in E).$$

From the definition of the product topology on K, this means that $\iota(\phi_\alpha) \to \iota(\phi)$.

Conversely, let $(\phi_\alpha)_\alpha$ be a net in $\operatorname{Ball}(E^*)$, and let $\phi \in \operatorname{Ball}(E^*)$ be such that $\iota(\phi_\alpha) \to \iota(\phi)$. By the definition of the product topology on K, this means that

$$\langle x, \phi_\alpha \rangle \to \langle x, \phi \rangle \qquad (x \in E),$$

i.e., $\phi_\alpha \overset{\sigma(E^*,E)}{\longrightarrow} \phi$.

This proves the claim.

Given the compactness of K, it is therefore sufficient to show that $\iota(\mathrm{Ball}(E^*))$ is closed in K.

Let $(\phi_\alpha)_\alpha$ be a net in $\mathrm{Ball}(E^*)$, and let $\phi \in K$ be such that $\iota(\phi_\alpha) \to \phi$. By the definition of K, we have $\phi\colon E \to \mathbb{F}$ with $|\phi(x)| \leq \|x\|$. We need to show that ϕ is linear. Let $x, y \in E$, let $\lambda \in \mathbb{F}$, and note that

$$\phi(x+y) = \lim_\alpha \langle x+y, \phi_\alpha \rangle = \lim_\alpha (\langle x, \phi_\alpha \rangle + \langle y, \phi_\alpha \rangle) = \phi(x) + \phi(y)$$

and

$$\phi(\lambda x) = \lim_\alpha \langle \lambda x, \phi_\alpha \rangle = \lambda \lim_\alpha \langle x, \phi_\alpha \rangle = \lambda \phi(x).$$

This proves the linearity of ϕ, i.e., $\phi \in \iota(\mathrm{Ball}(E^*))$. □

Of course, the Alaoğlu–Bourbaki Theorem remains true if we replace $\mathrm{Ball}(E^*)$ by $\mathrm{Ball}_r(E^*)$ for any $r > 0$. This implies the following:

Corollary 2.1.2. *Let E be a normed space, and let $\varnothing \neq K \subset E^*$. Then the following are equivalent:*

(i) *K is $\sigma(E^*, E)$ compact;*

(ii) *K is norm bounded and $\sigma(E^*, E)$ closed.*

Proof. (i) $\Longrightarrow$ (ii): For $x \in E$, the $\sigma(E^*, E)$ compactness of K yields $\sup_{\phi \in K} |\langle x, \phi \rangle| < \infty$. The Uniform Boundedness Principle entails that K is norm bounded. Also, as $\sigma(E^*, E)$ is a Hausdorff topology, the $\sigma(E^*, E)$ compactness of K entails that K is $\sigma(E^*, E)$ closed.

(ii) $\Longrightarrow$ (i): As K is norm bounded, there is $r > 0$ such that $K \subset \mathrm{Ball}_r(E^*)$. By the Alaoğlu–Bourbaki Theorem, $\mathrm{Ball}_r(E^*)$ is $\sigma(E^*, E)$ compact, and so is its $\sigma(E^*, E)$ closed subset K. □

Corollary 2.1.3. *The following are equivalent for a normed space E:*

(i) $\dim E < \infty$*;*

(ii) *$\sigma(E^*, E)$ and the norm topology coincide.*

Corollary 2.1.4. *There is no normed space E such that $E^* = L^1([0,1])$*

Proof. Assume that there is such a space E. Then $\mathrm{Ball}(E^*)$ is $\sigma(E^*, E)$ compact by the Alaoğlu–Bourbaki Theorem and therefore has extremal points by the Krein–Milman Theorem. However, we previously saw that $\mathrm{ext}\,\mathrm{Ball}(L^1([0,1])) = \varnothing$. □

Proposition 2.1.5. . *The following are equivalent for a Banach space E:*

(i) *E is reflexive;*

(ii) *E^* is reflexive;*

(iii) *$\sigma(E^*, E) = \sigma(E^*, E^{**})$;*

(iv) *$\mathrm{Ball}(E)$ is $\sigma(E, E^*)$ compact.*

Proof. (i) $\Longrightarrow$ (iii) is obvious because in this case $E^{**} = E$.

(iv) $\Longrightarrow$ (i): As $\sigma(E^{**}, E^*)|_E = \sigma(E, E^*)$, the $\sigma(E, E^*)$ compactness of $\mathrm{Ball}(E)$ implies that $\mathrm{Ball}(E)$ is also $\sigma(E^{**}, E^*)$ compact and therefore, in particular, $\sigma(E^{**}, E^*)$ closed in E^{**}, i.e.,

$$\overline{\mathrm{Ball}(E)}^{\sigma(E^{**},E^*)} = \mathrm{Ball}(E).$$

On the other hand, Goldstine's Theorem asserts that

$$\overline{\mathrm{Ball}(E)}^{\sigma(E^{**},E^*)} = \mathrm{Ball}(E^{**}).$$

This means that $E^{**} = E$.

(iii) $\Longrightarrow$ (ii): By the Alaoğlu–Bourbaki Theorem, $\mathrm{Ball}(E^*)$ is $\sigma(E^*, E)$ compact and thus $\sigma(E^*, E^{**})$ compact. As in the proof of (iv) $\Longrightarrow$ (i), we see that E^* is reflexive.

(ii) $\Longrightarrow$ (i): Clearly, $\mathrm{Ball}(E)$ is norm closed in E^{**} and therefore $\sigma(E^{**}, E^{***})$ closed. As $E^{***} = E^*$, this means that $\mathrm{Ball}(E)$ is $\sigma(E^{**}, E)$ closed in E^{**}. As the in proof of (iv) $\Longrightarrow$ (i), Goldstine's Theorem yields $E^{**} = E$.

Finally, (i) $\Longrightarrow$ (iv) is obvious. $\square$

Corollary 2.1.6. *Let E be a reflexive Banach space, and let F be a closed subspace of E. Then F and E/F are reflexive.*

Proof. First, note that $\mathrm{Ball}(F)$ is $\sigma(E, E^*)$ closed in E and thus $\sigma(E, E^*)$ compact. By the Hahn–Banach Theorem, $\sigma(E, E^*)|_F = \sigma(F, F^*)$. Consequently $\mathrm{Ball}(F)$ is $\sigma(F, F^*)$ is compact, and F is reflexive by Proposition 2.1.5.

To prove the reflexivity, of E/F, note that $(E/F)^*$ is isometrically isomorphic to the subspace F° of E^*. As E is reflexive, so is E^* by Proposition 2.1.5, and so is its subspace F° by the foregoing. Consequently, $(E/F)^*$ is reflexive as is E/F. $\square$

Definition 2.1.7. Let E be a normed space. A sequence $(x_n)_{n=1}^\infty$ in E is called a *weak Cauchy sequence* if $(\langle x_n, \phi \rangle)_{n=1}^\infty$ is a Cauchy sequence for each $\phi \in E^*$. If every weak Cauchy sequence in E converges *weakly* in E, i.e., with respect to $\sigma(E, E^*)$, we call E *weakly sequentially complete.*

Corollary 2.1.8. *Let E be a reflexive Banach space. Then E is weakly sequentially complete.*

Proof. Let $(x_n)_{n=1}^\infty$ be a weak Cauchy sequence in E. In particular, $(\langle x_n, \phi\rangle)_{n=1}^\infty$ is bounded for each $x \in E$. By the Uniform Boundedness Principle, this means that $(x_n)_{n=1}^\infty$ is norm bounded in E and therefore contained in some closed ball centered at 0. As this ball is $\sigma(E, E^*)$ compact, there are $x \in E$ and a sub*net* $(x_{n_\alpha})_\alpha$ of $(x_n)_{n=1}^\infty$ such that $x_{n_\alpha} \overset{\sigma(E,E^*)}{\longrightarrow} x$. For each $\phi \in E^*$, the sequence $(\langle x_n, \phi\rangle)_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{F}$ and therefore convergent. It follows that

$$\lim_{n\to\infty} \langle x_n, \phi\rangle = \lim_\alpha \langle x_{n_\alpha}, \phi\rangle = \langle x, \phi\rangle \qquad (\phi \in E^*),$$

i.e., $x_n \overset{\sigma(E,E^*)}{\longrightarrow} x$. $\square$

Corollary 2.1.9. *Let E be a reflexive Banach space, and let F be a closed subspace of E. Then, for each $x \in E$, there is $y \in F$ such that $\|x + y\| = \|x + F\|$.*

Proof. For each $n \in \mathbb{N}$, let $y_n \in F$ be such that

$$\|x + y_n\| \leq \|x + F\| + \frac{1}{n},$$

so that

$$\|y_n\| \leq \|x\| + \|x + y_n\| \leq 2\|x\| + \frac{1}{n}.$$

It follows that $(y_n)_{n=1}^\infty$ is bounded. Let $(y_{n_\alpha})_\alpha$ be a subset of $(y_n)_{n=1}^\infty$ that converges to some $y \in E$ with respect to $\sigma(E, E^*)$. By the Hahn–Banach Theorem, there is $\phi \in E^*$ with $\|\phi\| = 1$ and $\langle x + y, \phi\rangle = \|x + y\|$. It follows that

$$\begin{aligned}
\|x + y\| &= \langle x + y, \phi\rangle \\
&= \lim_\alpha \langle x + y_{n_\alpha}, \phi\rangle \\
&\leq \limsup_\alpha \|x + y_{n_\alpha}\| \\
&\leq \limsup_\alpha \|x + F\| + \frac{1}{n_\alpha} \\
&= \|x + F\|.
\end{aligned}$$

The reversed inequality holds trivially. $\square$

Our next goal is to prove the Krein–Šmulian Theorem:

Theorem 2.1.10 (Krein–Šmulian Theorem)**.** *Let E be a Banach space, and let $\varnothing \neq C \subset E^*$ be convex. Then the following are equivalent:*

(i) *C is $\sigma(E^*, E)$ closed;*

(ii) *$\mathrm{Ball}_r(E^*) \cap C$ is $\sigma(E^*, E)$ closed for each $r > 0$.*

As $\mathrm{Ball}_r(E^*)$ is $\sigma(E^*, E)$ compact by the Alaoğlu–Bourbaki Theorem, (i) $\Longrightarrow$ (ii) is clear.

To prove (ii) $\Longrightarrow$ (i), we first prove two Lemmas.

Lemma 2.1.11. *Let E be a normed space, let $r > 0$. Then*

$$\bigcap\{F^\circ : F \textit{ is a finite subset of } \mathrm{Ball}_{r^{-1}}(E)\} = \mathrm{Ball}_r(E^*). \tag{2.1}$$

Proof. It is straightforward that $\mathrm{Ball}_r(E^*)$ is contained in the left hand side of (2.1)

For the converse inclusion, let $\phi \in E^*$ be such that $\|\phi\| > r$. Then there is $x \in \mathrm{Ball}(E)$ with $|\langle x, \phi\rangle| > r$, so that $|\langle r^{-1}x, \phi\rangle| > 1$, i.e., $\phi \notin \{r^{-1}x\}^\circ$. □

Lemma 2.1.12. *Let E be a Banach space, let $\varnothing \neq C \subset E^*$ be convex such that $\mathrm{Ball}_r(E^*) \cap C$ is $\sigma(E^*, E)$ closed for each $r > 0$, and suppose that $\mathrm{Ball}(E^*) \cap C = \varnothing$. Then there is $x \in E$ such that*

$$\mathrm{Re}\langle x, \phi\rangle \geq 1 \qquad (\phi \in C).$$

Proof. Inductively, we define a sequence $F_0, F_1, F_2, \ldots$ of finite subsets of E such that

(a) $nF_n \subset \mathrm{Ball}(E)$, and

(b) $\mathrm{Ball}_n(E^*) \cap \bigcap_{k=1}^{n-1} F_k^\circ \cap C = \varnothing$

for all $n \in \mathbb{N}_0$.

Set $F_0 = \{0\}$.

Suppose that $F_0, F_1, \ldots, F_{n-1}$ have already been chosen satsifying (a) and (b). Set

$$Q := \mathrm{Ball}_{n+1}(E^*) \cap \bigcap_{k=1}^{n-1} F_k^\circ \cap C,$$

so that Q is $\sigma(E^*, E)$ compact. Assume that $Q \cap F^\circ \neq \varnothing$ for each finite subset F of $\mathrm{Ball}_{n^{-1}}(E)$. By Lemma 2.1.11, this means that

$$\varnothing \neq \mathrm{Ball}_n(E^*) \cap Q = \mathrm{Ball}_n(E^*) \cap \bigcap_{k=1}^{n-1} F_k^\circ \cap C,$$

which contradicts (b). It follows that there is a finite subset F_n of $\mathrm{Ball}_{n^{-1}}(E)$ with $Q \cap F_n^\circ = \varnothing$.

It is clear that $C \cap \bigcap_{n=1}^\infty F_n^\circ = \varnothing$. As $F_n \subset \mathrm{Ball}_{n^{-1}}(E)$ for $n \in \mathbb{N}$, we can arrange the elements of $\bigcup_{n=1}^\infty F_n$ as a sequence $(x_n)_{n=1}^\infty$ with $\lim_{n\to\infty} x_n = 0$; in particular, $\lim_{n\to\infty}\langle x_n, \phi\rangle = 0$ for each $\phi \in E^*$.

Define

$$T\colon E^* \to c_0, \quad \phi \mapsto (\langle x_n, \phi\rangle)_{n=1}^\infty.$$

Then T is linear and bounded, and TC is a convex subset of c_0. By the construction of $F_1, F_2, \ldots$, it is clear that

$$\|T\phi\| = \sup_{n\in\mathbb{N}} |\langle x_n, \phi\rangle| > 1 \qquad (\phi \in C),$$

i.e.,

$$TC \cap \mathrm{Ball}(c_0) = \varnothing.$$

The non-empty subsets TC and $\mathrm{ball}(c_0)$ of c_0 are both convex with $\mathrm{ball}(c_0)$ being open. By Proposition 1.3.6, there are therefore find $f \in \ell^1 = c_0^*$ and $c \in \mathbb{R}$ such that

$$\mathrm{Re}\langle \alpha, f\rangle < c \leq \mathrm{Re}\langle T\phi, f\rangle \qquad (\alpha \in \mathrm{ball}(c_0),\ \phi \in C).$$

With $f = (f_n)_{n=1}^\infty$, this means that

$$\mathrm{Re}\sum_{n=1}^\infty \alpha_n f_n < c \leq \mathrm{Re}\sum_{n=1}^\infty \langle x_n, \phi\rangle f_n$$

for all $\alpha = (\alpha_n)_{n=1}^\infty \in c_0$ with $\|\alpha\|_\infty < 1$. Without loss of generality, suppose that $\|f\|_1 = 1$. Let $\alpha \in \mathrm{ball}(c_0)$, an let $\lambda \in \mathbb{F}$ with $|\lambda| = 1$ be such that $\lambda\langle \alpha, f\rangle = |\langle \alpha, f\rangle|$. It follows that

$$|\langle \alpha, f\rangle| = \lambda\langle \alpha, f\rangle = \langle \lambda\alpha, f\rangle = \mathrm{Re}\langle \lambda\alpha, f\rangle \leq \mathrm{Re}\sum_{n=1}^\infty \langle x_n, \phi\rangle f_n \qquad (\phi \in C)$$

and, consequently,

$$1 = \|f\|_1 = \sup_{\alpha\in\mathrm{ball}(c_0)} |\langle \alpha, f\rangle| \leq \mathrm{Re}\sum_{n=1}^\infty \langle x_n, \phi\rangle f_n \qquad (\phi \in C).$$

Letting $x := \sum_{n=1}^\infty f_n x_n$ completes the proof. $\square$

Proof of the Krein–Šmulian Theorem. As we stated before, only (ii) $\Longrightarrow$ (i) needs proof.

Let $\varnothing \neq C \subset E^*$ be convex such that $\mathrm{Ball}_r(E^*) \cap C$ is $\sigma(E^*, E)$ closed for all $r > 0$.

We claim that C is norm closed. Let $(\phi_n)_{n=1}^\infty$ be a norm convergent sequence in C. Then $(\phi_n)_{n=1}^\infty$ is bounded, and therefore there is $r > 0$ with $\phi_n \in \mathrm{Ball}_r(E^*) \cap C$ for all $n \in \mathbb{N}$. As $\mathrm{Ball}_r(E^*) \cap C$ is $\sigma(E^*, E)$ closed it is also norm close, so that $\lim_{n\to\infty} \phi_n \in \mathrm{Ball}_r(E^*) \cap C \subset C$. This proves that C is indeed norm closed.

Let $\phi_0 \in E^* \setminus C$. As C is norm closed, there is $r > 0$ such that

$$\{\phi \in E^* : \|\phi - \phi_0\| \leq r\} \cap C = \varnothing,$$

i.e.,

$$\mathrm{Ball}(E^*) \cap r^{-1}(C - \phi_0) == \varnothing.$$

By Lemma 2.1.12—with $r^{-1}(C - \phi_0)$ *en lieu* of C, there is $x \in E$ such that

$$\mathrm{Re}\langle x, \phi\rangle \geq 1 \qquad (\phi \in r^{-1}(C - \phi_0)),$$

so that, in particular, $0 \notin \overline{r^{-1}(C - \phi_0)}^{\sigma(E^*,E)}$ and therefore $\phi_0 \notin \overline{C}^{\sigma(E^*,E)}$. $\square$

Corollary 2.1.13. *Let E be a Banach space. Then a subspace F of E^* is $\sigma(E^*, E)$ closed if and only if* $\mathrm{Ball}(F)$ *is $\sigma(E^*, E)$ closed.*

2.2 Weak Compactness in Banach Spaces

We call a subset of a Banach space *weakly compact* it is compact in the weak topology.

Definition 2.2.1. Let E be a Banach space. Then $S \subset E$ is called *relatively weakly compact* if $\overline{S}^{\sigma(E,E^*)}$ is weakly compact.

Lemma 2.2.2. *Let E be a Banach space. Then the following are equivalent for $S \subset E$:*

(i) *S is relatively weakly compact;*

(ii) *S is bounded such that $\overline{S}^{\sigma(E^{**},E^*)} \subset E$.*

Proof. (i) $\Longrightarrow$ (ii): Without loss of generality, suppose that S is weakly compact. As $\sup_{x \in S} |\langle x, \phi \rangle| < \infty$ for all $\phi \in S$, the Uniform Boundedness Principle implies that $\sup_{x \in S} \|x\| < \infty$. Since S is $\sigma(E, E^*)$ compact and $\sigma(E^{**}, E^*)|_E = \sigma(E, E^*)$, it is also $\sigma(E^{**}, E^*)$ compact and therefore $\sigma(E^{**}, E^*)$ closed.

(ii) $\Longrightarrow$ (i): Choose $r > 0$ such that $S \subset \mathrm{Ball}_r(E)$. By the Alaoğlu–Burbaki Theorem, $\overline{S}^{\sigma(E^{**},E^*)}$ is $\sigma(E^{**}, E^*)$ compact. Since $\overline{S}^{\sigma(E^{**},E^*)} \subset E$ and $\sigma(E^{**}, E^*)|_E = \sigma(E, E^*)$, this means that $\overline{S}^{\sigma(E,E^*)}$ is $\sigma(E, E^*)$ compact. $\square$

Theorem 2.2.3 (Eberlein–Šmulian Theorem)**.** *Let E be a Banach space. Then the following are equivalent for $S \subset E$:*

(i) *S is relatively weakly compact;*

(ii) *every sequence in S has a weakly convergent sub*sequence*.*

Proof. (i) $\Longrightarrow$ (ii): Let $(x_n)_{n=1}^\infty$ be a sequence in S, and set $F := \overline{\mathrm{lin}}\{x_n : n \in \mathbb{N}\}$, so that F is a separable subspace of E. Let $F_0 := \{y_m : m \in \mathbb{N}\}$ be a dense subset of F. For each $m \in \mathbb{N}$, find $\phi_m \in E^*$ be such that $\|\phi_m\| = 1$ and $\langle y_m, \phi_m \rangle = \|y_m\|$. It follows that $x = 0$ if $x \in F$ and $\langle x, \phi_m \rangle = 0$ for all $m \in \mathbb{N}$. Using a diagonal argument, we can find a subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that $(\langle x_{n_k}, \phi_m \rangle)_{k=1}^\infty$ converges for each $m \in \mathbb{N}$. Without loss of generality, suppose that $(\langle x_n, \phi_m \rangle)_{n=1}^\infty$ converges for each $m \in \mathbb{N}$.

Since S is relatively weakly compact, there are $x \in \overline{S}^{\sigma(E,E^*)}$ and a sub*net* $(x_{n_\alpha})_\alpha$ of $(x_n)_{n=1}^\infty$ such that $x_{n_\alpha} \overset{\sigma(E,E^*)}{\longrightarrow} x$. It follows that

$$\langle x, \phi_m \rangle = \lim_\alpha \langle x_{n_\alpha}, \phi_m \rangle = \lim_{n \to \infty} \langle x_n, \phi_m \rangle \qquad (m \in \mathbb{N}). \tag{2.2}$$

We claim that $\langle x_n, \phi \rangle \to \langle x, \phi \rangle$ for all $\phi \in E^*$. Assume towards a contradiction that there is $\phi_0 \in E^*$ such that $\langle x_n, \phi_0 \rangle \not\to \langle x, \phi_0 \rangle$. Pick a subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$

such that $(\langle x_{n_k}, \phi_0\rangle)_{k=1}^\infty$ converges to some $\lambda_0 \in \mathbb{F} \setminus \{\langle x, \phi_0\rangle\}$. Since S is relatively weakly compact, there are $\tilde{x} \in \overline{S}^{\sigma(E,E^*)}$ and a sub*net* $(x_{n_{k_\beta}})_\beta$ of $(x_{n_k})_{k=1}^\infty$ such that $x_{n_{k_\beta}} \overset{\sigma(E,E^*)}{\longrightarrow} \tilde{x}$, so that

$$\langle \tilde{x}, \phi_m\rangle = \lim_\beta \langle x_{n_{k_\beta}}, \phi_m\rangle = \lim_{n\to\infty} \langle x_n, \phi_m\rangle \qquad (m \in \mathbb{N}). \tag{2.3}$$

Combined, (2.2) and (2.3) yield $\langle x, \phi_m\rangle = \langle \tilde{x}, \phi_m\rangle$ for all $m \in \mathbb{N}$. Since $x, \tilde{x} \in F$, this means that $x = \tilde{x}$, which contradicts $\langle x, \phi_0\rangle \neq \langle \tilde{x}, \phi_0\rangle$.

(ii) $\Longrightarrow$ (i): Assume that S is *not* relatively compact.

If $\sup_{x\in S} \|x\| = \infty$, the Uniform Boundedness Principle, implies the existence of $\phi \in E^*$ such that $\sup_{x\in S} |\langle x, \phi\rangle| = \infty$. Choose $(x_n)_{n=1}^\infty$ in S such that $|\langle x_n, \phi\rangle| \geq n$ for $n \in \mathbb{N}$. Then $(\langle x_n, \phi\rangle)_{n=1}^\infty$ has no bounded subsequence and, consequently, $(x_n)_{n=1}^\infty$ has no weakly convergent subsequence.

We can therefore suppose that S is bounded. In this case, Lemma 2.2.2 yields $X \in \overline{S}^{\sigma(E^{**},E^*)} \setminus E$. Let $\theta := \operatorname{dist}(X, E) > 0$.

We shall inductively construct sequences $(x_n)_{n=1}^\infty$ in S and $(\phi_n)_{n=1}^\infty$ in E^* with the following properties:

$$\|\phi_n\| = 1 \qquad (n \in \mathbb{N}), \tag{2.4}$$

$$\operatorname{Re}\langle \phi_n, X\rangle > \frac{3}{4}\theta \qquad (n \in \mathbb{N}), \tag{2.5}$$

$$|\langle x_k, \phi_n\rangle| < \frac{1}{4}\theta \qquad (n \in \mathbb{N},\, k = 1, \ldots, n-1), \tag{2.6}$$

and

$$\operatorname{Re}\langle x_k, \phi_n\rangle > \frac{3}{4}\theta \qquad (n, k \in \mathbb{N},\, k \geq n). \tag{2.7}$$

We claim that $(x_n)_{n=1}^\infty$ does not have a weakly convergent subsequence.

Assume towards a contradiction that there are $x \in E$ and a subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that $x_{n_k} \overset{\sigma(E,E^*)}{\longrightarrow} x$; in particular,

$$x \in \overline{\operatorname{conv}\{x_{n_k} : k \in \mathbb{N}\}}^{\sigma(E,E^*)} = \overline{\operatorname{conv}\{x_{n_k} : k \in \mathbb{N}\}}^{\|\cdot\|}.$$

This means that there are $k_1, \ldots, k_\nu \in \mathbb{N}$ and $t_1, \ldots, t_\nu \geq 0$ with $t_1 + \cdots + t_\nu = 1$ such that

$$\left\| \sum_{j=1}^\nu t_j x_{n_{k_j}} - x \right\| < \frac{1}{4}\theta. \tag{2.8}$$

For $n > \max_{j=1,\ldots,\nu} n_{k_j}$, (2.6) implies

$$\left| \left\langle \sum_{j=1}^\nu t_j x_{n_{k_j}}, \phi_n \right\rangle \right| < \frac{1}{4}\theta. \tag{2.9}$$

Together (2.8) and (2.9) imply that $|\langle x, \phi_n \rangle| < \frac{1}{2}\theta$ for sufficiently large n. This, however, contradicts (2.7)

We complete the proof with the construction of the sequences $(x_n)_{n=1}^{\infty}$ and $(\phi_n)_{n=1}^{\infty}$.

As $\|X\| \geq \theta$, there is $\phi_1 \in E^*$ with $\|\phi_1\| = 1$ and $|\langle \phi_1, X \rangle| > \frac{3}{4}\theta$. Multiplying ϕ_1 with a suitable scalar of modulus 1 yields (2.5). As $X \in \overline{S}^{\sigma(E^{**}, E^*)}$, there is $x_1 \in S$ such that

$$|\langle \phi_1, X \rangle - \langle x_1, \phi_1 \rangle| < \operatorname{Re}\langle \phi_1, X \rangle - \frac{3}{4}\theta.$$

It follows that

$$\operatorname{Re}\langle x_1, \phi_1 \rangle = \operatorname{Re}\langle \phi_1, X \rangle - \underbrace{\operatorname{Re}(\langle \phi_1, X \rangle - \langle x_1, \phi_1 \rangle)}_{< \operatorname{Re}\langle \phi_1, X \rangle - \frac{3}{4}\theta} > \frac{3}{4}\theta,$$

so that (2.7) is satisfies.

Suppose that $x_1, \ldots, x_n \in S$ and $\phi_1, \ldots, \phi_n \in E^*$ have been constructed such that (2.4), (2.5), (2.6), and (2.7) are satisfied. As θ is the norm of the coset $X + E$ in E^{**}/E, the Hahn–Banach Theorem yields $\Phi \in E^{***}$ with $\Phi|_E \equiv 0$, $\|\Phi\| = 1$ and $\operatorname{Re}\langle X, \Phi \rangle > \frac{3}{4}\theta$. Using Goldstine's Theorem, we obtain $\phi_{n+1} \in E^*$ such that (2.4), (2.5), and (2.6) hold. Set $\epsilon := \min_{j=1,\ldots,n+1} \left(\operatorname{Re}\langle X, \phi_j \rangle - \frac{3}{4}\theta \right)$ and choose $x_{n+1} \in S$ such that $|\langle \phi_j, X \rangle - \langle x_{n+1}, \phi_j \rangle| < \epsilon$ for $j = 1, \ldots, n+1$. This guarantees that (2.7) holds as well. $\square$

Corollary 2.2.4. *Let E be a Banach space. Then $S \subset E$ is relatively weakly compact if and only if $F \cap F$ is relatively weakly compact for each separable subspace F of E.*

Theorem 2.2.5. *Let E be a Banach space, and let $K \subset E$ be weakly compact. Then $\overline{\operatorname{conv}}\, K$ is weakly compact.*

Proof. Suppose first that E is separable.

Let K be equipped with the relative topology inherited from $\sigma(E, E^*)$ turning into a compact Hausdorff space. The dual space of $\mathcal{C}(K)$ can be identified with the space $M(K)$ of all finite, regular, signed or complex measure on the Borel σ-algebra of K. For $\mu \in M(K)$, define $F_\mu \in E^{**}$ by letting

$$\langle \lambda, F_\mu \rangle := \int_K \langle x, \phi \rangle \, d\mu(x) \qquad (\phi \in E^*).$$

We claim that F_μ is weak* continuous. We will prove that $\ker F_\mu$ is $\sigma(E^*, E)$ closed. By the Aloğlu–Bourbaki Theorem, it is enough to show that $\operatorname{Ball}(E^{**}) \cap \ker F_\mu$ is weak*-closed, and we will do this by showing that $F_\mu|_{\operatorname{Ball}(E^{**})}$ is continuous. As E is separable, $\sigma(E^*, E)$ is metrizable on all norm bounded subsets of E^*. We may therefore use sequences to prove the continuity of $F_\mu|_{\operatorname{Ball}(E^*)}$. Let $(\phi_n)_{n=1}^{\infty}$ be a sequence in $\operatorname{Ball}(E^*)$, and let $\phi \in \operatorname{Ball}(E^*)$

be such that $\phi_n \overset{\sigma(E^*,E)}{\longrightarrow} \phi$, i.e., in particular, $\langle x, \phi_n \rangle \to \langle x, \phi \rangle$ for all $x \in K$. The Dominated Convergence Theorem, yields that

$$\langle \phi_n, F_\mu \rangle = \int_K \langle x, \phi_n \rangle \, d\mu(x) \to \int_K \langle x, \phi \rangle \, d\mu(x) = \langle \phi_n, F_\mu \rangle.$$

It follows that F_μ is weak* continuous. Consequently, there is a unique $x_\mu \in E$ such that

$$\langle \phi, F_\mu \rangle = \langle x_\mu, \phi \rangle \qquad (\phi \in E^*).$$

Define

$$T \colon M(K) \to E, \quad \mu \mapsto x_\mu;$$

it is clear that T is linear and bounded. We claim that T is $\sigma(M(K), \mathcal{C}(K))$-$\sigma(E, E^*)$ continuous. To see this, let $(\mu_\alpha)_\alpha$ be a net in $M(K)$ such that $\mu_\alpha \overset{\sigma(M(K),\mathcal{C}(K))}{\longrightarrow} 0$, and let $\phi \in E^*$. We obtain that

$$\langle T\mu_\alpha, \phi \rangle = \int_K \langle x, \phi \rangle \, d\mu_\alpha(x) \to 0,$$

which proves the claim. Let $M_1^+(K)$ denote the probability measures in $M(K)$. By the Alaoğlu–Bourbaki Theorem, $M_1^+(K)$ is $\sigma(M(K), \mathcal{C}(K))$ compact, and it is obviously convex. We conclude that $T(M_1^+(K)) \subset E$ is weakly compact and convex. For $x \in K$, let $\delta_x \in M(K)$ denote the corresponding Dirac measure. We observe that $\langle T\delta_x, \phi \rangle = \langle x, \phi$ for all $x \in K$ and $\phi \in E^*$, so that $T\delta_x = x$ for all $x \in K$. This means that $K \subset T(M_1^+(K))$ and, consequently, $\overline{\operatorname{conv}}\, K \subset T(M_1^+(K))$. This proves the case for separable E.

Let now E be arbitrary, and let $(x_n)_{n=1}^\infty$ be a sequence in $\operatorname{conv} K$. For each $n \in \mathbb{N}$, there is a finite set $F_n \subset K$ such that $x_n \in \operatorname{conv} F_n$. Set $F := \bigcup_{n=1}^\infty F_n$ and $E_0 := \overline{\operatorname{lin}}\, F$. Then E_0 is a separable Banach space, and $K_0 := E_0 \cap K$ is weakly compact. By construction $(x_n)_{n=1}^\infty$ is contained in $\operatorname{conv} K_0$. As K_0 is relatively weakly compact, in E_0 and therefore in in E, this means by the Eberlein–Šmulian Theorem that $(x_n)_{n=1}^\infty$ has a weakly convergent subsequence. As $(x_n)_{n=1}^\infty$ is an arbitrary sequence in $\operatorname{conv} K$, this means—again by the Eberlein–Šmulian Theorem—that $\operatorname{conv} K$ is relatively weakly compact. $\square$

2.3 Weakly Compact Operators

Definition 2.3.1. Let E and F be Banach spaces. A linear operator $T \colon E \to F$ is called *weakly compact* if $T(\operatorname{Ball}(E))$ is relatively weakly compact in F.

Remarks. 1. Weakly compact operators are necessarily bounded, and compact operators are necessarily weakly compact.

2. If E or F are reflexive, then every bounded linear operator from E to F is weakly compact

3. id_E is weakly compact if and only if E is reflexive.

4. Linear combinations of weakly compact operators are again weakly compact.

Proposition 2.3.2. *Let E, F, and G be Banach spaces, and let $T \colon E \to F$ and $S \colon F \to G$ be bounded, linear operators. Then ST is weakly compact if one of T or S is weakly compact.*

Proof. Suppose that T is weakly compact. Then $\overline{T(\mathrm{Ball}(E))}^{\sigma(F,F^*)}$ is weakly compact. As S is $\sigma(F,F^*)$-$\sigma(G,G^*)$ continuous, this means that $S\left(\overline{T(\mathrm{Ball}(E))}^{\sigma(F,E^*)}\right)$ is $\sigma(G,G^*)$ compact, so that $(ST)(\mathrm{Ball}(E))$ is relatively weakly compact.

Suppose now that S is weakly compact. Choose $r > 0$ such that $T(\mathrm{Ball}(E)) \subset \mathrm{Ball}(F)$. Then $S(\mathrm{Ball}_r(F))$ is relatively weakly compact as is $(ST)(\mathrm{Ball}(E))$. $\square$

Corollary 2.3.3. *Let E and F be Banach spaces, and let $T \colon E \to F$ be a bounded linear operator such that there are a reflexive Banach space R and bounded linear operators $B \colon E \to R$ and $A \colon R \to F$ with $T = AB$. Then T is weakly compact.*

Our main goal in this section is to prove that the sufficient condition for the weak compactness of a bounded linear operator in Corollary 2.3.3 is also necessary: every weakly compact operator between Banach spaces factors through a reflexive Banach space.

We proceed by first proving a few lemmas.

Lemma 2.3.4. *Let E be a Banach space, let $\varnothing \neq W \subset E$ be bounded and absolutely convex, let $n \in \mathbb{N}$, and set*

$$U_n := 2^n W + \mathrm{ball}_{2^{-n}}(E)$$

and $p_n := p_{U_n}$. Then p_n is a norm on E that is equivalent to the given norm.

Proof. Clearly, U_n is an open, convex, and balanced subset of E, so that p_n is well defined and then, of course, a seminorm.

As W is bounded, so is U_n. Chose $C > \sup_{x \in U_n} \|x\|$. Let $x \in E$ be such that $p_n(x) < 1$, i.e., $x \in U_n$. It follows that $\|x\| < C$, and we conclude that

$$\|x\| \leq C\, p_n(x) \qquad (x \in E).$$

In particular, p_n is a norm.

On the other hand, if $\|x\| < 1$, then $2^{-n}x \in \mathrm{ball}_{2^{-n}}(E) \subset U_n$, so that $p_n(2^{-n}x) < 1$ and therefore $p_n(x) < 2^n$. It follows that

$$p_n(x) \leq 2^n \|x\| \qquad (x \in E),$$

which completes the proof. $\square$

Lemma 2.3.5. *Let $((E_n, p_n))_{n=1}^\infty$ be a sequence of Banach spaces, and define their* ℓ^2-sum *as*

$$\ell^2\text{-}\bigoplus_{n=1}^\infty (E_n, p_n) := \left\{ (x_n)_{n=1}^\infty : x_n \in E_n \textit{ for } n \in \mathbb{N} \textit{ and } \sum_{n=1}^\infty p_n(x_n)^2 < \infty \right\}.$$

For $(x_n)_{n=1}^\infty \in \ell^2\text{-}\bigoplus_{n=1}^\infty (E_n, p_n)$, set

$$|||(x_n)_{n=1}^\infty||| := \left(\sum_{n=1}^\infty p_n(x_n)^2 \right)^{\frac{1}{2}}.$$

Then $\left(\ell^2\text{-}\bigoplus_{n=1}^\infty (E_n, p_n), ||| \cdot |||\right)$ is a Banach space.

Proof. This is very much routine—just adapt the proof that ℓ^2 is a Banach space—, and we omit it. $\square$

Lemma 2.3.6. *In the setting of Lemma* 2.3.4*, define*

$$R := \left\{ x \in E : \sum_{n=1}^\infty p_n(x)^2 < \infty \right\},$$

and set

$$|||x||| := \left(\sum_{n=1}^\infty p_n(x)^2 \right)^{\frac{1}{2}}.$$

Then:

(i) $(R, ||| \cdot |||)$ *is a Banach space;*

(ii) *the inclusion map $J \colon R \to E$ is continuous;*

(iii) $J^{**} \colon R^{**} \to E^{**}$ *is injective and satisfies $(J^{**})^{-1}(E) = R$;*

(iv) $W \subset \operatorname{ball}(R)$*;*

(v) *R is reflexive if and only if W is relatively weakly compact in E.*

Proof. (i): It is routine to see that $||| \cdot |||$ defines a norm on R. Consider

$$\iota \colon R \to \ell^2\text{-}\bigoplus_{n=1}^\infty (E, p_n), \quad x \mapsto (x, x, \ldots).$$

Then ι is a linear isometry such that

$$\iota(R) = \left\{ (x_n)_{n=1}^\infty \in \ell^2\text{-}\bigoplus_{n=1}^\infty (E, p_n) : x_1 = x_2 = \cdots \right\}. \tag{2.10}$$

By Lemma 2.3.5, $\ell^2\text{-}\bigoplus_{n=1}^{\infty}(E,p_n)$ is a Banach space, and clearly, the right hand side of (2.10) is a closed subspace of it and therefore itself a Banach space. As ι is an isometry, this means that $(R, |||\cdot|||)$ is a Banach space.

(ii): The projection $P_1 \colon \ell^2\text{-}\bigoplus_{n=1}^{\infty}(E,p_n) \to E$ onto the first coordinate is continuous, as is, consequently, $J = P_1 \circ \iota$.

(iii): Consider

$$\iota^{**} \colon R^{**} \to \left(\ell^2\text{-}\bigoplus_{n=1}^{\infty}(E,p_n)\right)^{**} = \ell^2\text{-}\bigoplus_{n=1}^{\infty}(E,p_n)^{**}.$$

As ι is an isometry, so is ι^{**} and therefore injective. Note that

$$\iota^{**}(R^{**}) = \left\{(X_n)_{n=1}^{\infty} \in \ell^2\text{-}\bigoplus_{n=1}^{\infty}(E,p_n)^{**} : X_1 = X_2 = \cdots\right\}$$

As P_1^{**} is just the projection onto the first coordinate of $\ell^2\text{-}\bigoplus_{n=1}^{\infty}(E,p_n)^{**}$, this means that $P_1^{**}|_{\iota^{**}(R^{**})}$ is injective as is, consequently, $J^{**} := P_1^{**} \circ \iota^{**}$.

Let $X \in (J^{**})^{-1}(E)$, i.e.,

$$\begin{aligned}
\iota^{**}(X) &\in \left\{(X_n)_{n=1}^{\infty} \in \ell^2\text{-}\bigoplus_{n=1}^{\infty}(E,p_n)^{**} : X_1 = X_2 = \cdots \in E\right\} \\
&= \left\{(x_n)_{n=1}^{\infty} \in \ell^2\text{-}\bigoplus_{n=1}^{\infty}(E,p_n) : x_1 = x_2 = \cdots\right\} \\
&= \iota(R).
\end{aligned}$$

As ι^{**} is injective, this means that there is $x \in R$ such that $\iota(x) = \iota^{**}(X)$ and therefore $x = X$. This yields $(J^{**})^{-1}(E) \subset R$; the reversed inclusion is obvious.

(iv): Let $x \in W$. For each $n \in \mathbb{N}$, then have $2^n x \in U_n$ and therefore $p_n(2^n x) < 1$, i.e., $p_n(x) < 2^{-n}$, so that

$$\sum_{n=1}^{\infty} p_n(x)^2 < \sum_{n=1}^{\infty} \frac{1}{4^n} < 1,$$

i.e., $x \in \mathrm{ball}(R)$.

(v): Suppose that R is reflexive. Then $\mathrm{Ball}(R)$ is weakly compact in R and, consequently, $J(\mathrm{Ball}(R))$ is weakly compact in E. By (iv), we have $\overline{W}^{\sigma(E,E^*)} \subset J(\mathrm{Ball}(R))$, so that W is relatively weakly compact in E.

For the converse, we first claim that

$$J^{**}(\mathrm{Ball}(R^{**}) = \overline{J(\mathrm{Ball}(R))}^{\sigma(E^{**},E)}. \tag{2.11}$$

To see this, let $X \in J^{**}(\mathrm{Ball}(R^{**})$. By Goldstine's Theorem, there is a net $(x_\alpha)_\alpha$ in $\mathrm{Ball}(R)$ with $x_\alpha \overset{\sigma(R^{**},R^*)}{\longrightarrow} X$. It follows that

$$J^{**}X = J^{**}(\sigma(R^{**},R^*)\text{-}\lim_\alpha Jx_\alpha) = \sigma(E^{**},E^*)\text{-}\lim_\alpha Jx_\alpha \in \overline{J(\mathrm{Ball}(R))}^{\sigma(E^{**},E)}$$

As $J^{**}(\mathrm{Ball}(R^{**}))$ is $\sigma(E^{**}, E)$ compact, the reversed inclusion also follows.

Now, suppose that W is relatively weakly compact in E. As $\mathrm{ball}(R) \subset U_n$ for each $n \in \mathbb{N}$, it is clear that

$$J(\mathrm{Ball}(R)) \subset 2^n \overline{W}^{\sigma(E^{**},E)} + \mathrm{Ball}_{2^{-n}}(E^{**}) \qquad (n \in \mathbb{N}) \tag{2.12}$$

The right hand side of (2.11) is $\sigma(E^{**}, E^*)$ compact, and from (2.11), we conclude that

$$J^{**}(\mathrm{Ball}(R^{**})) \subset 2^n \overline{W}^{\sigma(E^{**},E)} + \mathrm{Ball}_{2^{-n}}(E^{**}) \qquad (n \in \mathbb{N})$$

as well. It follows that

$$\begin{aligned} J^{**}(\mathrm{Ball}(R^{**})) &\subset \bigcap_{n=1}^{\infty} \left(2^n \overline{W}^{\sigma(E^{**},E)} + \mathrm{Ball}_{2^{-n}}(E^{**}) \right) \\ &\subset \bigcap_{n=1}^{\infty} (E + \mathrm{Ball}_{2^{-n}}(E^{**})), \qquad \text{because } \overline{W}^{\sigma(E^{**},E)} \subset E \text{ by Lemma 2.2.2,} \\ &= E, \end{aligned}$$

so that $J^{**}R^{**} \subset E$. From (iii), we conclude that $R^{**} = R$. $\square$

We can now prove:

Theorem 2.3.7. *Let E and F be Banach spaces. Then the following are equivalent for a linear map $T \colon E \to F$:*

(i) *T is weakly compact;*

(ii) *there are a reflexive Banach R and bounded linear operators $B \colon E \to R$ and $A \colon R \to F$ with $T = AB$.*

Proof. (ii) $\Longrightarrow$ (i) was observed in Corollary 2.3.3.

(i) $\Longrightarrow$ (ii): Set $W := T(\mathrm{Ball}(E))$, and construct a Banach space R—from F this time—as in Lemma 2.3.6. As W is relatively weakly compact, R is reflexive by Lemma 2.3.6(v). Let $A : R \to F$ be the inclusion map—denoted by J in Lemma 2.3.6. Let $x \in \mathrm{Ball}(E)$, so that $2^n Tx \in 2^n W \subset U_n$ and, consequently, $2^n p_n(Tx) = p(2^n Tx) < 1$, i.e., $p_n(x) < 2^{-n}$ for all $n \in \mathbb{N}$. It follows that

$$|||Tx|||^2 = \sum_{n=1}^{\infty} p_n(x)^2 < \sum_{n=1}^{\infty} \frac{1}{4^n},$$

so that the linear operator

$$B \colon E \to R, \quad x \mapsto Tx$$

is bounded. It is clear by construction that $T = AB$. $\square$

Schauder's Theorem asserts that the adjoint of a compact operator is also compact. With the help of Theorem 2.3.7, it is almost effortless to prove analog for weakly compact operators:

Corollary 2.3.8 (Gantmacher's Theorem)**.** *Let E and F be Banach spaces, and let $T\colon E \to F$ be a bounded linear operator. Then T is weakly compact if and only if $T^*\colon F^* \to E^*$ is weakly compact.*

Proof. Suppose that T is weakly compact, and let R, A and a B be as in the Theorem 2.3.7(ii). It follows that $T^* = B^* A^*$. As R^* is also reflexive, this proves the weak compactness of T^*.

Suppose that T^* is weakly compact. By the foregoing, this means that T^{**} is weakly compact, as is therefore $T = T^{**}|_E$. □

Lemma 2.3.9. *Let E be a Banach space. Then the following are equivalent for $\varnothing \neq W \subset E$:*

(i) *W is relatively weakly compact;*

(ii) *for each $\epsilon > 0$ there is a relatively compact $W_\epsilon \subset E$ such that $W \subset W_\epsilon + \mathrm{Ball}_\epsilon(E)$.*

Proof. (i) $\Longrightarrow$ (ii) is trivial.

(ii) $\Longrightarrow$ (i): It is clear that W has to be bounded. We will use Lemma 2.2.2 to prove that $\overline{W}^{\sigma(E^{**},E^*)} \subset E$. First note that $\overline{W_\epsilon}^{\sigma(E^{**},E^*)} \subset E$ is $\sigma(E^{**},E^*)$ compact for each $\epsilon > 0$, so that $\overline{W_\epsilon}^{\sigma(E^{**},E^*)} + \mathrm{Ball}_\epsilon(E^{**})$ is $\sigma(E^{**},E^*)$ compact. We conclude that

$$\overline{W}^{\sigma(E^{**},E^*)} \subset \overline{W_\epsilon}^{\sigma(E^{**},E^*)} + \mathrm{Ball}_\epsilon(E^{**}) \qquad (\epsilon > 0),$$

i.e.,

$$\overline{W}^{\sigma(E^{**},E^*)} \subset \bigcap_{\epsilon>0} \left(\overline{W_\epsilon}^{\sigma(E^{**},E^*)} + \mathrm{Ball}_\epsilon(E^{**}) \right) \subset \bigcap_{\epsilon>0} (E + \mathrm{Ball}_\epsilon(E^{**})) \subset E,$$

so that W is relatively weakly compact. □

Theorem 2.3.10. *Let E and F be a Banach spaces, let $(T_n)_{n=1}^\infty$ be a sequence of weakly compact operators from E to F, and let $T\colon E \to F$ be a bounded linear operator such that $\|T_n - T\| \to 0$. Then T is weakly compact.*

Proof. Let $\epsilon > 0$, and choose $n \in \mathbb{N}$ such that $\|T_n - T\| \leq \epsilon$. It follows that

$$T(\mathrm{Ball}(E)) \subset T_n(\mathrm{Ball}(E)) + (T - T_n)(\mathrm{Ball}(E)) \subset T_n(\mathrm{Ball}(E)) + \mathrm{Ball}_\epsilon(E).$$

As $T_n(\mathrm{Ball}(E))$ is relatively weakly compact, and since $\epsilon > 0$ is arbitrary, this means that $T(\mathrm{Ball}(E))$ is relatively weakly compact by Lemma 2.3.9. □

Chapter 3

Bases in Banach Spaces

3.1 Schauder Bases

Definition 3.1.1. Let E be a Banach space. A sequence $(x_n)_{n=1}^\infty$ in E is called a *Schauder basis*—or short: *basis*—for E if, for each $x \in E$, there is a unique sequence $(\lambda_n)_{n=1}^\infty$ in $\mathbb{F}$ such that $x = \sum_{n=1}^\infty \lambda_n x_n$.

Remarks. 1. By $x = \sum_{n=1}^\infty \lambda_n x_n$, we mean $x = \lim_{N\to\infty} \sum_{n=1}^N \lambda_n x_n$; in particular, we do *not* suppose absolute convergence.

2. The notion of a (Schauder) basis in a Banach space must not be confused with that of a basis in the context of linear algebra, i.e., a maximal linearly independent family of vectors in a vector space, which necessarily spans the whole space. To avoid confusion, we shall refer to such a basis as a *Hamel basis*. It is not difficult to see that a Banach space E contains a sequence that is both a Schauder and a Hamel basis if and only if $\dim E < \infty$.

3. It was an open problem for several decades whether or not any separable, infinite-dimensional Banach space had a basis. A counterexample was eventually constructed by the Swedish mathematician Per Enflo in 1972.

Example. Let $E = c_0$ or $E = \ell^p$ with $p \in [1, p)$. Then the canonical unit vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots$, i.e.,

$$\boldsymbol{e}_n(k) = \begin{cases} 1, & k = n, \\ 0, & \text{otherwise}, \end{cases} \qquad (n, k \in \mathbb{N})$$

form a basis for E.

Proposition 3.1.2. *Let E be a Banach space, and let $(x_n)_{n=1}^\infty$ be basis for E. For $N \in \mathbb{N}$ and $x = \sum_{n=1}^\infty \lambda_n x_n \in E$, define*

$$P_N x := \sum_{n=1}^N \lambda_n x_n.$$

Then $P_1, P_2, \ldots$ are continuous projections such that $\sup_{N\in\mathbb{N}} \|P_N\| < \infty$.

Proof. For $x = \sum_{n=1}^\infty \lambda_n x_n \in E$, the sequence $\left(\sum_{n=1}^N \lambda_n x_n\right)_{N=1}^\infty$ converges and therefore is bounded. It follows that

$$|||x||| := \sup_{N\in\mathbb{N}} \left\| \sum_{n=1}^N \lambda_n x_n \right\| = \sup_{N\in\mathbb{N}} \|P_N x\| < \infty.$$

Obviously, $|||\cdot|||$ is a norm on E with

$$|||x||| \geq \lim_{N\to\infty} \left\| \sum_{n=1}^N \lambda_n x_n \right\| = \|x\| \qquad (x \in E). \tag{3.1}$$

We claim that $(E, |||\cdot|||)$ is a Banach space.

To see this, let $(y_k)_{k=1}^\infty$ be a Cauchy sequence in $(E, |||\cdot|||)$. From the definition of $|||\cdot|||$, it is clear that $(P_N y_k)_{k=1}^\infty$ is a Cauchy sequence in $(E, \|\cdot\|)$ for each $N \in \mathbb{N}$. Consequently, for each $N \in \mathbb{N}$, there is $z_N \in E$ such that $\|P_N y_k - z_N\| \overset{k\to\infty}{\longrightarrow} 0$. Let $\epsilon > 0$, and let $k_\epsilon \in \mathbb{N}$ be such that $|||y_k - y_l||| < \epsilon$ for all $k, l \geq k_\epsilon$. For all $N \in \mathbb{N}$ and all $k \geq k_\epsilon$, we obtain

$$\|P_N y_k - z_N\| = \lim_{l\to\infty} \|P_N y_k - P_N y_l\| \leq \limsup_{l\to\infty} |||y_k - y_l||| \leq \epsilon.$$

It follows that

$$\sup_{N\in\mathbb{N}} \|P_N y_k - z_N\| \overset{k\to\infty}{\longrightarrow} 0.$$

Let $\epsilon > 0$, and fix $k_\epsilon \in \mathbb{N}$ such that $\sup_{N\in\mathbb{N}} \|P_N y_{k_\epsilon} - z_N\| < \frac{\epsilon}{3}$. As $y_{k_\epsilon} = \lim_{N\to\infty} P_N y_{k_\epsilon}$, there is $N_\epsilon \in \mathbb{N}$ such that $\|P_N y_{k_\epsilon} - P_M y_{k_\epsilon}\| < \frac{\epsilon}{3}$ for all $N, M \geq N_\epsilon$. For $N, M \geq N_\epsilon$, this yields

$$\|z_N - z_M\| \leq \|z_N - P_N y_{k_\epsilon}\| + P_N y_{k_\epsilon} - P_M y_{k_\epsilon}\| + \|P_M y_{k_\epsilon} - z_M\| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

i.e., $(z_N)_{N=1}^\infty$ is a Cauchy sequence in $(E, \|\cdot\|)$ and therefore has a limit, say z. Let $N, M \in \mathbb{N}$. As $\dim P_M E < \infty$, the restriction of P_N to $P_M E$ is continuous. It follows that

$$\begin{aligned} P_N z_M &= P\left(\lim_{k\to\infty} P_M y_k\right) \\ &= \lim_{k\to\infty} P_N P_M y_k \\ &= \lim_{k\to\infty} P_{\min\{N,M\}} y_k \\ &= z_{\min\{N,M\}}. \end{aligned}$$

Therefore, there is a sequence $(\mu_n)_{n=1}^\infty$ in $\mathbb{F}$ such that $z_N = \sum_{n=1}^N \mu_n x_n$ for $N \in \mathbb{N}$, so that $z = \sum_{n=1}^\infty \mu_n x_n$ and therefore $P_N z = z_N$ for $N \in \mathbb{N}$. We conclude that

$$|||y_k - z||| = \sup_{N\in\mathbb{N}} \|P_N y_k - P_N z\| = \sup_{N\in\mathbb{N}} \|P_N y_k - z_N\| \overset{k\to\infty}{\longrightarrow} 0,$$

i.e., z is the limit of $(y_k)_{k=1}^\infty$ in $(E, |||\cdot|||)$.

In view of (3.1), the Inverse Mapping Theorem yields that $\|\cdot\|$ and $|||\cdot|||$ are equivalent, i.e., there is a constant $C > 0$ such that

$$|||x||| = \sup_{N \in \mathbb{N}} \|P_N x\| \leq C\|x\| \qquad (x \in E).$$

In view of this, it is clear that $P_1, P_2, \ldots$ are continuous with $\sup_{N \in \mathbb{N}} \|P_N\| < \infty$. $\square$

The projections $P_1, P_2, \ldots$ are called the *basic projections* of $(x_n)_{n=1}^\infty$, and the supremum $\sup_{N \in \mathbb{N}} \|P_N\|$ is called its *basic constant*.

Our next proposition complements Proposition 3.1.2:

Proposition 3.1.3. *Let E be a Banach space, and let $(P_n)_{n=1}^\infty$ be a sequence of bounded projections on E with the following properties:*

(a) $x = \lim_{n\to\infty} P_n x$ *for all* $x \in E$*;*

(b) $P_n P_m = P_{\min\{n,m\}}$ *for all* $n, m \in \mathbb{N}$*;*

(c) $\dim P_n E = n$ *for* $n \in \mathbb{N}$.

Then any sequence $(x_n)_{n=1}^\infty$ in $E \setminus \{0\}$ with $x_1 \in P_1 E$ and $x_n \in P_n E \cap \ker P_{n-1}$ for $n \geq 2$ is a basis for E.

Proof. Let $(x_n)_{n=1}^\infty$ be a sequence as described; it is immediate that $x_1, x_2, \ldots$ are linearly independent.

Let $x \in E$. For each $n \in \mathbb{N}$, (c) yields unique $\lambda_1^{(n)}, \ldots, \lambda_n^{(n)} \in \mathbb{F}$ such that $P_n x = \sum_{k=1}^n \lambda_k^{(n)} x_k$. For $m \geq n$, we have by (b)

$$\sum_{k=1}^n \lambda_k^{(n)} x_k = P_n x = P_n P_m x = P_n\left(\sum_{k=1}^m \lambda_k^{(m)} x_k\right) = \sum_{k=1}^n \lambda_k^{(m)} x_k,$$

so that $\lambda_k^{(n)} = \lambda_k^{(m)}$ for $k = 1, \ldots, n$. Consequently, there is a sequence $(\lambda_n)_{n=1}^\infty$ in $\mathbb{F}$ such that $P_n x = \sum_{k=1}^n \lambda_k x_k$. From (a), we conclude that

$$x = \lim_{n\to\infty} P_n x = \lim_{n\to\infty} \sum_{k=1}^n \lambda_k x_k = \sum_{n=1}^\infty \lambda_n x_n.$$

Let $\lambda_1, \lambda_2, \ldots \in \mathbb{F}$ be such that $\sum_{n=1}^\infty \lambda_n x_n = 0$. It follows that

$$0 = P_n 0 = \sum_{k=1}^n \lambda_k x_k \qquad (n \in \mathbb{N}).$$

As $x_1, x_2, \ldots$ are linearly independent, this means that $\lambda_1 = \lambda_2 = \cdots = 0$. This proves the uniqueness part in the definition of a Schauder basis. $\square$

We will now use Proposition 3.1.3 to show that certain well known Banach spaces do have a basis.

Examples. 1. Consider the space $\mathcal{C}([0,1])$.

Let $(t_n)_{n=1}^\infty$ be a sequence in $[0,1]$ such that: $t_1 = 0$, $t_2 = 1$, $t_n \neq t_m$ for $n \neq m$, and $\overline{\{t_n : n \in \mathbb{N}\}} = [0,1]$. For $n \in \mathbb{N}$ and $f \in \mathcal{C}([0,1])$, define $P_n f := f(0)$ if $n = 1$ and $P_n f$ for $n \geq 2$ as the piecewise linear map with nodes $t_1, \ldots, t_n$ and values $f(t_j)$ at t_j for $j = 1, \ldots, n$. It is immediate that $P_1, P_2, \ldots \colon \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$ form a sequence of contractive projections satisfying Proposition 3.1.3(b) and (c). We will show that $(P_n)_{n=1}^\infty$ satisfies Proposition 3.1.3(a) as well.

Let $f \in \mathcal{C}([0,1])$, and let $\epsilon > 0$. By the uniform continuity of f, there is $\delta > 0$ such that $|f(s) - f(t)| < \frac{\epsilon}{2}$ for all $s, t \in [0,1]$ with $|s - t| < \delta$. Let $0 = s_0 < s_1 < \cdots < s_m = 1$ be a partition such that $\max_{j=1,\ldots,m} s_j - s_{j-1} < \frac{\delta}{2}$. Choose $n_\epsilon \geq 2$ so large that $(s_{j-1}, s_j) \cap \{t_1, \ldots, t_{n_\epsilon}\} \neq \varnothing$ for each $j = 1, \ldots, m$, and let $n \geq n_\epsilon$. Let $t_1' < \ldots < t_n'$ be a rearrangement of $t_1, \ldots, t_n$ according to size. Let $t \in [0,1]$. If $t \in \{t_1, \ldots, t_n\}$, it is clear that $(P_n f)(t) = f(t)$, so suppose that $t \notin \{t_1, \ldots, t_n\}$. Choose $k \in \{2, \ldots, n\}$ such that $t \in (t_{k-1}', t_k')$. Assume that $t_k' - t_{k-1}' \geq \delta$. As $\max_{j=1,\ldots,m} s_j - s_{j-1} < \frac{\delta}{2}$, this means that there is $j_0 \in \{1, \ldots, m\}$ with $(s_{j_0-1}, s_{j_0}) \subset (t_{k-1}', t_k')$ and thus $(s_{j_0-1}, s_{j_0}) \cap \{t_1, \ldots, n\} = \varnothing$, which contradicts the choice of n_ϵ. It follows $t_k' - t_{k-1}' < \delta$ and therefore $|f(t_k') - f(t_{k-1}')| < \frac{\epsilon}{2}$; as $t \in (t_{k-1}', t_k')$, we also have $|t_{k-1}' - t| < \delta$ and therefore $|f(t_{k-1}') - f(t)| < \frac{\epsilon}{2}$ as well. We conclude that

$$\begin{aligned}
|(P_n f)(t) - f(t)| &= \left| f(t_{k-1}') + \frac{t - t_{k-1}'}{t_k' - t_{k-1}'}(f(t_k') - f(t_{k-1}')) - f(t) \right| \\
&\leq \underbrace{|f(t_{k-1}') - f(t)|}_{<\frac{\epsilon}{2}} + \underbrace{\left| \frac{t - t_{k-1}'}{t_k' - t_{k-1}'} \right|}_{<1} \underbrace{|f(t_k') - f(t_{k-1}')|}_{<\frac{\epsilon}{2}} \\
&< \epsilon.
\end{aligned}$$

As $t \in [0,1]$ was arbitrary, this means that $\|P_n f - f\|_\infty \leq \epsilon$.

All in all, the sequence $(P_n)_{n=1}^\infty$ satisfies Proposition 3.1.3(a), (b), and (c), so that $\mathcal{C}([0,1])$ has a basis.

2. We now consider the spaces $L^p([0,1])$ with $p \in [1, \infty)$.

Define $h_0, h_1, h_2, \ldots \colon [0,1] \to \mathbb{F}$ as follows. Set $h_0 :\equiv 1$. For $n \in \mathbb{N}$, there are unique $j \in \mathbb{N}_0$ and $k \in \{0, 1, \ldots, 2^j - 1\}$ such that $n = 2^j + k$; define

$$h_n(t) := \begin{cases} 1, & t \in [k2^{-j}, (2k+1)2^{-j-1}), \\ -1, & t \in [(2k+1)2^{-j-1}, (k+1)2^{-j}), \\ 0, & \text{otherwise}. \end{cases}$$

For example, if $n = 1$, then $j = k = 0$ and

$$h_1(t) = \begin{cases} 1, & t \in \left[0, \frac{1}{2}\right), \\ -1, & t \in \left[\frac{1}{2}, 1\right), \\ 0, & t = 1, \end{cases}$$

if $n = 2$, then $j = 1$, $k = 0$, and

$$h_2(t) = \begin{cases} 1, & t \in \left[0, \frac{1}{4}\right), \\ -1, & t \in \left[\frac{1}{4}, \frac{1}{2}\right), \\ 0, & t \in \left[\frac{1}{2}, 1\right], \end{cases}$$

and, if $n = 3$, then $j = 1$, $k = 1$, and

$$h_3(t) = \begin{cases} 1, & t \in \left[\frac{1}{2}, \frac{3}{4}\right), \\ -1, & t \in \left[\frac{3}{4}, 1\right), \\ 0, & t \in \left[0, \frac{1}{2}\right) \cup \{1\}. \end{cases}$$

For $n \in \mathbb{N}$, let $j \in \mathbb{N}_0$ and $k \in \{0, 1, \ldots, 2^j - 1\}$ be such that $n = 2^j + k$, set

$$\begin{aligned} S_n := \{(\nu 2^{-j-1}, (\nu+1)2^{-j-1}) : \nu = 0, 1, \ldots 2k+1\} \\ \cup \{(\nu 2^{-j}, (\nu+1)2^{-j}) : \nu = k+1, \ldots, 2^j - 1\}, \end{aligned}$$

and define F_n to consist of those functions on $[0, 1]$ that are constant on the intervals in S_n. As there are $n+1$ intervals contained in in S_n, it is clear that $\dim F_n = n+1$ if F_n is viewed as a subspace of $L^p([0,1])$ (so that functions are identified if they coincide outside a set of measure zero). It is obvious that $h_0, h_1, \ldots, h_n \in F_n$, and since $h_0, h_1, h_2, \ldots$ are obviously linearly independent, this means that $F_n = \operatorname{lin}\{h_0, h_1, \ldots, h_n\}$.

For $f \in L^p([0,1])$ define $P_0 f := \int_{[0,1]} f$, and for $n \in \mathbb{N}$, set

$$P_n f := \sum_{I \in S_n} \frac{1}{|I|} \left(\int_I f \right) \chi_I,$$

so that $\|P_n f\|_p \leq \|f\|_p$ if $p = 1$ and

$$\begin{aligned} \|P_n f\|_p^p &= \sum_{I \in S_n} \frac{1}{|I|^p} \left| \int_I f \right|^p |I| \\ &= \sum_{I \in S_n} |I|^{1-p} \left| \int_I f \right|^p \\ &\leq \sum_{I \in S_n} |I|^{1-p} |I|^{\frac{p}{q}} \int_I |f|^p, \qquad \text{by Hölder's Inequality with } \frac{1}{p} + \frac{1}{q} = 1, \\ &= \sum_{I \in S_n} \int_I \int_I |f|^p \\ &= \|f\|_p^p. \end{aligned}$$

if $p > 1$. All in all $P_0, P_1, P_2, \ldots \colon L^p([0,1]) \to L^p([0,1])$ are contractive projections with $P_0 L^p([0,1]) = \mathbb{F}1$, $P_n L^p([0,1]) = F_n$ for $n \in \mathbb{N}$, and

$$P_n P_m = P_{\min\{n,m\}} \qquad (n, m \in \mathbb{N}_0),$$

i.e., satisfying Proposition 3.1.3(b) and (c).

Let $f \in L^p([0,1])$, and let $\epsilon > 0$. It is not difficult to show that $\bigcup_{n=1}^{\infty} F_n$ is norm dense in $L^p([0,1])$. Choose $n_\epsilon \in \mathbb{N}$ and $g \in F_{n_\epsilon}$ such that $\|f - g\|_p < \frac{\epsilon}{2}$. For $n \geq n_\epsilon$, we obtain

$$\|f - P_n f\|_p \leq \underbrace{\|f - g\|}_{<\frac{\epsilon}{2}} + \underbrace{\|g - P_n g\|_p}_{=0} + \underbrace{\|P_n g - P_n f\|_p}_{\leq \|g - f\|_p < \frac{\epsilon}{2}} < \epsilon.$$

Consequently, $(P_n)_{n=0}^{\infty}$ satisfies Proposition 3.1.3(a) as well.

From Proposition 3.1.3, it follows that $(h_n)_{n=0}^{\infty}$ for $L^p([0,1])$.

Definition 3.1.4. A sequence $(x_n)_{n=1}^{\infty}$ in a Banach space E is called *basic* if it is a basis for $\overline{\operatorname{lin}}\{x_n : n \in \mathbb{N}\}$.

Lemma 3.1.5. *Let E be an infinite-dimensional Banach space, and let F be a finite dimensional subspace of E. Then, for each $\epsilon > 0$, there is $x \in E$ with $\|x\| = 1$ such that*

$$\|y\| \leq (1 + \epsilon)\|y + \lambda x\| \qquad (y \in F,\ \lambda \in \mathbb{F}).$$

Proof. Without loss of generality, suppose that $0 < \epsilon < 1$.

Let $y_1, \ldots, y_n \in F$ be unit vectors such that, for each $y \in F$ with $\|y\| = 1$, there is $j \in \{1, \ldots, n\}$ such that $\|y_j - y\| < \frac{\epsilon}{2}$. For $j = 1, \ldots, n$, choose $\phi_j \in E^*$ with $\|\phi_j\| = 1$ and $\langle y_j, \phi_j \rangle = 1$. As $\dim E = \infty$, there is $x \in E$ with $\|x\| = 1$ such that $\langle x, \phi_j \rangle = 0$ for $j = 1, \ldots, n$. Let $y \in F$ be such that $\|y\| = 1$, and let $\lambda \in \mathbb{F}$. Choose $j \in \{1, \ldots, n\}$ such that $\|y_j - y\| < \frac{\epsilon}{2}$. It follows that

$$\|y + \lambda x\| \geq \|y_j - \lambda x\| - \|y - y_j\| \geq |\langle y_j - \lambda x, \phi_j \rangle| - \frac{\epsilon}{2} = 1 - \frac{\epsilon}{2} \geq \frac{1}{1 + \epsilon}.$$

This proves the claim for $y \in F$ with $\|y\| = 1$; division by $\|y\|$ proves it for general non-zero $y \in F$. $\square$

Lemma 3.1.6. *Let E be a Banach space. Then a sequence $(x_n)_{n=1}^{\infty}$ in $E \setminus \{0\}$ is a basic sequence if and only if there is $C \geq 0$ such that*

$$\left\| \sum_{k=1}^{n} \lambda_k x_k \right\| \leq C \left\| \sum_{k=1}^{m} \lambda_k x_k \right\| \tag{3.2}$$

for all $\lambda_1, \lambda_2, \ldots \in \mathbb{F}$ and $n, m \in \mathbb{N}$ with $m \geq n$.

Proof. Suppose that $(x_n)_{n=1}^\infty$ is basis. Without loss of generality suppose that $(x_n)_{n=1}^\infty$ is a Schauder basis for E. Let $C \geq 0$ be its basic constant. For $\lambda_1, \lambda_2, \ldots \in \mathbb{F}$ and $n, m \in \mathbb{N}$ with $m \geq n$, we get

$$\left\| \sum_{k=1}^{n} \lambda_k x_k \right\| \leq \left\| P_n \left(\sum_{k=1}^{m} \lambda_k x_k \right) \right\| \leq C \left\| \sum_{k=1}^{m} \lambda_k x_k \right\|.$$

Conversely, suppose that (3.2) holds for all $\lambda_1, \lambda_2, \ldots \in \mathbb{F}$ and $n, m \in \mathbb{N}$ with $m \geq n$.

We claim that $x_1, x_2, \ldots$ are linearly independent. Let $m \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$ such that $\sum_{k=1}^m \lambda_k x_k = 0$. Applying (3.1) for $n = 1$ yields, $\|\lambda_1 x_1\| = 0$ and thus $\lambda_1 = 0$; applying, (3.2) with $n = 2$ implies $\|\lambda_1 x_1 + \lambda_2 x_2\| = 0$ and thus $\lambda_2 = 0$ because $\lambda_1 = 0$ already. Continuing in this fashion, we obtain that $\lambda_1 = \cdots = \lambda_m = 0$.

For $n \in \mathbb{N}$, define

$$P_n \colon \operatorname{lin}\{x_n : n \in \mathbb{N}\} \to \operatorname{lin}\{x_1, \ldots, x_n\}, \quad \sum_{k=1}^{<\infty} \lambda_k x_k \mapsto \sum_{k=1}^{n} \lambda_k x_k;$$

due to the linear independence of $x_1, x_2, \ldots$, this is well defined. From (3.2) it follows that $P_1, P_2, \ldots$ are bounded with $\sup_{n \in \mathbb{N}} \|P_n\| \leq C$. Therefore, $P_1, P_2, \ldots$ extend to $\overline{\operatorname{lin}}\{x_n : n \in \mathbb{N}\}$ as bounded projections. It is straightforward that $(P_n)_{n=1}^\infty$ satisfies Proposition 3.1.3(a), (b), and (c), and Proposition 3.1.3 therefore yields that $(x_n)_{n=1}^\infty$ is a basis for $\overline{\operatorname{lin}}\{x_n : n \in \mathbb{N}\}$. $\square$

Theorem 3.1.7. *Let E be an infinite-dimensional Banach space. Then E contains a basic sequence.*

Proof. Let $(\epsilon_n)_{n=1}^\infty$ be a sequence of strictly positive reals such that $C := \prod_{n=1}^\infty (1 + \epsilon_n) < \infty$.

Choose $x_1 \in E$ with $\|x_1\| = 1$. Invoking Lemma 3.1.5, we inductively obtain unit vectors $x_2, x_3, \ldots \in E$ such that

$$\|y\| \leq (1 + \epsilon_n)\|y + \lambda x_{n+1}\| \qquad (n \in \mathbb{N},\ y \in \operatorname{lin}\{x_1, \ldots, x_n\}, \lambda \in \mathbb{F}).$$

Let $m > n$, and let $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$, and note that

$$\begin{aligned}
\left\| \sum_{k=1}^{n} \lambda_k x_k \right\| &\leq (1+\epsilon_n) \left\| \sum_{k=1}^{n} \lambda_k x_k + \lambda_{n+1} x_{n+1} \right\| \\
&= (1+\epsilon_n) \left\| \sum_{k=1}^{n+1} \lambda_k x_k \right\| \\
&\leq (1+\epsilon_n)(1+\epsilon_{n+1}) \left\| \sum_{k=1}^{n+2} \lambda_k x_k \right\| \\
&\vdots \\
&\leq \underbrace{(1+\epsilon_n)(1+\epsilon_{n+1}) \cdots (1+\epsilon_{m-1})}_{\leq C} \left\| \sum_{k=1}^{m} \lambda_k x_k \right\|.
\end{aligned}$$

By Lemma 3.1.6, this means that $(x_n)_{n=1}^{\infty}$ is a basic sequence. $\square$

If $(x_n)_{n=1}^{\infty}$ is a basic sequence in a Banach space E, then refer to its *basic constant* as the basic constant of the Schauder basis $(x_n)_{n=1}^{\infty}$ of the subspace $\overline{\mathrm{lin}}\{x_n : n \in \mathbb{N}\}$ of E.

Definition 3.1.8. Let E be a Banach space with Schauder basis $(x_n)_{n=1}^{\infty}$. For $n \in \mathbb{N}$, define $x_n^* \colon E \to \mathbb{F}$ by letting

$$\langle x, x_n^* \rangle := \lambda_n$$

for $x = \sum_{k=1}^{\infty} \lambda_k x_k$.

The functionals $x_1^*, x_2^*, \ldots \colon E \to \mathbb{F}$ are called the *coefficient functionals* of $(x_n)_{n=1}^{\infty}$.

Proposition 3.1.9. *Let E be a Banach space with Schauder basis $(x_n)_{n=1}^{\infty}$ with basic constant C. Then $x_1^*, x_2^*, \ldots \in E^*$ such that $\sup_{n \in \mathbb{N}} \|x_n\| \|x_n^*\| \leq 2C$.*

Proof. Note that

$$P_1 x = \langle x, x_1^* \rangle x_1 \qquad (x \in E)$$

and

$$P_n x - P_{n-1} x = \langle x, x_n^* \rangle x_n \qquad (x \in E)$$

for $n \geq 2$. It follows that

$$\|x_1\| \|x_1^*\| \leq C$$

and

$$\begin{aligned}
\|x_n^*\| &= \sup\{|\langle x, x_n^* \rangle| : x \in \mathrm{Ball}(E)\} \\
&= \frac{1}{\|x_n\|} \sup\{\|\langle x, x_n^* \rangle x_n\| : x \in \mathrm{Ball}(E)\} \\
&= \frac{1}{\|x_n\|} \sup\{\|P_n x - P_{n-1} x\| : x \in \mathrm{Ball}(E)\} \\
&\leq \frac{2C}{\|x_n\|}
\end{aligned}$$

for $n \geq 2$, which proves the claim. $\square$

Definition 3.1.10. Let E and F be Banach spaces, and let $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ be basic sequences in E and F, respectively. Then $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ are called *equivalent* if $\sum_{n=1}^\infty \lambda_n x_n$ converges if and only if $\sum_{n=1}^\infty \lambda_n y_n$ converges.

Proposition 3.1.11. *Let $(x_n)_{n=1}^\infty$ be a basic sequence in a Banach space E, and let $(y_n)_{n=1}^\infty$ be a sequence in a Banach space F. The the following are equivalent:*

(i) *$(y_n)_{n=1}^\infty$ is a basic sequence equivalent to $(x_n)_{n=1}^\infty$;*

(ii) *there is an isomorphism $T\colon \overline{\operatorname{lin}}\{x_n : n \in \mathbb{N}\} \to \overline{\operatorname{lin}}\{y_n : n \in \mathbb{N}\}$ such that*

$$Tx_n = y_n \qquad (n \in \mathbb{N}); \tag{3.3}$$

(iii) *there are $C_1, C_2 > 0$ such that*

$$\frac{1}{C_1}\left\|\sum_{j=1}^n \lambda_j x_j\right\| \leq \left\|\sum_{j=1}^n \lambda_j y_j\right\| \leq C_2 \left\|\sum_{j=1}^n \lambda_j x_j\right\| \qquad (n \in \mathbb{N},\ \lambda_1, \ldots, \lambda_n \in \mathbb{F}).$$

Proof. (i) $\Longrightarrow$ (ii): Define $T\colon \overline{\operatorname{lin}}\{x_n : n \in \mathbb{N}\} \to \overline{\operatorname{lin}}\{y_n : n \in \mathbb{N}\}$ by letting

$$T\left(\sum_{n=1}^\infty \lambda_n x_n\right) := \sum_{n=1}^\infty \lambda_n y_n$$

for $\sum_{n=1}^\infty \lambda_n x_n \in \overline{\operatorname{lin}}\{x_n : n \in \mathbb{N}\}$. Then T is well defined, linear, and bijective and satisfies (3.3). We show that T is continuous using the Closed Graph Theorem. Let $(z_k)_{k=1}^\infty$ be a sequence in $\overline{\operatorname{lin}}\{x_n : n \in \mathbb{N}\}$ such that

$$z_k \to 0 \qquad \text{and} \qquad Tz_k \to z \in \overline{\operatorname{lin}}\{y_n : n \in \mathbb{N}\}.$$

For all $n \in \mathbb{N}$, it follows that

$$0 = \lim_{k\to\infty} \langle z_k, x_n^* \rangle = \lim_{k\to\infty} \langle Tz_k, y_n^* \rangle = \langle z, y_n^* \rangle.$$

so that $z = 0$.

(ii) $\Longrightarrow$ (iii): Set $C_2 := \|T\|$ and $C_1 := \|T^{-1}\|$.

(iii) $\Longrightarrow$ (i): Let C be the basic constant of $(x_n)_{n=1}^\infty$. For $m \geq n$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{N}$, (ii) yields

$$\left\|\sum_{j=1}^n \lambda_j y_j\right\| \leq C_1 C_2 C \left\|\sum_{j=1}^m \lambda_j y_j\right\|.$$

By Lemma 3.1.6, this means that $(y_n)_{n=1}^\infty$ is a basic sequence. Also, (ii) yields that $\left(\sum_{k=1}^n \lambda_k x_k\right)_{n=1}^\infty$ is a Cauchy sequence if and only if $\left(\sum_{k=1}^n \lambda_k y_k\right)_{n=1}^\infty$ is, i.e., $\sum_{n=1}^\infty \lambda_n x_n$ converges if and only if $\sum_{n=1}^\infty \lambda_n y_n$ converges. $\square$

Theorem 3.1.12. *Let E be a Banach space, and let $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ be sequences in E such that:*

(a) $(x_n)_{n=1}^\infty$ *is basic;*

(b) $\sum_{n=1}^\infty \|x_n^*\|\|x_n - y_n\| < 1$.

Then:

(i) $(y_n)_{n=1}^\infty$ *is a basic sequence equivalent to* $(x_n)_{n=1}^\infty$*;*

(ii) *if* $\overline{\text{lin}}\{x_n : n \in \mathbb{N}\}$ *is complemented in E, then so is* $\overline{\text{lin}}\{y_n : n \in \mathbb{N}\}$*;*

(iii) *if* $(x_n)_{n=1}^\infty$ *is a Schauder basis for E, then so is* $(y_n)_{n=1}^\infty$.

Proof. To prove (i), use the Hahn–Banach Theorem to extend $x_1^*, x_2^*, \ldots$ to all of E, preserving the norms. Note that

$$\sum_{n=1}^\infty |\langle x, x_n^*\rangle| \|x_n - y_n\| < \infty,$$

so that

$$S\colon E \to E, \quad x \mapsto \sum_{n=1}^\infty \langle x, x_n^*\rangle (x_n - y_n).$$

is well defined. As

$$\|Sx\| = \left\| \sum_{n=1}^\infty \langle x, x_n^*\rangle (x_n - y_n) \right\| \leq \|x\| \underbrace{\sum_{n=1}^\infty \|x_n^*\|\|x_n - y_n\|}_{<1} \qquad (x \in E),$$

it is clear that S is bounded with $\|S\| < 1$. Set $T := \text{id}_E - S$, and note that

$$\|x - Tx\| = \|Sx\| \leq \|S\|\|x\| \qquad (x \in E) \tag{3.4}$$

and therefore

$$\|Tx\| \geq \|x\| - \|x - Tx\| \geq \underbrace{(1 - \|S\|)}_{>0} \|x\| \qquad (x \in E).$$

This means that T is injective with closed range. We claim that $TE = E$. Assume that $TE \subsetneq E$. Choose $\theta \in (\|S\|, 1)$. By Riesz' Lemma, there is $x \in E$ with $\|x\| = 1$ such that $\|x - y\| \geq \theta$ for all $y \in TE$; in particular, $\|x - Tx\| \geq \theta > \|S\|$, which contradicts (3.4). Consequently, $T : E \to E$ is an isomorphism. As $Tx_n = y_n$ for $n \in \mathbb{N}$, it is clear that T maps $\overline{\text{lin}}\{x_n : n \in \mathbb{N}\}$ injectively into $\overline{\text{lin}}\{y_n : n \in \mathbb{N}\}$. Moreover, since T is bounded below, $T\left(\overline{\text{lin}}\{x_n : n \in \mathbb{N}\}\right)$ is closed in $\overline{\text{lin}}\{y_n : n \in \mathbb{N}\}$. Consequently, T is an isomorphism from $\overline{\text{lin}}\{x_n : n \in \mathbb{N}\}$ onto $\overline{\text{lin}}\{y_n : n \in \mathbb{N}\}$.

For (ii), let F be a closed subspace of E such that

$$E = \overline{\operatorname{lin}}\{x_n : n \in \mathbb{N}\} \oplus F,$$

and let $T \colon E \to E$ be as in the proof of (i). It follows that

$$E = TE = T\left(\overline{\operatorname{lin}}\{x_n : n \in \mathbb{N}\}\right) \oplus TF = \overline{\operatorname{lin}}\{y_n : n \in \mathbb{N}\} \oplus TF,$$

which yields the claim.

Finally, with T as in the proof of (i), we obtain

$$E = TE = T\left(\overline{\operatorname{lin}}\{x_n : n \in \mathbb{N}\}\right) = \overline{\operatorname{lin}}\{y_n : n \in \mathbb{N}\},$$

so that (iii) holds. □

3.2 Bases in Classical Banach Spaces

Definition 3.2.1. Let $(x_n)_{n=1}^\infty$ be a basic sequence in a Banach space E. A sequence $(y_n)_{n=1}^\infty$ in $E \setminus \{0\}$ is called a *block basic sequence* of $(x_n)_{n=1}^\infty$ if there are $\lambda_1, \lambda_2, \ldots \in \mathbb{F}$ and $0 = p_0 < p_1 < p_2 < \cdots$ in $\mathbb{N}_0$ such that

$$y_n = \sum_{k=p_{n-1}+1}^{p_n} \lambda_k x_k \qquad (n \in \mathbb{N}).$$

Remark. Any block basic sequence $(y_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ is again a basic sequence, with a basic constant not larger than that of $(x_n)_{n=1}^\infty$. To see this, let C be the basic constant of $(x_n)_{n=1}^\infty$, let $m \geq n$, and let $\mu_1, \ldots, \mu_m \in \mathbb{F}$. It follows that

$$\begin{aligned}
\left\| \sum_{k=1}^n \mu_k y_k \right\| &= \left\| \sum_{k=1}^n \mu_k \sum_{j=p_{k-1}+1}^{p_k} \lambda_j x_j \right\| \\
&= \left\| \sum_{k=1}^n \sum_{j=p_{k-1}+1}^{p_k} \mu_k \lambda_j x_j \right\| \\
&\leq C \left\| \sum_{k=1}^m \sum_{j=p_{k-1}+1}^{p_k} \mu_k \lambda_j x_j \right\| \\
&= C \left\| \sum_{k=1}^m \mu_k y_k \right\|.
\end{aligned}$$

Theorem 3.2.2. *Let E be a Banach space with Schauder basis $(x_n)_{n=1}^\infty$, and let F be an infinite-dimensional closed subspace of E. Then F contains an infinite-dimensional subspace G with a Schauder basis equivalent to a block basic sequence of $(x_n)_{n=1}^\infty$.*

Proof. Let C be the basic constant of $(x_n)_{n=1}^\infty$. Let $y_1 = \sum_{n=1}^\infty \lambda_n^{(1)} x_n \in F$ be arbitrary with $\|y_1\| > 1$. Choose $p_1 \in \mathbb{N}$ such that

$$\left\| y_1 - \sum_{n=1}^{p_1} \lambda_n^{(1)} x_n \right\| < \frac{1}{4C} \qquad \text{and} \qquad \left\| \sum_{n=1}^{p_1} \lambda_n^{(1)} x_n \right\| \geq 1,$$

and set $z_1 := \sum_{n=1}^{p_1} \lambda_n^{(1)} x_n$. As $\dim F = \infty$, the finite-codimensional subspace $\overline{\text{lin}}\{x_n : n \geq p_1 + 1\}$ has non-zero intersection with F. Let $y_2 = \sum_{n=p_1+1}^\infty \lambda_n^{(2)} x_n \in F$ be such that $\|y_2\| > 1$. Choose $p_2 > p_1$ such that

$$\left\| y_2 - \sum_{n=p_1+1}^{p_2} \lambda_n^{(2)} x_n \right\| < \frac{1}{8C} \qquad \text{and} \qquad \left\| \sum_{n=p_1+1}^{p_2} \lambda_n^{(2)} x_n \right\| \geq 1,$$

and set $z_2 := \sum_{n=p_1+1}^{p_2} \lambda_n^{(2)} x_n$. Continuing in this fashion, we obtain sequences $(y_k)_{k=1}^\infty$ of vectors in F and $(z_k)_{n=1}^\infty$ in E along with $\lambda_1, \lambda_2, \ldots$ in $\mathbb{F}$ and $0 = p_0 < p_1 < p_2 < \cdots$ in $\mathbb{N}$ such that

$$z_k = \sum_{n=p_{k-1}+1}^{p_k} \lambda_n x_n, \quad \|z_k\| \geq 1, \quad \text{and} \quad \|y_k - z_k\| < \frac{1}{2^{k+1}C} \qquad (k \in \mathbb{N}).$$

It is then clear by construction that $(z_k)_{k=1}^\infty$ is a block basic sequence of $(x_n)_{n=1}^\infty$ and, in particular, a basic sequence. Let C' be the basic constant of $(z_k)_{k=1}^\infty$. By the remark following Definition 3.2.1, $C' \leq C$, so that, by Proposition 3.1.9,

$$\sup_{k \in \mathbb{N}} \|z_k\| \|z_k^*\| \leq 2C' \leq 2C.$$

It follows that

$$\sum_{k=1}^\infty \|z_k^*\| \|z_k - y_k\| \leq \sum_{k=1}^\infty \|z_k\| \|z_k^*\| \|z_k - y_k\| \leq 2C \sum_{k=1}^\infty \|z_k - y_k\| < 2C \sum_{k=1}^\infty \frac{1}{2^{k+1}C} < 1.$$

From Theorem 3.1.12, we conclude that $(y_k)_{k=1}^\infty$ is a basic sequence equivalent to $(z_k)_{k=1}^\infty$. Set $G := \overline{\text{lin}}\{y_k : k \in \mathbb{N}\}$. □

Proposition 3.2.3. *Let $E = c_0$ or $E = \ell^p$ with $p \in [1, \infty)$, and let $(x_n)_{n=1}^\infty$ be a block basic sequence of the standard basis $(\boldsymbol{e}_n)_{n=1}^\infty$ such that*

$$\sup_{n \in \mathbb{N}} \|x_n\| < \infty \qquad \textit{and} \qquad \inf_{n \in \mathbb{N}} \|x_n\| > 0$$

Then $(\boldsymbol{e}_n)_{n=1}^\infty$ and $(x_n)_{n=1}^\infty$ are equivalent. Moreover, $\overline{\text{lin}}\{x_n : n \in \mathbb{N}\}$ is complemented in E.

Proof. We only treat the case where $E = \ell^p$ with $p \in [1, \infty)$.

Set $C := \sup_{n\in\mathbb{N}} \|x_n\|$ and $c := \inf_{n\in\mathbb{N}} \|x_n\|$.

Let $\lambda_1, \lambda_2, \ldots$ in $\mathbb{F}$ and $0 = p_0 < p_1 < p_2 < \cdots$ in $\mathbb{N}_0$ be such that

$$x_n = \sum_{k=p_{n-1}+1}^{p_n} \lambda_k \boldsymbol{e}_k \qquad (n \in \mathbb{N}),$$

so that

$$\|x_n\|_p^p = \sum_{k=p_{n-1}+1}^{p_n} |\lambda|_k^p \qquad (n \in \mathbb{N}).$$

Let $n \in \mathbb{N}$, let $\mu_1, \ldots, \mu_n \in \mathbb{F}$, and note that

$$\left\|\sum_{k=1}^n \mu_k x_k\right\|_p^p = \sum_{k=1}^n \sum_{j=p_{k-1}+1}^{p_k} |\mu_k \lambda_j|^p = \sum_{k=1}^n \left(|\mu_k|^p \sum_{j=p_{k-1}+1}^{p_k} |\lambda_j|^p\right) = \sum_{k=1}^n |\mu_k|^p \|x_n\|_p^p.$$

It follows that

$$c^p \left\|\sum_{k=1}^n \mu_k \boldsymbol{e}_k\right\|_p^p \leq \left\|\sum_{k=1}^n \mu_k x_k\right\|_p^p \leq C^p \left\|\sum_{k=1}^n \mu_k \boldsymbol{e}_k\right\|_p^p,$$

so that $(\boldsymbol{e}_n)_{n=1}^\infty$ and $(x_n)_{n=1}^\infty$ are equivalent by Proposition 3.1.11.

For the "moreover" part, choose, for each $n \in \mathbb{N}$, a norm one functional

$$\phi_n \in \operatorname{lin}\{\boldsymbol{e}^*_{p_{n-1}+1}, \ldots, \boldsymbol{e}^*_{p_n}\} \subset (\ell^p)^*$$

such that $\langle x_n, \phi_n\rangle = \|x_n\|$; it follows that

$$\langle x_m, \phi_n\rangle = 0 \qquad (n, m \in \mathbb{N},\, n \neq m).$$

Let $x = \sum_{n=1}^\infty \mu_n \boldsymbol{e}_n$, and note that

$$|\langle x, \phi_n\rangle|^p \leq \sum_{k=p_{n-1}+1}^{p_n} |\mu_k|^p \qquad (n \in \mathbb{N}).$$

We obtain

$$\begin{aligned}
\left\|\sum_{n=1}^\infty \langle x, \phi_n\rangle x_n\right\|_p^p &= \sum_{n=1}^\infty \sum_{k=p_{n-1}+1}^{p_n} |\langle x, \phi_n\rangle|^p |\lambda_k|^p \\
&= \sum_{n=1}^\infty |\langle x, \phi_n\rangle|^p \|x_n\|_p^p \\
&\leq C^p \sum_{n=1}^\infty |\langle \phi_n, x\rangle|^p \\
&\leq C^p \sum_{n=1}^\infty \sum_{k=p_{n-1}+1}^{p_n} |\mu_k|^p \\
&= C^p \|x\|_p^p.
\end{aligned}$$

Consequently,

$$P\colon \ell^p \to \overline{\operatorname{lin}}\{x_n : n \in \mathbb{N}\}, \quad x \mapsto \sum_{n=1}^{\infty} \langle x, \phi_n \rangle x_n$$

is a bounded projection. $\square$

Remark. If $(x_n)_{n=1}^{\infty}$ consists of unit vectors, then the isomorphism $T\colon E \to \overline{\operatorname{lin}}\{x_n : n \in \mathbb{N}\}$ with $T\boldsymbol{e}_n = x_n$ for $n \in \mathbb{N}$ is an isometry and the projection from E onto $\overline{\operatorname{lin}}\{x_n : n \in \mathbb{N}\}$ has norm one: this follows from an inspection of the proof of Proposition 3.2.3 shows.

Theorem 3.2.4. *Let $E = c_0$ or $E = \ell^p$ with $p \in [1, \infty)$, and let F be an infinite-dimensional subspace of E. Then F contains a subspace G which is complemented in E and isomorphic to E.*

Proof. By Theorem 3.2.2, there are a sequence $(y_n)_{n=1}^{\infty}$ in F and a block basic sequence $(z_n)_{n=1}^{\infty}$ of $(\boldsymbol{e}_n)_{n=1}^{\infty}$ such that $(y_n)_{n=1}^{\infty}$ is a basic sequence equivalent to $(z_n)_{n=1}^{\infty}$. An inspection of the proof of Theorem 3.2.2 shows that $(z_n)_{n=1}^{\infty}$ can be chosen to satisfy $\sup_{n\in\mathbb{N}} \|z_n\| < \infty$ and $\inf_{n\in\mathbb{N}} \|z_n\| > 0$. Therefore, Proposition 3.2.3 applies. $\square$

Corollary 3.2.5. *Let $p \in (1, \infty)$. Then c_0 and ℓ^1 do not contain an isomorphic copy of ℓ^p.*

Proof. We only formulate the proof for c_0—for ℓ^1, it carries over verbatim.

Assume that there is a closed subspace F of c_0 isomorphic to ℓ^p. As ℓ^p is reflexive, so is F. By Theorem 3.2.4, F contains a subspace G isomorphic to c_0. As F is reflexive, so are G and—consequently—c_0, which is a contradiction. $\square$

We introduce some notation, part of which we already encountered earlier (see Lemma 2.3.5).

Given two normed spaces E and F and $p \in [1, \infty]$, their *ℓ^p-direct sum* $E \oplus_{\ell^p} F$ the vector space $E \oplus F$ equipped with the norm $\|\cdot\|_p$ given by

- $\|(x, y)\|_p := (\|x\|^p + \|y\|^p)^{\frac{1}{p}}$ for $(x, y) \in E \oplus F$ if $p \in [1, \infty)$ and
- $\|(x, y)\|_p := \max\{\|x\|, \|y\|\}$ for $(x, y) \in E \oplus F$ if $p = \infty$.

It is easy to see that $E \oplus_{\ell^p} F$ is again a Banach space if E and F are.

More generally, let $E_1, E_2, \ldots$ be a sequence of normed spaces. For $p \in [1, \infty)$, we define their ℓ^p-direct sum as

$$\ell^p\text{-}\bigoplus_{n=1}^{\infty} E_n := \left\{ (x_n)_{n=1}^{\infty} : x_n \in E_n \text{ for } n \in \mathbb{N} \text{ and } \sum_{n=1}^{\infty} \|x_n\|^p < \infty \right\}$$

equipped with the norm

$$\|(x_n)_{n=1}^\infty\|_p := \left(\sum_{n=1}^\infty \|x_n\|^p\right)^{\frac{1}{p}} \qquad \left((x_n)_{n=1}^\infty \in \ell^p\text{-}\bigoplus_{n=1}^\infty E_n\right);$$

for $p = \infty$, we set

$$\ell^\infty\text{-}\bigoplus_{n=1}^\infty E_n := \left\{(x_n)_{n=1}^\infty : x_n \in E_n \text{ for } n \in \mathbb{N} \text{ and } \sup_{n\in\mathbb{N}} \|x_n\| < \infty\right\}$$

equipped with

$$\|(x_n)_{n=1}^\infty\|_\infty := \sup_{n\in\mathbb{N}} \|x_n\| \qquad \left((x_n)_{n=1}^\infty \in \ell^\infty\text{-}\bigoplus_{n=1}^\infty E_n\right).$$

Furthermore, we define the c_0*-direct sum* of $E_1, E_2, \ldots$ as

$$c_0\text{-}\bigoplus_{n=1}^\infty E_n := \left\{(x_n)_{n=1}^\infty : x_n \in E_n \text{ for } n \in \mathbb{N} \text{ and } \lim_{n\to\infty} \|x_n\| = 0\right\};$$

it is straightforward that $c_0\text{-}\bigoplus_{n=1}^\infty E_n$ is a closed subspace of $\ell^\infty\text{-}\bigoplus_{n=1}^\infty E_n$. It is routinely verified that, if $E_1, E_2, \ldots$ are all Banach spaces, then so are $\ell^p\text{-}\bigoplus_{n=1}^\infty E_n$ for $p \in [1,\infty]$ and $c_0\text{-}\bigoplus_{n=1}^\infty E_n$.

As $\mathbb{N}\times\mathbb{N}$ and $(\mathbb{N}\times\{1\})\cup(\mathbb{N}\times\{2\})$ have the same cardinality as $\mathbb{N}$, we immediately obtain isometric isomorphisms

$$\ell^p \oplus_{\ell^p} \ell^p \cong \ell^p((\mathbb{N}\times\{1\})\cup(\mathbb{N}\times\{2\})) \cong \ell^p \cong \ell^p(\mathbb{N}\times\mathbb{N}) \cong \ell^p\text{-}\bigoplus_{n=1}^\infty \ell^p$$

for $p \in [1,\infty]$ as well as

$$c_0 \oplus_{\ell^\infty} c_0 \cong c_0 \cong c_0\text{-}\bigoplus_{n=1}^\infty c_0.$$

Theorem 3.2.6. *Let $E = c_0$ or $E = \ell^p$ with $p \in [1,\infty)$ and let F be an infinite-dimensional, complemented subspace of E. Then F is isomorphic to E.*

Proof. Let $\cong$ stand for isomorphism of Banach spaces.

Let E_0 be a closed subspace of E such that $E = F \oplus E_0$. By Theorem 3.2.4, there is a closed subspace G of F that is complemented in E such that $G \cong E$. Let F_0 be a closed subspace of F such that $F = G \oplus F_0$. It follows that

$$\begin{aligned} E \oplus F &\cong E \oplus (G \oplus F_0) \\ &\cong (E \oplus G) \oplus F_0 \\ &\cong (E \oplus E) \oplus F_0 \\ &\cong E \oplus F_0 \\ &\cong G \oplus F_0 \\ &\cong F. \end{aligned}$$

On the other hand, with $\bigoplus_{n=1}^\infty E$ denoting the ℓ^p- or c_0-direct sum depending on whether $E = c_0$ or $E = \ell^p$, we have

$$\begin{aligned} E \oplus F &\cong \left(\bigoplus_{n=1}^\infty E\right) \oplus F \\ &\cong \left(\bigoplus_{n=1}^\infty (F \oplus E_0)\right) \oplus F \\ &\cong \left(\bigoplus_{n=1}^\infty F\right) \oplus \left(\bigoplus_{n=1}^\infty E_0\right) \oplus F \\ &\cong \left(\bigoplus_{n=1}^\infty F\right) \oplus \left(\bigoplus_{n=1}^\infty E_0\right) \\ &\cong \bigoplus_{n=1}^\infty (F \oplus E_0) \\ &\cong \bigoplus_{n=1}^\infty E \\ &\cong E, \end{aligned}$$

so that $E \cong F$. □

Definition 3.2.7. For $n \in \mathbb{N}$, the n^{th} *Rademacher function* r_n is defined through

$$r_n(t) := \operatorname{sgn}(\sin(2^n \pi t)) \qquad (t \in [0,1]).$$

Remarks. 1. It is obvious that $r_n \in L^p([0,1])$ for all $n \in \mathbb{N}$ and $p \in [1, \infty]$ with $\|r_n\|_p = 1$.

2. For $m > n$, we have

$$\int_0^1 r_m(t) r_n(t)\, d = \sum_{k=1}^{2^{m-1}} \int_{(k-1)2^{-m+1}}^{k2^{-m+1}} r_m(t) \underbrace{r_n(t)}_{=\text{const}}\, dt = 0,$$

so that $(r_n)_{n=1}^\infty$ is an orthonormal sequence in $L^2([0,1])$. This means that

$$\int_0^1 \left| \sum_{k=1}^n \lambda_k r_k(t) \right|^2 dt = \sum_{k=1}^n |\lambda_k|^2 \tag{3.5}$$

for all $n \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$. This can be used to embed ℓ^2 isometrically into $L^2([0,1])$.

We will use Rademacher functions to embed ℓ^2 into $L^p([0,1])$ for $p \in [1, \infty)$. The key is the following:

Theorem 3.2.8 (Khintchine's Inequality)**.** *For every $p \in [1,\infty)$, there are $A_p, B_p > 0$ such that*

$$A_p \left(\sum_{k=1}^n |\lambda_k|^2 \right)^{\frac{1}{2}} \leq \left(\int_0^1 \left| \sum_{k=1}^n \lambda_k r_k(t) \right|^p \right)^{\frac{1}{p}} \leq B_p \left(\sum_{k=1}^n |\lambda_k|^2 \right)^{\frac{1}{2}}$$

for all $n \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$.

Proof. As $L^r([0,1]) \subset L^s([0,1])$ contractively for $r > s$, it is sufficient to show the existence of A_1 and of B_{2m} for arbitrary $m \in \mathbb{N}$.

Existence of B_{2m}. Let $n \in \mathbb{N}$, and let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. The Multinomial Theorem yields

$$\begin{aligned} \int_0^1 \left| \sum_{k=1}^n \lambda_k r_k(t) \right|^{2m} dt &= \int_0^1 \left(\sum_{k=1}^n \lambda_k r_k(t) \right)^{2m} dt \\ &= \sum_{\substack{\alpha_1,\ldots,\alpha_\nu \in \mathbb{N}_0 \\ \alpha_1+\cdots+\alpha_\nu = 2m \\ 1 \leq n_1 < \cdots < n_\nu \leq n}} \underbrace{\frac{(2m)!}{\alpha_1! \cdots \alpha_\nu!}}_{=: C_{\alpha_1,\ldots,\alpha_\nu}} \lambda_{n_1}^{\alpha_1} \cdots \lambda_{n_\nu}^{\alpha_\nu} \int_0^1 r_{n_1}(t)^{\alpha_1} \cdots r_{n_\nu}(t)^{\alpha_\nu}\, dt \end{aligned}$$

As

$$\int_0^1 r_{n_1}(t)^{\alpha_1} \cdots r_{n_\nu}(t)^{\alpha_\nu}\, dt = \begin{cases} 1, & \alpha_1, \ldots, \alpha_\nu \text{ are all even}, \\ 0, & \text{otherwise}, \end{cases}$$

we conclude that

$$\int_0^1 \left| \sum_{k=1}^n \lambda_k r_k(t) \right|^{2m} dt = \sum_{\substack{\beta_1,\ldots,\beta_\nu \in \mathbb{N}_0 \\ \beta_1+\cdots+\beta_\nu = m \\ 1 \leq n_1 < \cdots < n_\nu \leq n}} C_{2\beta_1,\ldots,2\beta_\nu} \lambda_{n_1}^{2\beta_1} \cdots \lambda_{n_\nu}^{2\beta_\nu}. \tag{3.6}$$

The set

$$S := \left\{ \frac{C_{\beta_1,\ldots,\beta_\nu}}{C_{2\beta_1,\ldots,2\beta_\nu}} : \beta_1, \ldots, \beta_\nu \in \mathbb{N}_0,\ \beta_1 + \cdots + \beta_\nu = m \right\}$$

is finite. Set

$$B_{2m} := (\min S)^{-\frac{1}{2m}}.$$

It follows that

$$
\begin{aligned}
\left(\sum_{k=1}^{n}|\lambda_k|^2\right)^m &= \sum_{\substack{\beta_1,\ldots,\beta_\nu\in\mathbb{N}_0\\ \beta_1+\cdots+\beta_\nu=m\\ 1\le n_1<\cdots<n_\nu\le n}} C_{\beta_1,\ldots,\beta_\nu}\lambda_{n_1}^{2\beta_1}\cdots\lambda_{n_\nu}^{2\beta_\nu}\\
&= \sum_{\substack{\beta_1,\ldots,\beta_\nu\in\mathbb{N}_0\\ \beta_1+\cdots+\beta_\nu=m\\ 1\le n_1<\cdots<n_\nu\le n}} \frac{C_{\beta_1,\ldots,\beta_\nu}}{C_{2\beta_1,\ldots,2\beta_\nu}} C_{2\beta_1,\ldots,2\beta_\nu}\lambda_{n_1}^{2\beta_1}\cdots\lambda_{n_\nu}^{2\beta_\nu}\\
&\ge B_{2m}^{-2m} \sum_{\substack{\beta_1,\ldots,\beta_\nu\in\mathbb{N}_0\\ \beta_1+\cdots+\beta_\nu=m\\ 1\le n_1<\cdots<n_\nu\le n}} C_{2\beta_1,\ldots,2\beta_\nu}\lambda_{n_1}^{2\beta_1}\cdots\lambda_{n_\nu}^{2\beta_\nu}\\
&= B_{2m}^{-2m}\int_0^1\left|\sum_{k=1}^{n}\lambda_k r_k(t)\right|^{2m} dt, \qquad \text{by (3.6)},
\end{aligned}
$$

and therefore

$$
\left(\left|\sum_{k=1}^{n}\lambda_k r_k(t)\right|^{2m} dt\right)^{\frac{1}{2m}} \le B_{2m}\left(\sum_{k=1}^{n}|\lambda_k|^2\right)^{\frac{1}{2}}.
$$

Existence of A_1. Let $n\in\mathbb{N}$, let $\lambda_1,\ldots,\lambda_n\in\mathbb{R}$, and set $f := \sum_{k=1}^{n}\lambda_k r_k$ for the sake of notational simplicity. We obtain

$$
\begin{aligned}
\int_0^1 |f(t)|^2\,dt &= \int_0^1 |f(t)|^{\frac{2}{3}}|f(t)|^{\frac{4}{3}}\,dt\\
&\le \left(\int_0^1 |f(t)|\,dt\right)^{\frac{2}{3}}\left(\int_0^1 |f(t)|^4\,dt\right)^{\frac{1}{3}},\\
&\qquad\qquad \text{by Hölder's Inequality with } p=\frac{3}{2} \text{ and } q=3,\\
&\le \left(\int_0^1 |f(t)|\,dt\right)^{\frac{2}{3}} B_4^{\frac{4}{3}}\left(\sum_{k=1}^{n}|\lambda|_k^2\right)^{\frac{2}{3}}\\
&= \left(\int_0^1 |f(t)|\,dt\right)^{\frac{2}{3}} B_4^{\frac{4}{3}}\left(\int_0^1 |f(t)|^2\,dt\right)^{\frac{2}{3}}, \qquad \text{by (3.5)},
\end{aligned}
$$

so that

$$
\left(\int_0^1 |f(t)|\,dt\right)^{\frac{2}{3}} \ge B_4^{-\frac{4}{3}}\left(\int_0^1 |f(t)|^2\,dt\right)^{\frac{1}{3}}.
$$

Finally, we obtain

$$
\int_0^1\left|\sum_{k=1}^{n}\lambda_k r_k(t)\right| dt = \int_0^1 |f(t)|\,dt \ge B_4^{-2}\left(\int_0^1 |f(t)|^2\,dt\right)^{\frac{1}{2}} = B_4^{-2}\left(\sum_{k=1}^{n}|\lambda|_k^2\right)^{\frac{1}{2}}.
$$

Set $A_1 := B_4^{-2}$. $\square$

Theorem 3.2.9. *Let $p \in (1, \infty)$. Then $L^p([0,1])$ has a complemented subspace isomorphic to ℓ^2.*

Proof. We first deal with the case where $\mathbb{F} = \mathbb{R}$.

Define

$$T\colon \ell^2 \to L^p([0,1]), \quad (\lambda_1, \lambda_2, \ldots) \mapsto \sum_{k=1}^{\infty} \lambda_n r_n.$$

By Khintchine's Inequality, T is an isomorphism onto its range. We need to show that $T\ell^2$ is complemented in $L^p([0,1])$.

Case 1: $p \geq 2$.

In this case, $L^p([0,1]) \subset L^2([0,1])$ holds contractively. By Khintchine's Inequality, we have

$$\overline{\operatorname{lin}\{r_n : n \in \mathbb{N}\}}^{L^p([0,1])} = \overline{\operatorname{lin}\{r_n : n \in \mathbb{N}\}}^{L^2([0,1])}.$$

Let P be the orthogonal projection from $L^2([0,1])$ onto $\overline{\operatorname{lin}\{r_n : n \in \mathbb{N}\}}^{L^2([0,1])}$. Again from Khintchine's Inequality, we obtain

$$\|Pf\|_p \leq B_p \|Pf\|_2 \leq B_p \|f\|_2 \leq B_p \|f\|_p \qquad (f \in L^p([0,1])),$$

which proves that $P|_{L^p([0,1])}$ is a bounded projection onto $\overline{\operatorname{lin}\{r_n : n \in \mathbb{N}\}}^{L^p([0,1])}$.

Case 2: $p \in (1,2)$.

Let q be conjugate to p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Then $q > 2$, and Case 1 yields that there are a closed subspace F of $L^q([0,1])$ isomorphic to ℓ^2 and a bounded projection P from $L^q([0,1])$ onto F. Then $P^* F^*$ is a complemented subspace of $L^p([0,1])$ isomorphic to ℓ^2.

The case where $\mathbb{F} = \mathbb{C}$ is easily reduced to the real case. □

Our next result has a proof reminiscent of that of Theorem 3.2.2.

Theorem 3.2.10 (Bessaga–Pełczyński Selection Principle)**.** *Let E be a Banach space with a Schauder basis $(x_n)_{n=1}^{\infty}$, and let $(y_n)_{n=1}^{\infty}$ be a sequence in E such that $y_n \overset{\sigma(E,E^*)}{\longrightarrow} 0$ and $\inf_{n\in\mathbb{N}} \|y_n\| > 0$. Then there is a subsequence of $(y_n)_{n=1}^{\infty}$ that is a basic sequence equivalent to a block basic sequence of $(x_n)_{n=1}^{\infty}$.*

Proof. Let C be the basic constant of $(x_n)_{n=1}^{\infty}$, and set $\delta := \inf_{n\in\mathbb{N}} \|y_n\|$.

For each $n \in \mathbb{N}$, there are $\lambda_1^{(n)}, \lambda_2^{(n)}, \ldots \in \mathbb{F}$ such that

$$y_n = \sum_{k=1}^{\infty} \lambda_k^{(n)} x_k.$$

As $y_n \overset{\sigma(E,E^*)}{\longrightarrow} 0$, it is clear that $\lim_{n\to\infty} \lambda_k^{(n)} = 0$ for all $k \in \mathbb{N}$.

Let $n_1 = 1$, and choose $p_1 \in \mathbb{N}$ so large that

$$\left\| y_{n_1} - \sum_{k=1}^{p_1} \lambda_k^{(n_1)} x_k \right\| < \frac{\delta}{8C} \qquad \text{and} \qquad \left\| \sum_{k=1}^{p_1} \lambda_k^{(n_1)} x_k \right\| > \frac{\delta}{2};$$

set $z_1 := \sum_{k=1}^{p_1} \lambda_k^{(n_1)} x_k$.

As $\lim_{n\to\infty} \lambda_k^{(n)} = 0$ for $k = 1, \ldots, p_1$, we have

$$\lim_{n\to\infty} \left\| \sum_{k=1}^{p_1} \lambda_k^{(n)} x_k \right\| = 0.$$

Since $\inf_{n\in\mathbb{N}} \|y_n\| > \frac{\delta}{2}$, we can therefore ensure, for sufficiently large $n_2 > n_1$, that

$$\left\| \sum_{k=1}^{p_1} \lambda_k^{(n_2)} x_k \right\| < \frac{\delta}{32C} \qquad \text{and} \qquad \left\| \sum_{k=p_1+1}^{\infty} \lambda_k^{(n_2)} x_k \right\| > \frac{\delta}{2}.$$

Choose $p_2 > p_1$ so large that

$$\left\| \sum_{k=p_1+1}^{\infty} \lambda_k^{(n_2)} x_k - \sum_{k=p_1+1}^{p_2} \lambda_k^{(n_2)} x_k \right\| < \frac{\delta}{32C} \qquad \text{and} \qquad \left\| \sum_{k=p_1+1}^{p_2} \lambda_k^{(n_2)} x_k \right\| > \frac{\delta}{2}$$

and define $z_2 := \sum_{k=p_1+1}^{p_2} \lambda_k^{(n_2)} x_k$. It follows that

$$\|y_{n_2} - z_2\| \le \left\| y_{n_2} - \sum_{k=p_1+1}^{\infty} \lambda_k^{(n_2)} x_k \right\| + \left\| \sum_{k=p_1+1}^{\infty} \lambda_k^{(n_2)} x_k - z_2 \right\| < \frac{2\delta}{32C} = \frac{\delta}{16C}.$$

Continuing in this fashion, we obtain $n_1 < n_2 < \cdots$ and $0 = p_0 < p_1 < p_2 < \cdots$ such that

$$\left\| y_{n_j} - \sum_{k=p_{j-1}+1}^{p_j} \lambda_k^{(n_j)} x_k \right\| < \frac{\delta}{2^{j+2}C} \qquad \text{and} \qquad \left\| \sum_{k=p_{j-1}+1}^{p_j} \lambda_k^{(n_j)} x_k \right\| > \frac{\delta}{2}$$

for $j \in \mathbb{N}$. Letting $z_j := \sum_{k=p_{j-1}+1}^{p_j} \lambda_k^{(n_j)} x_k$ for $j \in \mathbb{N}$, we obtain a block basic sequence $(z_j)_{j=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that

$$\|y_{n_j} - z_j\| < \frac{\delta}{2^{j+2}C} \quad \text{and} \quad \|z_j\| \ge \frac{\delta}{2} \qquad (j \in \mathbb{N}).$$

for $j = 1, 2, \ldots$. As $(z_j)_{j=1}^\infty$ is a block basic sequence of $(x_n)_{n=1}^\infty$, it is itself a basic sequence with basic constant at most C. Finally, as

$$\begin{aligned}
\sum_{j=1}^{\infty} \|z_j^*\| \|z_j - y_{n_j}\| &\le \frac{2}{\delta} \sum_{j=1}^{\infty} \|z_j^*\| \|z_j\| \|z_j - y_{n_j}\| \\
&\le \frac{4C}{\delta} \sum_{j=1}^{\infty} \|z_j - y_{n_j}\|, \qquad \text{by Proposition 3.1.9,} \\
&< \frac{4C}{\delta} \sum_{j=1}^{\infty} \frac{\delta}{2^{j+2}C} \\
&= \frac{4C}{\delta} \frac{\delta}{4C} \\
&= 1,
\end{aligned}$$

it follows from Theorem 3.1.12 that $(y_{n_j})_{j=1}^\infty$ is a basic sequence equivalent to $(z_j)_{j=1}^\infty$. $\square$

Theorem 3.2.11 (Pitt's Theorem)**.** *Let $1 \leq p < q < \infty$. Then every bounded linear operator from ℓ^q to ℓ^p is compact.*

Proof. Let $T : \ell^q \to \ell^p$ be a bounded linear operator, and assume that T is not compact. Then there is a sequence $(x_n)_{n=1}^\infty$ in $\mathrm{Ball}(\ell^q)$ such that $(Tx_n)_{n=1}^\infty$ has no convergent subsequence. As ℓ^q is reflexive, the Eberlein–Šmulian Theorem yields that $(x_n)_{n=1}^\infty$ has a weakly convergent subsequence. We can therefore suppose without loss of generality that there is $x \in \ell^q$ such $x_n \overset{\sigma(\ell^q,(\ell^q)^*)}{\longrightarrow} x$. As $(Tx_n)_{n=1}^\infty$ has no convergent subsequence, we have $\inf_{n\in\mathbb{N}} \|Tx_n - Tx\| > 0$. Replacing each x_n by $x_n - x$, we therefore obtain a sequence $(x_n)_{n=1}^\infty$ in ℓ^q such that

$$x_n \overset{\sigma(\ell^q,(\ell^q)^*)}{\longrightarrow} 0 \qquad \text{and} \qquad \inf_{n\in\mathbb{N}} \|Tx_n\| > 0.$$

This implies that

$$Tx \overset{\sigma(\ell^p,(\ell^p)^*)}{\longrightarrow} 0 \qquad \text{and} \qquad \inf_{n\in\mathbb{N}} \|x_n\| > 0,$$

so that we can apply the Bessaga–Pełczyński Seclection Principle to both $(x_n)_{n=1}^\infty$ and $(Tx_n)_{n=1}^\infty$. An iterated application of selection principle yields a subsequence $(x_{n_k})_{n=1}^\infty$ such that $(x_{n_k})_{k=1}^\infty$ and $(Tx_{n_k})_{k=1}^\infty$ are equivalent to block basic sequences of the canonical bases of ℓ^q and ℓ^p, respectively. Arguing as in the proof of Theorem 3.2.4, we see that $(x_{n_k})_{k=1}^\infty$ and $(Tx_{n_k})_{k=1}^\infty$ are equivalent to the respective canonical bases of ℓ^q and ℓ^p.

Let $(\lambda_1, \lambda_2, \ldots) \in \ell^q \setminus \ell^p$. Then $\sum_{k=1}^\infty \lambda_k x_k$ converges, which entails that $\sum_{k=1}^\infty \lambda_k T x_k = T\left(\sum_{k=1}^\infty \lambda_k x_k\right)$ converges, which yields $(\lambda_1, \lambda_2, \ldots) \in \ell^p$, which is impossible. □

Corollary 3.2.12. *Let $p > 1$. Then every bounded linear operator from c_0 to ℓ^p is compact.*

Proof. Let $T : c_0 \to \ell^p$ be linear and bounded, and let $q > 1$ be conjugate to p. Then $T^* \colon \ell^q \to \ell^1$ is compact by Pitt's Theorem, as is T by Schauder's Theorem. □

Corollary 3.2.13. *Let $p, q \in [1, \infty)$. Then $\ell^p \cong \ell^q$ if and only if $p = q$.*

Corollary 3.2.14. *Let $p \in (1, \infty)$. Then $L^p([0,1]) \cong \ell^p$ if and only if $p = 2$.*

Proof. Assume that there is an isomorphism $T \colon L^p([0,1]) \to \ell^p$. Let F be a complemented subspace of $L^p([0,1])$ isomorphic to ℓ^2. Then TF is a complemented subspace of ℓ^p isomorphic to ℓ^2. By Theorem 3.2.6, this means that $\ell^2 \cong \ell^p$ and therefore $p = 2$. □

3.3 Unconditional Bases

Definition 3.3.1. A sequence $(x_n)_{n=1}^\infty$ in a Banach space E is called *boundedly complete* if $\sum_{n=1}^\infty \lambda_n x_n$ converges whenever $\sup_{n\in\mathbb{N}} \left\|\sum_{k=1}^n \lambda_k x_k\right\| < \infty$ for all $\lambda_1, \lambda_2, \ldots \in \mathbb{F}$..

Theorem 3.3.2. *Let E be a Banach space with a boundedly complete Schauder basis $(x_n)_{n=1}^\infty$. Then E is isomorphic to $\overline{\mathrm{lin}}\{x_n^* : n \in \mathbb{N}\}^*$.*

Proof. Set $F := \overline{\mathrm{lin}}\{x_n^* : n \in \mathbb{N}\} \subset E^*$. Then there is a canonical contraction $J \colon E \to F^*$. We claim that J is an isomorphism.

Let $P_1, P_2, \ldots$ be the basic projections of $(x_n)_{n=1}^\infty$, and let $C \geq 1$ be its basic constant.

Let $x \in E$ and $n \in \mathbb{N}$. Pick $\phi \in E^*$ with $\|\phi\| = 1$ and $\langle P_n x, \phi \rangle = \|P_n x\|$. Noting that $P_n^* \phi \in F$, we obtain from the definition of J that

$$\langle P_n^* \phi, J(P_n x) \rangle = \langle P_n x, P_n^* \phi \rangle = \langle P_n x, \phi \rangle = \|P_n x\|,$$

so that

$$\|J(P_n x)\|_{F^*} \geq \left| \left\langle \frac{P_n^* \phi}{\|P_n^* \phi\|}, J(P_n x) \right\rangle \right| = \frac{\|P_n x\|}{\|P_n^* \phi\|} \geq \frac{1}{C} \|P_n x\|.$$

Letting $n \to \infty$ yields $\|Jx\|_{F^*} \geq \frac{1}{C}\|x\|$. All in all, J is injective and has closed range.

By Problem 5 on Assignment #3, $(x_n^*)_{n=1}^\infty$ is a basic sequence in E^*. Let $Q_1, Q_2, \ldots$ be the corresponding projections on F, and let D be the basic constant of $(x_n^*)_{n=1}^\infty$. For $\phi \in F^*$ and $n \in \mathbb{N}$, we have

$$\left\| J \left(\sum_{k=1}^n \langle x_k^*, \phi \rangle x_k \right) \right\|_{F^*} = \left\| \sum_{k=1}^n \langle x_k^*, \phi \rangle J x_k \right\|_{F^*} = \|Q_n^* \phi\| \leq D \|\phi\|$$

and, consequently,

$$\left\| \sum_{k=1}^n \langle x_k^*, \phi \rangle x_k \right\| \leq C \left\| J \left(\sum_{k=1}^n \langle x_k^*, \phi \rangle x_k \right) \right\|_{F^*} \leq CD \|\phi\|.$$

As $(x_n)_{n=1}^\infty$ is boundedly complete, $\sum_{k=1}^\infty \langle x_k^*, \phi \rangle x_k$ converges to some $x \in E$. We have

$$Jx = \lim_{n \to \infty} J \left(\sum_{k=1}^n \langle x_k^*, \phi \rangle x_k \right) = \lim_{n \to \infty} \sum_{k=1}^n \langle x_k^*, \phi \rangle J x_k = \lim_{n \to \infty} Q_n^* \phi$$

with convergence in the norm topology. On the other hand

$$\langle y, Q_n^* \phi \rangle = \langle Q_n y, \phi \rangle \to \langle y, \phi \rangle \qquad (y \in F),$$

i.e., $Q_n^* \phi \xrightarrow{\sigma(F^*, F)} \phi$, so that $Jx = \phi$. $\square$

Corollary 3.3.3. $L^1([0,1])$ *does not have a boundedly complete Schauder basis.*

Theorem 3.3.4. *Let E be a Banach space. Then the following are equivalent for a sequence $(x_n)_{n=1}^\infty$ in E:*

(i) $\sum_{n=1}^\infty \epsilon_n x_n$ *converges for any $\epsilon_1, \epsilon_2, \ldots \in \{-1, 1\}$;*

(ii) $\sum_{n=1}^{\infty} \lambda_n x_n$ *converges for all* $\lambda_1, \lambda_2, \ldots \in \mathbb{F}$ *with* $\sup_{n\in\mathbb{N}} |\lambda_n| < \infty$*;*

(iii) *there is a compact linear map* $T\colon c_0 \to E$ *with* $T\boldsymbol{e}_n = x_n$ *for* $n \in \mathbb{N}$*;*

(iv) $\sum_{n=1}^{\infty} x_{\sigma(n)}$ *converges for each bijection* $\sigma\colon \mathbb{N} \to \mathbb{N}$*;*

(v) $\sum_{k=1}^{\infty} x_{n_k}$ *converges for any* $n_1 < n_2 < \cdots$ *in* $\mathbb{N}$*.*

Proof. (iii) $\Longrightarrow$ (ii): Let $\lambda_1, \lambda_2, \ldots \in \mathbb{F}$ with $\sup_{n\in\mathbb{N}} |\lambda_n| < \infty$. Let $\alpha_1, \alpha_2, \ldots \in \mathbb{F}$ be such that $\sum_{n=1}^{\infty} |\alpha_n| < \infty$; then the sequence $(\sum_{k=1}^{n} \alpha_k \lambda_k)_{n=1}^{\infty}$ is Cauchy. By the duality $(c_0)^* = \ell^1$, this means that $(\sum_{k=1}^{n} \lambda_k \boldsymbol{e}_n)_{n=1}^{\infty}$ is weakly Cauchy in c_0. As $T\boldsymbol{e}_n = x_n$ for $n \in \mathbb{N}$, it is clear $(\sum_{k=1}^{n} \lambda_k x_n)_{n=1}^{\infty}$ is weakly Cauchy in E. As T is compact, $(\sum_{k=1}^{n} \lambda_k x_n)_{n=1}^{\infty}$ has a norm convergent subsequence. It follows that $(\sum_{k=1}^{n} \lambda_k x_n)_{n=1}^{\infty}$ is weakly convergent. As the weak limit of $(\sum_{k=1}^{n} \lambda_k x_n)_{n=1}^{\infty}$ is unique, this means that $(\sum_{k=1}^{n} \lambda_k x_n)_{n=1}^{\infty}$ has only one norm accumulation point, i.e., converges.

(ii) $\Longrightarrow$ (v): Choose $\lambda_n = 1$ if $n \in \{n_1, n_2, \ldots\}$ and $\lambda_n = 0$ otherwise.

(v) $\Longrightarrow$ (i): Let $\epsilon_1, \epsilon_2, \ldots$ in $\{-1, 1\}$. Define $n_1 < n_2 < \cdots$ such that

$$\epsilon_n = 1 \quad \Longleftrightarrow \quad n \in \{n_1, n_2, \ldots\}.$$

We then have

$$\sum_{n=1}^{\infty} \epsilon_n x_n = 2\sum_{k=1}^{\infty} x_{n_k} - \sum_{n=1}^{\infty} x_n.$$

As $\sum_{k=1}^{\infty} x_{n_k}$ and $\sum_{n=1}^{\infty} x_n$ converge, so does $\sum_{n=1}^{\infty} \epsilon_n x_n$.

(i) $\Longrightarrow$ (iii): Let c_{00} denote the space of all finitely supported sequences in c_0. Then we can define a linear map $T : c_{00} \to E$ by letting $T\boldsymbol{e}_n := x_n$. We will show that T is compact and, consequently, extends to all of c_0.

Let $\{-1, 1\}^{\mathbb{N}}$ be equipped with the product topology, turning it into a compact Hausdorff space. Then the map

$$\{-1, 1\}^{\mathbb{N}} \to E, \quad (\epsilon_1, \epsilon_2, \ldots) \mapsto \sum_{n=1}^{\infty} \epsilon_n x_n$$

is continuous. Therefore, its range, the set $\{\sum_{n=1}^{\infty} \epsilon_n x_n : \epsilon_1, \epsilon_2 \ldots. \in \{-1, 1\}\}$, is compact in E, as is

$$K := \overline{\operatorname{absconv}\left\{\sum_{n=1}^{\infty} \epsilon_n x_n : \epsilon_1, \epsilon_2 \ldots. \in \{-1, 1\}\right\}}.$$

As

$$\operatorname{Ball}(c_{00}) = \operatorname{absconv}\{(\epsilon_n)_{n=1}^{\infty} \in c_{00} : \epsilon_n \in \{-1, , 0, 1\} \text{ for } n \in \mathbb{N}\},$$

we see that T maps $\operatorname{Ball}(c_{00})$ into the compact set K and therefore is compact and extends to all of c_0.

(iii) $\Longrightarrow$ (iv): Let $\sigma\colon \mathbb{N} \to \mathbb{N}$ be a bijection. It induces an invertible isometry $J_\sigma\colon c_0 \to c_0$ by letting $J_\sigma \boldsymbol{e}_n := \boldsymbol{e}_{\sigma(n)}$ for $n \in \mathbb{N}$. Then TJ_σ is compact such that $TJ_\sigma \boldsymbol{e}_n = x_{\sigma(n)}$. The same argument as for (iii) $\Longrightarrow$ (ii) yields the convergence of $\sum_{n=1}^\infty x_{\sigma(n)}$.

(iv) $\Longrightarrow$ (v): Assume that (v) does not hold. Then there are $\epsilon_0 > 0$ and sequences $n_1 < n_2 < \cdots$ in $\mathbb{N}$ and $0 = N_0 < N_1 < N_2 < \cdots$ in $\mathbb{N}_0$ such that

$$\left\| \sum_{k=N_\nu+1}^{N_{\nu+1}} x_{n_k} \right\| \geq \epsilon_0 \qquad (\nu \in \mathbb{N}_0).$$

Let $m_1 < m_2 < \cdots$ be such that $\{m_1, m_2, \ldots\} = \mathbb{N} \setminus \{n_1, n_2, \ldots\}$. Define a bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$ via

$$(1, 2, 3, \ldots) \mapsto (n_1, n_2, \ldots, n_{k_{N_1}}, m_1, n_{k_{N_1}+1}, \ldots, n_{k_{N_2}}, m_2, n_{k_{N_2}+1}, \ldots).$$

By construction, $\sum_{n=1}^\infty x_{\sigma(n)}$ does not converge. $\square$

Definition 3.3.5. A series $\sum_{n=1}^\infty x_n$ in a Banach space E is said to converge *unconditionally* if it satisfies the equivalent conditions of Theorem 3.3.4.

Remarks.

1. As in the scalar case, an absolutely convergent series is unconditionally convergent.

2. Unlike in the scalar case, a unconditionally convergent sequence need not converge absolutely. Let $(\boldsymbol{e}_n)_{n=1}^\infty$ be the canonical basis of c_0. Then

$$\sum_{n=1}^\infty \frac{1}{\sigma(n)} \boldsymbol{e}_{\sigma(n)} = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots\right) \in c_0,$$

for each bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$, but $\sum_{n=1}^\infty \left\| \frac{1}{n} \boldsymbol{e}_n \right\|_\infty = \sum_{n=1}^\infty \frac{1}{n} = \infty$.

Definition 3.3.6. A Schauder basis of a Banach space E is called *unconditional* if, for each $x \in E$, the series $x = \sum_{n=1}^\infty \lambda_n x_n$ converges unconditionally.

Likewise, we call a basic sequence in a Banach space *unconditional* if it is an unconditional basis of its closed linear span.

Examples.

1. The standard bases of c_0 and ℓ^p with $p \in [1, \infty)$ are unconditional.

2. Let $(\boldsymbol{e}_n)_{n=1}^\infty$ be the standard basis of c_0. For $n \in \mathbb{N}$, set $x_n := \sum_{k=1}^n \boldsymbol{e}_k$. We claim that $(x_n)_{n=1}^\infty$ is a Schauder basis for c_0 that fails to be unconditional.

   Let $x = \sum_{n=1}^\infty \lambda_n \boldsymbol{e}_n \in c_0$. For $n \in \mathbb{N}$, set $\mu_n := \lambda_n - \lambda_{n+1}$, and note that

$$\sum_{k=n}^\infty \mu_k = \lambda_n \qquad (n \in \mathbb{N}).$$

We obtain

$$
\begin{aligned}
x &= \sum_{n=1}^{\infty} \lambda_n \boldsymbol{e}_n \\
&= \sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} \mu_k \right) \boldsymbol{e}_n \\
&= \sum_{n=1}^{\infty} \mu_n \sum_{k=1}^{n} \boldsymbol{e}_k \\
&= \sum_{n=1}^{\infty} \mu_n x_n.
\end{aligned}
$$

Let $\mu_1', \mu_2', \ldots \in \mathbb{F}$ be such that $x = \sum_{n=1}^{\infty} \mu_n' x_n$. It follows that

$$\lambda_n = \sum_{k=n}^{\infty} \mu_k' \qquad (n \in \mathbb{N})$$

and therefore

$$\mu_n' = \lambda_n - \lambda_{n+1} = \mu_n \qquad (n \in \mathbb{N}).$$

All in all, $(x_n)_{n=1}^{\infty}$ is a Schauder basis for c_0.

Now, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x_n = \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n}, \sum_{n=2}^{\infty} \frac{(-1)^n}{n}, \sum_{n=3}^{\infty} \frac{(-1)^n}{n}, \ldots \right) \in c_0$$

converges whereas $\sum_{n=1}^{\infty} (-1)^n \frac{(-1)^n}{n} x_n = \sum_{n=1}^{\infty} \frac{1}{n} x_n$ doesn't. So, the basis $(x_n)_{n=1}^{\infty}$ is not unconditional.

We state the following without proof:

Proposition 3.3.7. *The following are equivalent for a sequence $(x_n)_{n=1}^{\infty}$ in a Banach space E:*

(i) *$(x_n)_{n=1}^{\infty}$ is an unconditional basic sequence;*

(ii) *there is $C > 0$ such that, for all $m \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$, and $\epsilon_1, \ldots, \epsilon_m \in \{-1, 1\}$, we have*

$$\left\| \sum_{k=1}^{m} \epsilon_k \lambda_k x_k \right\| \leq C \left\| \sum_{k=1}^{m} \lambda_k x_k \right\|;$$

(iii) *there is $C' > 0$ such that, for all $m \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$, and $\boldsymbol{\sigma} \subset \{1, \ldots, m\}$, we have*

$$\left\| \sum_{k \in \boldsymbol{\sigma}} \lambda_k x_k \right\| \leq C' \left\| \sum_{k=1}^{m} \lambda_k x_k \right\|.$$

Lemma 3.3.8. *Let E be a Banach space, let $(x_n)_{n=1}^\infty$ be an unconditional basic sequence in E, let $C \geq 0$ be as in Proposition* 3.3.7(ii)*, and let $\lambda_1, \lambda_2, \ldots$ in $\mathbb{F}$ such that $\sum_{n=1}^\infty \lambda_n x_n$ converges. Then, for each bounded sequence $(\mu_n)_{n=1}^\infty$ in $\mathbb{F}$, the series $\sum_{n=1}^\infty \mu_n \lambda_n x_n$ converges with*

$$\left\| \sum_{n=1}^\infty \mu_n \lambda_n x_n \right\| \leq C \sup_{n\in\mathbb{N}} |\mu_n| \left\| \sum_{n=1}^\infty \lambda_n x_n \right\|$$

Proof. We only prove the case where $\mathbb{F} = \mathbb{R}$.

Let $m \in \mathbb{N}$. It is enough to show that

$$\left\| \sum_{k=1}^m \mu_k \lambda_k x_k \right\| \leq C \sup_{n\in\mathbb{N}} |\mu_n| \left\| \sum_{k=1}^m \lambda_k x_k \right\|.$$

Let $\phi \in E^*$ with $\|\phi\| = 1$ be such that

$$\left\langle \sum_{k=1}^m \mu_k \lambda_k x_k, \phi \right\rangle = \left\| \sum_{k=1}^m \mu_k \lambda_k x_k \right\|.$$

For $k = 1, \ldots, m$, set

$$\epsilon_k := \begin{cases} 1, & \lambda_k \langle x_k, \phi \rangle \geq 0, \\ -1, & \lambda_k \langle x_k, \phi \rangle < 0. \end{cases}$$

It follows that

$$\begin{aligned} \left\| \sum_{k=1}^m \mu_k \lambda_k x_k \right\| &\leq \sum_{k=1}^m |\mu_k| |\lambda_k \langle x_k, \phi \rangle| \\ &\leq \sup_{n\in\mathbb{N}} |\mu_n| \sum_{k=1}^m |\lambda_k \langle x_k, \phi \rangle| \\ &= \sup_{n\in\mathbb{N}} |\mu_n| \sum_{k=1}^m \epsilon_k \lambda_k \langle x_k, \phi \rangle \\ &= \sup_{n\in\mathbb{N}} |\mu_n| \left\langle \sum_{k=1}^m \epsilon_k \lambda_k x_k, \phi \right\rangle \\ &\leq \sup_{n\in\mathbb{N}} |\mu_n| \left\| \sum_{k=1}^m \epsilon_k \lambda_k x_k \right\| \\ &\leq C \sup_{n\in\mathbb{N}} |\mu_n| \left\| \sum_{k=1}^m \lambda_k x_k \right\|, \end{aligned}$$

which completes the proof. □

Theorem 3.3.9. *Let E be a Banach space with an unconditional Schauder basis that is not boundedly complete. Then E contains a copy of c_0.*

Proof. Let $(x_n)_{n=1}^\infty$ be an unconditional Schauder basis for E that is not boundedly complete. Consequently, there are $\lambda_1, \lambda_2, \ldots$ in $\mathbb{F}$ such that

$$\left\| \sum_{k=1}^{n} \lambda_k x_k \right\| \leq 1 \qquad (n \in \mathbb{N}),$$

but with $\sum_{n=1}^\infty \lambda_n x_n$ diverging. Then there are $\epsilon_0 > 0$ and $p_1 < q_1 < p_2 < q_2 < \cdots$ such that

$$\left\| \sum_{k=p_\nu}^{q_\nu} \lambda_k x_k \right\| \geq \epsilon_0 \qquad (\nu \in \mathbb{N}).$$

Set $y_\nu := \sum_{k=p_\nu}^{q_\nu} \lambda_k x_k$ for $\nu \in \mathbb{N}$. With $C' > 0$ as in Proposition 3.3.7(iii), we obtain

$$\left\| \sum_{\nu=1}^{m} y_\nu \right\| = \left\| \sum_{\nu=1}^{m} \sum_{k=p_\nu}^{q_\nu} \lambda_k x_k \right\| \leq C' \left\| \sum_{k=1}^{q_m} \lambda_k x_k \right\| \leq C'.$$

For every bounded sequence $(\mu_n)_{n=1}^\infty$ in $\mathbb{F}$, we therefore have by the previous lemma

$$\left\| \sum_{\nu=1}^{m} \mu_\nu y_\nu \right\| \leq C \sup_{n \in \mathbb{N}} |\mu_n| \left\| \sum_{\nu=1}^{m} y_\nu \right\| \leq CC' \leq \sup_{n \in \mathbb{N}} |\mu_n|.$$

On the other hand, Proposition 3.3.7(iii) implies that

$$\left\| \sum_{\nu=1}^{m} \mu_\nu y_\nu \right\| \geq \frac{1}{C'} \|\mu_\nu y_\nu\| \geq \frac{\epsilon_0}{C'} |\mu_\nu| \qquad (\nu = 1, \ldots, m).$$

and therefore

$$\left\| \sum_{\nu=1}^{m} \mu_\nu y_\nu \right\| \geq \frac{\epsilon_0}{C'} \max\{|\mu_1|, \ldots, |\mu_m|\}.$$

All in all $(y_\nu)_{\nu=1}^\infty$ is equivalent to the standard basis of c_0. $\square$

For the proof of the following corollary, we require the fact that $L^1([0,1])$ is weakly sequentially complete.

Corollary 3.3.10. $L^1([0,1])$ *does not have an unconditional basis.*

Proof. Assume that $L^1([0,1])$ has an unconditional basis. By Corollary 3.3.3, this basis cannot be boundedly complete. Therefore, $L^1([0,1])$ contains a copy of c_0. Let $(\boldsymbol{e}_n)_{n=1}^\infty$ be the standard basis of c_0. Then $\left(\sum_{k=1}^n \boldsymbol{e}_k\right)_{n=1}^\infty$ is weakly Cauchy, but not weakly convergent. As $L^1([0,1])$ is weakly sequentially complete, this is impossible. $\square$

Chapter 4

Local Structure and Geometry of Banach Spaces

4.1 Finite-dimensional Structure

Definition 4.1.1. Let E be a normed space. An n-tuple $(x_1, x_1^*), \ldots, (x_n, x_n^*)$ in $E \times E^*$ is called an *Auerbach basis* for E if:

(a) $x_1, \ldots, x_n$ is a basis for E;

(b) $\langle x_j, x_k^* \rangle = \delta_{j,k}$ for $j, k = 1, \ldots, n$;

(c) $\|x_j\| = \|x_j^*\| = 1$ for $j = 1, \ldots, n$.

Theorem 4.1.2 (Auerbach's "Lemma")**.** *Let E be a finite-dimensional normed space. Then E has an Auerbach basis.*

Proof. Set $n := \dim E$, and let $x_1, \ldots, x_n$ be a Hamel basis for E.

Let $y_1, \ldots, y_n \in \operatorname{Ball}(E)$ with

$$y_k = \sum_{j=1}^{n} \lambda_{k,j} x_j$$

and define

$$v(y_1, \ldots, y_n) := \det \begin{bmatrix} \lambda_{1,1}, & \ldots, & \lambda_{1,n} \\ \vdots & \ddots & \vdots \\ \lambda_{n,1}, & \ldots, & \lambda_{n,n} \end{bmatrix}$$

Then

$$v \colon \operatorname{Ball}(E)^n \to \mathbb{F}, \quad (y_1, \ldots, y_n) \mapsto v(y_1, \ldots, y_n)$$

is continuous. As $\operatorname{Ball}(E)^n$ is compact, there is $(e_1, \ldots, e_n) \in \operatorname{Ball}(E)^n$ such that

$$|v(e_1, \ldots, e_n)| = \sup\{|v(y_1, \ldots, y_n)| : y_1, \ldots, y_n \in \operatorname{Ball}(E)\}.$$

Assume towards a contradiction that $\|e_k\| < 1$ for some $k \in \{1, \ldots, n\}$. Without loss of generality, suppose that $k = 1$. As

$$\left| v\Bigg(\underbrace{\frac{1}{\|e_1\|} e_1, e_2, \ldots, e_n}_{\in \mathrm{Ball}(E)^n} \Bigg) \right| = \frac{1}{\|e_1\|} |v(e_1, \ldots, e_n)| > |v(e_1, \ldots, e_n)|,$$

we obtain a contradiction, so that $\|e_1\| = \cdots = \|e_n\| = 1$.

Obviously, $e_1, \ldots, e_n$ are linearly independent. For $k = 1, \ldots, n$, define e_k^* via

$$e_k^* \colon E \to \mathbb{F}, \quad x \mapsto \frac{v(e_1, \ldots, e_{k-1}, x, e_{k+1}, \ldots, e_n)}{v(e_1, \ldots, e_n)}.$$

Then $e_1^*, \ldots, e_n^*$ are linear such that $\langle e_j, e_k^* \rangle = \delta_{j,k}$ for $j, k = 1, \ldots, n$ and

$$|\langle x, e_k^* \rangle| = \frac{|v(e_1, \ldots, e_{k-1}, x, e_{k+1}, \ldots, e_n)|}{|v(e_1, \ldots, e_n)|} \leq \frac{|v(e_1, \ldots, e_n)|}{|v(e_1, \ldots, e_n)|} \qquad (x \in \mathrm{Ball}(E)).$$

All in all, $(e_1, e_1^*), \ldots, (e_n, e_n^*)$ is an Auerbach basis for E. □

Corollary 4.1.3. *Let E be a normed space, and let F be a subspace of E with $n := \dim F < \infty$. Then there is a projection P from E onto F such that $\|P\| \leq n$.*

Proof. Let $(f_1, f_1^*), \ldots, (f_n, f_n^*)$ be an Auerbach basis for F. Use the Hahn–Banach Theorem to extend $f_1^*, \ldots, f_n^*$ to E as norm one functionals. Define

$$P \colon E \to E, \quad x \mapsto \sum_{k=1}^{n} \langle x, f_k^* \rangle f_k.$$

Then $PE \subset F$ and $P|_F = \mathrm{id}_F$. Moreover,

$$\|Px\| \leq \sum_{k=1}^{n} |\langle x, f_k^* \rangle| \|f_k\| \leq n\|x\|$$

holds. □

Given normed spaces E and F, we denote

$$\mathcal{B}(E, F) := \{T \colon E \to F : T \text{ is linear and bounded}\};$$

if $E = F$, we simply write $\mathcal{B}(E)$ instead of $\mathcal{B}(E, F)$.

Lemma 4.1.4. *Let F be a Banach space, then there is a natural isometric isomorphism from $\mathcal{B}(\ell_N^1, F)^{**}$ to $\mathcal{B}(\ell_N^1, F^{**})$.*

Proof. Let $e_1, \ldots, e_N$ be the canonical basis vectors of ℓ^1_N. Define

$$\mathcal{B}(\ell^1_N, F) \to \ell^\infty_N(F), \quad T \mapsto (Te_1, \ldots, Te_N). \tag{4.1}$$

It is ovious that (4.1) is onto and that

$$\|(Te_1, \ldots, Te_N)\|_\infty = \max_{k=1,\ldots,N} \|Te_k\| \leq \|T\| \qquad (T \in \mathcal{B}(\ell^1_N, F))$$

We claim that (4.1) is an isometry. Let $\lambda_1, \ldots, \lambda_N \in \mathbb{F}$ such that $\sum_{k=1}^N |\lambda_k| \leq 1$. For $T \in \mathcal{B}(\ell^1_N, F)$, we therefore have

$$\|(Te_1, \ldots, Te_N)\|_\infty \geq \|(T(\lambda_1 e_1), \ldots, T(\lambda_N e_N))\|_\infty = \|T(\lambda_1, \ldots, \lambda_N)\|,$$

i.e.,

$$\|(Te_1, \ldots, Te_N)\|_\infty \geq \|T\|.$$

We therefore obtain the isometrical identifications

$$\mathcal{B}(\ell^1_N, F)^{**} \cong \ell^\infty_N(F)^{**} \cong \ell^1_N(F^*)^* \cong \ell^\infty_N(F^{**}) \cong \mathcal{B}(\ell^1_N, F^{**}),$$

which prove the claim. $\square$

Remark. Let $T \in \mathcal{B}(\ell^1_N, F)^{**}$. Then, by Goldstine's Theorem, there is a net $(T_\alpha)_\alpha$ in $\mathcal{B}(\ell^1_N, F)$ with $\|T_\alpha\| \leq \|T\|$ for all indices α and $T_\alpha \overset{\sigma(\mathcal{B}(\ell^1_N, F^{**}), \sigma(\mathcal{B}(\ell^1_N, F)^*)}{\longrightarrow} T$. For $k = 1, \ldots, N$, the net $(T_\alpha e_k)_\alpha$ is weak* convergent in F^{**} and therefore has a limit, say $y_k \in F^{**}$. Consequently, the desired map is

$$\mathcal{B}(\ell^1_N, F)^{**} \to \mathcal{B}(\ell^1_N, F^{**}), \quad T \mapsto (y_1, \ldots, y_N).$$

Theorem 4.1.5. *Let E and F be Banach spaces with $\dim E < \infty$. Then there is a natural isometric isomorphism between $\mathcal{B}(E, F)^{**}$ and $\mathcal{B}(E, F^{**})$.*

Proof. Let $T \in \mathcal{B}(E, F)^{**}$ and use Goldstine's Theorem to find a net $(T_\alpha)_\alpha$ in $\mathcal{B}(E, F)$ with $\|T_\alpha\| \leq \|T\|$ for all indices α and $T_\alpha \overset{\sigma(\mathcal{B}(E, F^{**}), \mathcal{B}(E, F)^*))}{\longrightarrow} T$.

Given $x \in E$ and $\phi \in F^*$, define $\omega_{x,\phi} \in \mathcal{B}(E, F)^*$ by letting

$$\langle S, \omega_{x,\phi} \rangle := \langle Sx, \phi \rangle \qquad (S \in \mathcal{B}(E, F)),$$

so that

$$\langle T, \omega_{x,\phi} \rangle = \lim_\alpha \langle T_\alpha, \omega_{x,\phi} \rangle = \lim_\alpha \langle T_\alpha x, \phi \rangle.$$

Consequently, $\hat{T} \in \mathcal{B}(E, F^{**})$ defined by $\hat{T}x := \sigma(F^{**}, F^*)\text{-}\lim_\alpha T_\alpha x$ for $x \in E$ is well defined and independent of the choice of the net $(T_\alpha)_\alpha$. Define

$$\Theta \colon \mathcal{B}(E, F)^{**} \to \mathcal{B}(E, F^{**}), \quad T \mapsto \hat{T}.$$

It is clear that Θ is a linear contraction (and if $E = \ell^1_N$, then Θ is just the map from the previous lemma).

Let $\epsilon \in (0,1)$, and choose $x_1, \dots, x_N \in E$ of norm one such that, for each $x \in E$ with $\|x\| = 1$, there is $k \in \{1, \dots, N\}$ such that $\|x - x_k\| < \epsilon$.

Consider

$$j \colon E^* \to \ell^\infty_N, \quad \phi \mapsto (\langle x_1, \phi \rangle, \dots, \langle x_N, \phi \rangle).$$

Clearly, j is linear such that

$$(1-\epsilon)\|\phi\| \leq \|j\phi\| \leq \|\phi\|$$

For *any* Banach space G, define

$$J_G \colon \mathcal{B}(E, G) \to \mathcal{B}(\ell^1_N, G), \quad T \mapsto T \circ j^*.$$

It follows that

$$\begin{aligned}
\|J_G T\| &= \sup\{|\langle J_G T x, \phi\rangle| : x \in \mathrm{Ball}(\ell^1_N),\ \phi \in \mathrm{Ball}(G^*)\} \\
&= \sup\{|\langle x, jT^*\phi\rangle| : x \in \mathrm{Ball}(\ell^1_N),\ \phi \in \mathrm{Ball}(G^*)\} \\
&= \sup\{\|jT^*\phi\| : \phi \in \mathrm{Ball}(G^*)\} \\
&\geq (1-\epsilon) \sup\{\|T^*\phi\| : \phi \in \mathrm{Ball}(G^*)\} \\
&= (1-\epsilon)\|T\|.
\end{aligned}$$

We therefore obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{B}(E,F)^{**} \arrow[r, "{J_F}^{**}"] \arrow[d, "\Theta"'] & \mathcal{B}(\ell^N, F)^{**} \arrow[d] \\
\mathcal{B}(E,F^{**}) \arrow[r, "J_{F^{**}}"] & \mathcal{B}(\ell^N, F^{**})
\end{tikzcd}$$

with the second column being an isometry. It follows that

$$(1-\epsilon)\|T\| \leq \|\Theta T\| \leq \|T\| \qquad (T \in \mathcal{B}(E,F)^{**}).$$

As $\epsilon \in (0,1)$ was arbitrary, this means that Θ is an isometry.

Finally, set $M := \dim E$, so that $E \cong \ell^1_M$, albeit not necessarily isometrically. We obtain isomorphisms

$$\mathcal{B}(E,F)^{**} \cong \mathcal{B}(\ell^1_M, F)^{**} \cong \mathcal{B}(\ell^1_M, F^{**}) \cong \mathcal{B}(E, F^{**}),$$

which proves that Θ is surjective. $\square$

Lemma 4.1.6 (Helly's Lemma)**.** *Let E be a Banach space, and let Φ be a finite-dimensional subspace of E^*. Then, for each $X \in E^{**}$ and $\delta > 0$, there is $x \in E$ such that $\|x\| \leq (1+\delta)\|X\|$ and*

$$\langle x, \phi \rangle = \langle \phi, X \rangle \qquad (\phi \in \Phi).$$

Proof. Consider the quotient map $\pi \colon E \to E/\Phi_\circ$, and note that $\dim E/\Phi_\circ < \infty$. Consider

$$\pi^{**} \colon E^{**} \to (E/\Phi_\circ)^{**} \cong E/\Phi_\circ$$

and note that $\ker \pi^{**} = \Phi^\circ$. It follows that π^{**} induces an isometric isomorphism of E^{**}/Φ° and $E/\Phi_\circ$. By the definition of the quotient norm there is therefore, for each $X \in E^{**}$ and $\delta > 0$, an element $x \in E$ with $\pi(x) = \pi^{**}(X)$ and $\|x\| \leq (1+\delta)\|x\|$. $\square$

Corollary 4.1.7. *Let E and F be Banach spaces with $\dim E < \infty$, and let Φ be a finite-dimensional subspace of F^*. Then, for each $T \in \mathcal{B}(E, F^{**})$ and $\delta > 0$, there is $S \in \mathcal{B}(E, F)$ with $\|S\| \leq (1+\delta)\|T\|$ such that*

$$\langle \phi, Tx \rangle = \langle Sx, \phi \rangle \qquad (x \in E,\ \phi \in \Phi).$$

Proof. Use the isometric identity $\mathcal{B}(E, F^{**}) \cong \mathcal{B}(E, F)^{**}$, and apply Helly's Lemma to the finite-dimensional subspace of $\mathcal{B}(E, F)^*$ spanned by $\{\omega_{x,\phi} : x \in E,\ \phi \in \Phi\}$. $\square$

Theorem 4.1.8 (Principle of Local Reflexivity)**.** *Let E be a Banach space, let Φ be a finite-dimensional subspace of E^*, let F be a finite-dimensional subspace of E^{**}, and let $\epsilon > 0$. Then there is an injective linear map $\tau \colon F \to E$ such that:*

(i) $\tau|_{E \cap F} = \mathrm{id}_{E \cap F}$;

(ii) $\|\tau\| \|\tau^{-1}|_{\tau(F)}\| < 1 + \epsilon$;

(iii) $\langle \tau(X), \phi \rangle = \langle \phi, X \rangle \qquad (X \in F,\ \phi \in \Phi)$.

Proof. As $\lim_{\delta \to 0} \frac{1+\delta}{1-3\delta} = 1$, we can fix $\delta \in \left(0, \frac{1}{3}\right)$ such that

$$\frac{1+\delta}{1-3\delta} < 1 + \epsilon.$$

Also, fix $X_1, \ldots, X_n \in F$ with $\|X_1\| = \cdots = \|X_n\| = 1$ such that, for each $X \in F$ with $\|X\| = 1$, there is $j \in \{1, \ldots, n\}$ such that $\|X - X_j\| < \delta$.

Choose $\phi_1, \ldots, \phi_n \in E^*$ with $\|\phi_1\| = \cdots = \|\phi_n\| = 1$ such that

$$\langle \phi_j, X_j \rangle \geq 1 - \delta \qquad (j = 1, \ldots, n).$$

Set $\Psi := \mathrm{lin}(\Phi \cup \{\phi_1, \ldots, \phi_n\})$. Apply Corollary 4.1.7, with

- E replaced by F,

- F replaced by E,
- Φ replaced by Ψ, and
- T being the canonical embedding $F \hookrightarrow E^{**}$.

The corollary, then yields $\tau \in \mathcal{B}(F, E)$ with $\|\tau\| \leq 1 - \delta$ such that

$$\langle \tau(X), \phi \rangle = \langle \phi, X \rangle \qquad (X \in F,\ \phi \in \Psi).$$

This means, in particular, that τ satisfies (iii).

Suppose that (i) is wrong, i.e., there is $x \in E \cap F$ such that $\tau(x) \neq x$. Then, by the choice of $X_1, \ldots, X_n \in F$, there is $j \in \{1, \ldots, n\}$ such that

$$\left\| \frac{\tau(x) - x}{\|\tau(x) - x\|} - X_j \right\| < \delta.$$

This implies

$$1 - \delta \leq \langle \phi_j, X_j \rangle = \left| \left\langle \phi_j, \frac{\tau(x) - x}{\|\tau(x) - x\|} - X_j \right\rangle \right| < \delta,$$

so that $\delta > \frac{1}{2}$. This contradicts the choice of δ. So, τ satisfies (i) as well.

Let $X \in F$ with $\|X\| = 1$, and choose $j \in \{1, \ldots, n\}$ such that $\|X - X_j\| < \delta$. It follows that

$$\|\tau(X_j)\| \leq \|\tau(X)\| + \|\tau(X - X_j)\| \leq \|\tau(X)\| + (1 + \delta)\delta \leq \|\tau(X)\| + 2\delta.$$

On the other hand,

$$\|\tau(X_j)\| \geq |\langle \tau(X_j), \phi_j \rangle| = |\langle \phi_j, X_j \rangle| \geq 1 - \delta,$$

so that

$$\|\tau(X)\| \geq \|\tau(X_j)\| - 2\delta \geq 1 - 3\delta. \tag{4.2}$$

This means that τ is bounded below and therefore injective. Morever, (4.2) yields immediately that $\|\tau^{-1}|_{\tau(F)}\| \leq \frac{1}{1-3\delta}$. As $\|\tau\| \leq 1 + \delta$, the choice of δ yields that

$$\|\tau\| \|\tau^{-1}|_{\tau(F)}\| \leq \frac{1 + \delta}{1 - 3\delta} < 1 + \epsilon.$$

This proves (ii). □

4.2 Ultraproducts

Definition 4.2.1. Let $\mathbb{I}$ be a set. A *filter* over $\mathbb{I}$ is a subset $\mathcal{F}$ of $\mathfrak{P}(\mathbb{I})$ such that:

(a) $\varnothing \notin \mathcal{F}$;

(b) if $F \in \mathcal{F}$ and $S \subset \mathbb{I}$ with $F \subset S$, then $S \in \mathcal{F}$;

(c) if $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$.

Example. Let X be a topological space, and let $x \in X$. Then

$$\mathcal{N}_x := \{N \subset X : N \text{ is a neighborhood of } x\}$$

is a filter over X; it is called the *neighborhood filter* of x.

Remark. Let $\mathbb{I}$ be a set. Then a *filter basis* over $\mathbb{I}$ is a subset $\mathcal{S}$ of $\mathfrak{P}(\mathbb{I})$ such that (a) $\varnothing \notin \mathcal{S}$ and (b), for all $S_1, S_2 \in \mathcal{S}$, there is $S_3 \in \mathcal{S}$ with $S_3 \subset S_1 \cap S_2$. If $\mathcal{S}$ is a filter basis, then

$$\mathcal{F} := \{F \subset \mathbb{I} : \text{there is } S \in \mathcal{S} \text{ such that } S \subset F\}$$

is a filter and, in fact, the smallest filter containing $\mathcal{S}$. We call $\mathcal{F}$ the *filter generated by* $\mathcal{S}$.

Examples.

1. Let $\mathbb{I}$ be a directed set with the order being denoted by $\preceq$. For $i \in \mathbb{I}$, set

   $$S_i := \{j \in \mathbb{I} : i \preceq j\}.$$

   Let $i_1, i_2 \in \mathbb{I}$. As $\mathbb{I}$ is directed, there is $i_3 \in \mathbb{I}$ such that $i_1 \preceq i_3$ and $i_2 \preceq i_3$, so that $S_{i_3} \subset S_{i_1} \cap S_{i_2}$. This means that $\{S_i : i \in \mathbb{I}\}$ is a filter basis over $\mathbb{I}$. The filter $\mathcal{F}_{\preceq}$ it generates is called the *order filter* over $\mathbb{I}$.

2. Let $\mathbb{N}$ be a equipped with its natural order $\leq$. Then it is easy to see that

   $$\mathcal{F}_{\leq} = \{F \subset \mathbb{N} : \mathbb{N} \setminus F \text{ is finite}\}.$$

   This filter is also called the *Fréchet filter* over $\mathbb{N}$.

Definition 4.2.2. Let $\mathbb{I}$ be a set. A filter $\mathcal{U}$ over $\mathbb{I}$ is called an *ultrafilter* if there is no filter $\mathcal{F}$ over $\mathbb{I}$ such that $\mathcal{U} \subsetneq \mathcal{F}$.

Example. Let $i \in \mathbb{I}$. Then

$$\mathcal{U}_i := \{S \subset \mathbb{I} : i \in S\}$$

is an ultrafilter over $\mathbb{I}$. Ultrafilters of this form are called *fixed*; all other ultrafilters are called *free*.

Proposition 4.2.3. *Let $\mathbb{I}$ be a set, and let $\mathcal{F}$ be a filter over $\mathbb{I}$. Then there is an ultrafilter $\mathcal{U}$ over $\mathbb{I}$ such that $\mathcal{F} \subset \mathcal{U}$.*

Proof. Zorn's Lemma. □

Proposition 4.2.4. *Let $\mathbb{I}$ be a set, and let $\mathcal{U}$ be a filter over $\mathbb{I}$. Then the following are equivalent:*

(i) $\mathcal{U}$ *is an ultrafilter;*

(ii) *for each* $S \subset \mathbb{I}$*, either* $S \in \mathcal{U}$ *or* $\mathbb{I} \setminus S \in \mathcal{U}$*.*

Proof. (i) $\Longrightarrow$ (ii): Let $S \subset \mathbb{I}$ be such that $S \notin \mathcal{U}$. Set

$$\mathcal{F} := \{T \subset \mathbb{I} : \text{there is } U \in \mathcal{U} \text{ such that } (\mathbb{I} \setminus S) \cap U \subset T\}.$$

It is clear that $\mathcal{U} \subset \mathcal{F}$ and that $\mathbb{I} \setminus S \in \mathcal{F}$.

We claim that $\mathcal{F}$ is a filter. Obviously, $\mathcal{F}$ is closed under taking supersets and finite intersections. Assume that $\varnothing \in \mathcal{F}$. Then there is $U \in \mathcal{U}$ such that $(\mathbb{I} \setminus S) \cap U = \varnothing$, i.e., $U \subset S$. But this means that $S \in \mathcal{U}$, which is a contradiction. All in all, $\mathcal{F}$ is a filter, so that $\mathcal{F} = \mathcal{U}$ and therefore $\mathbb{I} \setminus S \in \mathcal{U}$.

(ii) $\Longrightarrow$ (i): Assume that $\mathcal{U}$ is not an ultrafilter. Then, by Proposition 4.2.3, there is an ultrafilter $\mathcal{V}$ over $\mathbb{I}$ such that $\mathcal{U} \subsetneq \mathcal{V}$. Let $S \in \mathcal{V} \setminus \mathcal{U}$. Then $\mathbb{I} \setminus S \in \mathcal{U} \subset \mathcal{V}$, so that $\varnothing = S \cap (\mathbb{I} \setminus S) \in \mathcal{V}$, which is a contradiction. $\square$

Corollary 4.2.5. *Let* $\mathbb{I}$ *be a set, let* $\mathcal{U}$ *be an ultrafilter over* $\mathbb{I}$*, and let* $S_1, \ldots, S_n \subset \mathbb{I}$ *be such that* $S_1 \cup \cdots \cup S_n = \mathbb{I}$*. Then there is* $j \in \{1, \ldots, n\}$ *such that* $S_j \in \mathcal{U}$*.*

Proof. Assume otherwise, i.e., $S_j \notin \mathcal{U}$ for $j = 1, \ldots, n$. By Proposition 4.2.4, this means that $\mathbb{I} \setminus S_j \in \mathcal{U}$ for $j = 1, \ldots, n$. It follows that

$$\mathcal{U} \ni \bigcap_{j=1}^{n} (\mathbb{I} \setminus S_j) = \mathbb{I} \setminus \bigcup_{j=1}^{n} S_j = \varnothing,$$

which is a contradiction. $\square$

Definition 4.2.6. Let X be a topological space, let $\mathbb{I}$ be a set, and let $\mathcal{F}$ be a filter over $\mathbb{I}$. Then a family $(x_i)_{i \in \mathbb{I}}$ in X is said to *converge to* $x \in X$ *along* $\mathcal{F}$—in symbols: $x = \lim_{i \to \mathcal{F}} x_i$—if $\{i \in \mathbb{I} : x_i \in N\} \in \mathcal{F}$ for each $N \in \mathcal{N}_x$.

Remark. Let $\mathbb{I}$ be directed. Then $(x_i)_{i \in \mathbb{I}}$ is a net, and we can speak of $x = \lim_i x_i$. On the other hand, there is the order filter $\mathcal{F}_{\preceq}$ over $\mathbb{I}$, so that we can speak of $x = \lim_{i \to \mathcal{F}_{\preceq}} x_i$ as well.

Suppose that $x = \lim_i x_i$, and let $N \in \mathcal{N}_x$. Then there is $i_N \in \mathbb{I}$ such that $x_i \in N$ for all $i \in \mathbb{I}$ with $i_N \preceq i$. As $S_{i_N} = \{i \in \mathbb{I} : i_N \preceq i\}$ lies in $\mathcal{F}_{\preceq}$, so is its superset $\{i \in \mathbb{I} : x_i \in N\}$. It follows that $x = \lim_{i \to \mathcal{F}_{\preceq}} x_i$.

Conversely, suppose that $x = \lim_{i \to \mathcal{F}_{\preceq}} x_i$. Let $N \in \mathcal{N}_x$, so that $F := \{i \in \mathbb{I} : x_i \in N\} \in \mathcal{F}_{\preceq}$. By the definition of $\mathcal{F}_{\preceq}$, this means that there is $i_0 \in \mathbb{I}$ such that $S_{i_0} \subset F$, i.e., $x_i \in N$ for all $i \in \mathbb{I}$ with $i_0 \preceq i$. It follows that $x = \lim_i x_i$.

All in all,

$$x = \lim_i x_i \qquad \Longleftrightarrow \qquad x = \lim_{i \to \mathcal{F}_{\preceq}} x_i.$$

Theorem 4.2.7. *Let K be a compact topological space, let $\mathbb{I}$ be a set, let $(x_i)_{i\in\mathbb{I}}$ be a family in K, and let $\mathcal{U}$ be an ultrafilter over $\mathbb{I}$. Then $\lim_{i\to\mathcal{U}} x_i$ exists in K.*

Proof. Assume otherwise. Then, for each $x \in K$, there is $N_x \in \mathcal{N}_x$ such that $\{i \in \mathbb{I} : x_i \in N_x\} \notin \mathcal{U}$. For each $x \in K$, there is an open set $U_x \subset K$ with $x \in U_x \subset N_x$. For each $x \in K$, set

$$S_x := \{i \in \mathbb{I} : x_i \in U_x\},$$

so that $S_x \subset \{i \in \mathbb{I} : x_i \in N_x\} \notin \mathcal{U}$.

As K is compact, there are $x_1, \ldots, x_n \in K$ such that $K = U_{x_1} \cup \cdots \cup U_{x_n}$. It follows that $\mathbb{I} = S_{x_1} \cup \cdots \cup S_{x_n}$. By Corollary 4.2.5, this means that there is $j \in \{1, \ldots, n\}$ with $S_{x_j} \in \mathcal{U}$, which is a contradiction. $\square$

Given a family of Banach spaces $(E_i)_{i\in\mathbb{I}}$, we had defined

$$\ell^\infty\text{-}\bigoplus_{i\in\mathbb{I}} E_i = \left\{(x_i)_{i\in\mathbb{I}} : x_i \in E_i \text{ for } i \in \mathbb{I} \text{ and } \sup_{i\in\mathbb{I}} \|x_i\| < \infty\right\}.$$

Definition 4.2.8. Let $(E_i)_{i\in\mathbb{I}}$ be a family of Banach spaces, and let $\mathcal{U}$ be an ultrafilter over $\mathbb{I}$. Set

$$\mathcal{N}_{\mathcal{U}} := \left\{(x_i)_{i\in\mathbb{I}} \in \ell^\infty\text{-}\bigoplus_{i\in\mathbb{I}} E_i : \lim_{i\to\mathcal{U}} \|x_i\| = 0\right\}.$$

Then the quotient space

$$(E_i)_{\mathcal{U}} := \ell^\infty\text{-}\bigoplus_{i\in\mathbb{I}} E_i \Big/ \mathcal{N}_{\mathcal{U}}$$

is called the *ultraproduct* of $(E_i)_{i\in\mathbb{I}}$ with respect to $\mathcal{U}$. If $E_i = E$ for $i \in \mathbb{I}$, we call $(E)_{\mathcal{U}}$ an *ultrapower* of E.

Remark. Given $(x_i)_{i\in\mathbb{I}} \in \ell^\infty\text{-}\bigoplus_{i\in\mathbb{I}} E_i$, we write $(x_i)_{\mathcal{U}}$ for its equivalence class in $(E_i)_{\mathcal{U}}$.

Proposition 4.2.9. *Let $(E_i)_{i\in\mathbb{I}}$ be a family of Banach spaces, and let $\mathcal{U}$ be an ultrafilter over $\mathbb{I}$. Then:*

(i) *$\mathcal{N}_{\mathcal{U}}$ is a closed subspace of $\ell^\infty\text{-}\bigoplus_{i\in\mathbb{I}} E_i$;*

(ii) *for each $(x_i)_{i\in\mathbb{I}} \in \ell^\infty\text{-}\bigoplus_{i\in\mathbb{I}} E_i$, we have*

$$\|(x_i)_{\mathcal{U}}\| = \lim_{i\to\mathcal{U}} \|x_i\|.$$

Proof. Define

$$p\colon \ell^\infty\text{-}\bigoplus_{i\in\mathbb{I}} E_i \to [0,\infty), \quad (x_i)_{i\in\mathbb{I}} \mapsto \lim_{i\to\mathcal{U}} \|x_i\|.$$

Then p is a seminorm on $\ell^\infty\text{-}\bigoplus_{i\in\mathbb{I}} E_i$ with

$$p((x_i)_{i\in\mathbb{I}}) \leq \sup_{i\in\mathbb{I}} \|x_i\| \quad \left((x_i)_{i\in\mathbb{I}} \in \ell^\infty\text{-}\bigoplus_{i\in\mathbb{I}} E_i\right) \qquad \text{and} \qquad \ker p = \mathcal{N}_{\mathcal{U}}.$$

As p is, in particular, continuous, this proves (i).

Obviously, p drops to a norm—likewise denoted by p—on $(E_i)_{\mathcal{U}}$ such that

$$p((x_i)_{\mathcal{U}}) \leq \|(x_i)_{\mathcal{U}}\| \qquad ((x_i)_{\mathcal{U}} \in (E_i)_{\mathcal{U}}). \tag{4.3}$$

To see that the inequality (4.3) can be reversed, let $(x_i)_{\mathcal{U}}$, and set $r := p((x_i)_{\mathcal{U}})$. Let $\epsilon > 0$. Then

$$U_\epsilon := \{i \in \mathbb{I} : |\|x_i\| - r| < \epsilon\} \in \mathcal{U}.$$

Define $(y_i)_{i\in\mathbb{I}}$ by letting

$$y_i := \begin{cases} 0, & i \in U_\epsilon, \\ -x_i, & \text{otherwise.} \end{cases}$$

Then $(y_i)_{i\in\mathbb{I}} \in \mathcal{N}_{\mathcal{U}}$ such that

$$\sup_{i\in\mathbb{I}} \|x_i + y_i\| = \sup_{i\in U_\epsilon} \leq r + \epsilon,$$

so that $\|(x_i)_{\mathcal{U}}\| \leq r + \epsilon$ and therefore $\|(x_i)_{\mathcal{U}}\| \leq r$. $\square$

Theorem 4.2.10. *Let $(E_i)_{i\in\mathbb{I}}$ be a family of Banach spaces, and let $\mathcal{U}$ be an ultrafilter over $\mathbb{I}$. Then:*

(i) *if each E_i is a Hilbert space, then $(E_i)_{\mathcal{U}}$ is a Hilbert space;*

(ii) *if each E_i is of the form $L^p(X_i, \mathcal{S}_i, \mu_i)$ for some measure space $(X_i, \mathcal{S}_i, \mu)$ and $p \in [1, \infty)$ independent of i, then there is a measure space $(X, \mathcal{S}, \mu)$ such that $(E_i)_{\mathcal{U}} \cong L^p(X, \mathcal{S}, \mu)$ holds isometrically;*

(iii) *if each E_i is of the form $\mathcal{C}(K_i)$ for some compact Hausdorff space K_i, then there is a compact Hausdorff space K such that $(E_i)_{\mathcal{U}} \cong \mathcal{C}(K)$ holds isometrically.*

Proof. (of (i) only). Define

$$\langle\cdot|\cdot\rangle : (E_i)_{\mathcal{U}} \times (E_i)_{\mathcal{U}} \mapsto \mathbb{F}, \quad ((\xi_i)_{\mathcal{U}}, (\eta_i)_{\mathcal{U}}) \to \lim_{i\to\mathcal{U}} \langle \xi_i | \eta_i \rangle.$$

Then $\langle\cdot|\cdot\rangle$ is an inner product on $(E_i)_{\mathcal{U}}$ such that

$$\|(\xi_i)_{\mathcal{U}}\|^2 = \lim_{i\to\mathcal{U}} \|\xi_i\|^2 = \lim_{i\to\mathcal{U}} \langle \xi_i | \xi_i \rangle^2 = \langle (\xi_i)_{\mathcal{U}} | (\xi_i)_{\mathcal{U}} \rangle \qquad ((\xi_i)_{\mathcal{U}} \in (E_i)_{\mathcal{U}}).$$

This proves that $(E_i)_{\mathcal{U}}$ is a Hilbert space. $\square$

Remarks. 1. The proof of (ii) requires the fact that the L^p-spaces for $p \in [1, \infty)$ can be axiomatically characterized in the category of Banach lattices (Kakutani–Bohnenblust–Nakano Theorem). It can be shown that these axioms are inherited by ultraproducts, which then yields (ii).

2. The proof of (iii), requires that $\mathcal{C}(K)$-spaces are precisely the unital commutative C^*-algebras (commutative Gelfand–Naimark Theorem). As the ultraproduct of unital commutative C^*-algebras is again a unital commutative C^*-algebra, this yields (iii).

Given an Banach space E, there is a canonical embedding of E into any ultrapower $(E)_{\mathcal{U}}$: this is obvious. The following is less straightforward:

Theorem 4.2.11. *Let E be a Banach space. Then there are an index set $\mathbb{I}$, an ultrafilter $\mathcal{U}$ over $\mathbb{I}$, and a linear isometry $J \colon E^{**} \to (E)_{\mathcal{U}}$, which extends the canonical embedding of E into $(E)_{\mathcal{U}}$. Moreover, there is a norm one projection from $(E)_{\mathcal{U}}$ onto JE^{**}.*

Proof. Let $\mathbb{I}$ consist of all triples $i = (F_i, \Phi_i, \epsilon_i)$, where

- F_i is a finite-dimensional subspace of E^{**},
- Φ_i is a finite-dimensional subspace of E^*, and
- $0 < \epsilon_i \le 1$.

For each $i \in \mathbb{I}$, the Principle of Local Reflexivity yields an injective, linear map $\tau_i \colon F_i \to E$ such that

(A) $\tau_i|_{E \cap F_i} = \mathrm{id}_{E \cap F_i}$,

(B) $\|\tau_i\| \|\tau_i^{-1}|_{\tau_i(F_i)}\| < 1 + \epsilon_i$, and

(C) $\langle \tau_i(X), \phi \rangle = \langle \phi, X \rangle$ for $X \in F_i$ and $\phi \in \Phi_i$.

Define

$$\tilde{J} \colon E^{**} \to \ell^\infty\text{-}\bigoplus_{i \in \mathbb{I}} E, \quad X \mapsto (x_i)_{i \in \mathbb{I}}$$

where

$$x_i := \begin{cases} \tau_i(X), & X \in F_i, \\ 0, & \text{otherwise} \end{cases} \qquad (i \in \mathbb{I}).$$

Note that $\tilde{J}$ need *not* be linear.

Turn $\mathbb{I}$ into a directed set by defining for $i, j \in \mathbb{I}$ that

$$i \preceq j \quad :\Longleftrightarrow \quad F_i \subset F_j,\ \Phi_i \subset \Phi_j, \text{ and } \epsilon_i \ge \epsilon_j.$$

Let $\mathcal{U}$ be an ultrafilter over $\mathcal{I}$ such that $\mathcal{F}_{\preceq} \subset \mathcal{U}$, let $\pi_{\mathcal{U}} : \ell^\infty\text{-}\bigoplus_{i\in\mathbb{I}} E \to (E)_{\mathcal{U}}$ be the quotient map, and set $J := \pi_{\mathcal{U}} \circ \tilde{J}$. Then J is linear, is an isometry by (B), and extends the canonical embedding of E into $(E)_{\mathcal{U}}$ by (A).

Define

$$Q\colon (E)_{\mathcal{U}} \to E^{**}, \quad (x_i)_{\mathcal{U}} \mapsto \sigma(E^{**}, E^*)\text{-}\lim_{i\to\mathcal{U}} x_i.$$

This is well defined by Theorem 4.2.7 because $\mathrm{Ball}(E^{**})$ is $\sigma(E^{**}, E^*)$-compact by the Alaoğlu–Bourbaki Theorem. Clearly, Q is a contraction, and from (C), it follows that $QJ = \mathrm{id}_{E^{**}}$. Set $P := JQ$. Then P is a norm one projection onto JE^{**}. □

Definition 4.2.12. Let E and F be Banach spaces, and let $C \geq 1$. We say that E and F are *C-isomorphic* if there is an isomorphism $T\colon E \to F$ such that $\|T\|\|T^{-1}\| \leq C$. In this case, we call T a *C-isomorphism.*

Definition 4.2.13. Let E and F be Banach spaces, and let $C \geq 1$. We say that F is *C-representable* in E if, for each finite-dimensional subspace X and each $\epsilon > 0$, there is a finite-dimensional subspace Y of E that is $(C+\epsilon)$-isomorphic to X.

Example. By the Local Reflexivity Principle, E^{**} is 1-representable in E for each Banach space E.

Proposition 4.2.14. *Let E be a Banach space, let $\mathbb{I}$ be a set, and let $\mathcal{U}$ be an ultrafilter over $\mathbb{I}$. Then $(E)_{\mathcal{U}}$ is 1-representable in E.*

Proof. Let X be a finite-dimensional subspace of $(E)_{\mathcal{U}}$, and let $\epsilon > 0$. Let $x_1, \ldots, x_n$ be a basis of X consisting of unit vectors. For $k = 1, \ldots, n$, there are therefore $(x_{k,i})_{i\in\mathbb{I}} \in \ell^\infty\text{-}\bigoplus_{i\in\mathbb{I}} E$ with $(x_{k,i})_{\mathcal{U}} = x_k$.

For $i \in \mathbb{I}$, define linear $T_i\colon X \to E$ by letting $T_i x_k = x_{k,i}$ for $k = 1, \ldots, n$. We claim that there is $i \in \mathbb{I}$ such that

$$(1+\epsilon)^{-1}\|x\| \leq \|T_i x\| \leq (1+\epsilon)\|x\| \qquad (x \in X). \tag{4.4}$$

Let

$$x = \lambda_1 x_1 + \cdots + \lambda_n x_n, \tag{4.5}$$

and note that

$$\lim_{i\to\mathcal{U}} \|T_i x\| = \lim_{i\to\mathcal{U}} \left\| \sum_{k=1}^n \lambda_k x_{k,i} \right\| = \left\| \left(\sum_{k=1}^n \lambda_k x_{k,i} \right)_{\mathcal{U}} \right\| = \|x\|.$$

Therefore, there is, for each $x \in X$, a set $U_x \in \mathcal{U}$ such that

$$\left(1+\frac{\epsilon}{2}\right)^{-1} \|x\| \leq \|T_i x\| \leq \left(1+\frac{\epsilon}{2}\right) \|x\| \qquad (i \in U_x)$$

Now, $X \cong \ell^1_n$—even though not necessarily isometrically—, so that there is $C \geq 0$ such that $\sum_{k=1}^n |\lambda_k| \leq C\|x\|$ if x is as in (4.5). With $M := \sup_{\substack{k=1,\ldots,n \\ j \in \mathbb{I}}} \|x_{k,j}\|$, we obtain

$$\|T_i x\| = \left\| \sum_{k=1}^n \lambda_k x_{k,i} \right\| \leq \left(\max_{k=1,\ldots,n} \|x_{k,i}\| \right) \sum_{k=1}^n |\lambda_k| \leq CM\|x\| \qquad (x \in X,\ i \in \mathbb{I}).$$

Set

$$\delta := \frac{\epsilon}{CM(1+\epsilon)(2+\epsilon)}.$$

Let $y_1, \ldots, y_m \in X$ with $\|y_1\| = \cdots = \|y_m\| = 1$ be such that, for each $y \in X$ with $\|y\| = 1$, there is $k \in \{1, \ldots, m\}$ with $\|y - y_k\| < \delta$. Set $U := \bigcap_{k=1}^m U_{y_k}$, and note that $U \in \mathcal{U}$. Let $i \in U$, and let $y \in X$ with $\|y\| = 1$. Choose $k \in \{1, \ldots, m\}$ with $\|y - y_k\| < \delta$. Note that

$$\|T_i y\| \leq \|T_i(y - y_k)\| + \|T_i y_k\| \leq CM\delta + \left(1 + \frac{\epsilon}{2}\right) = \underbrace{\frac{\epsilon}{(1+\epsilon)(2+\epsilon)}}_{\leq \frac{\epsilon}{2}} + 1 + \frac{\epsilon}{2} \leq 1 + \epsilon$$

and

$$\begin{aligned}
\|T_i y\| &\geq \|T_i y_k\| - \|T_i(y - y_k)\| \geq \left(1 + \frac{\epsilon}{2}\right)^{-1} - CM\delta \\
&= \frac{2}{2+\epsilon} - \frac{\epsilon}{(1+\epsilon)(2+\epsilon)} = \frac{2+2\epsilon}{(1+\epsilon)(2+\epsilon)} - \frac{\epsilon}{(1+\epsilon)(2+\epsilon)} \\
&= \frac{2+\epsilon}{(1+\epsilon)(2+\epsilon)} = (1+\epsilon)^{-1}.
\end{aligned}$$

As (4.4) is homogeneous in x, this means that (4.4) holds for any $i \in U$.

Let $i \in U$, and set $Y := T_i X$. Then (4.4) implies that $T_i \colon X \to Y$ is an isomorphism satisfying $\|T_i\|\|T_i^{-1}\| \leq 1 + \epsilon$. ☐

Theorem 4.2.15. *Let E and F be Banach spaces, and let $C \geq 1$. Then the following are equivalent:*

(i) *F is C-representable in E;*

(ii) *there is an ultrafilter $\mathcal{U}$ over some index set $\mathbb{I}$ such that F is C-isomorphic to a subspace of $(E)_{\mathcal{U}}$.*

Proof. (ii) $\Longrightarrow$ (i): Let X be a finite-dimensional subspace of F, and let $T \colon F \to (E)_{\mathcal{U}}$ be a C-isomorphism onto its range. Let $\epsilon > 0$. By Proposition 4.2.14, there are a finite-dimensional subspace Y of E and a $\left(1 + \frac{\epsilon}{C}\right)$-isomorphism $S : TX \to Y$. Consequently, $ST \colon X \to Y$ is a $(C + \epsilon)$-isomorphism.

(i) $\Longrightarrow$ (ii): Let $\mathbb{I}$ consist of all pairs $i = (X_i, \epsilon_i)$, where

- X_i is a finite-dimensional subspace of F
- $0 < \epsilon_i \leq 1$.

For every $i = (X_i, \epsilon_i) \in \mathbb{I}$, there is an injective, linear $T_i \colon X_i \to E$ such that $\|T_i\| \leq C + \epsilon_i$ and $\|T_i^{-1}|_{TX_i}\| \leq 1$. Define

$$\tilde{T} \colon F \to \ell^\infty\text{-}\bigoplus_{i \in \mathbb{I}} E, \quad x \mapsto \begin{cases} T_i x, & x \in X_i, \\ 0, & \text{otherwise.} \end{cases}$$

Turn $\mathbb{I}$ into a directed set by defining

$$(X_i, \epsilon_i) \preceq (X_j, \epsilon_j) \quad :\Longleftrightarrow \quad X_1 \subset X_2,\, \epsilon_1 \geq \epsilon_2,$$

and let $\mathcal{U}$ be an ultrafilter over $\mathbb{I}$ with $\mathcal{F}_\preceq \subset \mathcal{U}$. Let $\pi_U : \ell^\infty\text{-}\bigoplus_{i \in \mathbb{I}} E \to (E)_\mathcal{U}$ be the quotient map, and set $T := \pi_U \circ \tilde{T}$. Then $T \colon F \to (E)_\mathcal{U}$ is linear such that

$$\|x\| \leq \|Tx\| \leq C\|x\| \qquad (x \in F)$$

i.e., is a C-isomorphism onto its range. □

Corollary 4.2.16. *Let $C \geq 1$, and let E be a Banach space that is C-representable in a Hilbert space. Then E is isomorphic to a Hilbert space. If $C = 1$, then E even* is *a Hilbert space.*

Corollary 4.2.17. *Let $C \geq 1$, let $p \in [1, \infty)$, and let E be a Banach space that is C-representable in an L^p-space. Then E is C-isomorphic to a subspace of an L^p-space.*

Proposition 4.2.18. *Let $(E_i)_{i \in \mathbb{I}}$ and $(F_i)_{i \in \mathbb{I}}$ be families of Banach spaces, and let $\mathcal{U}$ be an ultrafilter over $\mathbb{I}$. Then $\Theta \colon (\mathcal{B}(E_i, F_i))_\mathcal{U} \to \mathcal{B}((E_i)_\mathcal{U}, (F_i)_\mathcal{U})$ defined via*

$$\Theta((T_i)_\mathcal{U})(x_i)_\mathcal{U} := (T_i x_i)_\mathcal{U} \qquad ((T_i)_\mathcal{U} \in (\mathcal{B}(E_i, F_i))_\mathcal{U},\, (x_i)_\mathcal{U} \in (E_i)_\mathcal{U})$$

is an isometry.

Proof. Let $(x_i)_{i \in \mathbb{I}} \in \ell^\infty\text{-}\bigoplus_{i \in \mathbb{I}} E_i$, and let $(T_i)_{i \in \mathbb{I}} \in \ell^\infty\text{-}\bigoplus_{i \in \mathbb{I}} \mathcal{B}(E_i, F_i)$. It follows that $(T_i x_i)_{i \in \mathbb{I}} \in \ell^\infty\text{-}\bigoplus_{i \in \mathbb{I}} F_i$ such that

$$\|T_i x_i\| \leq \|T_i\|\|x_i\| \qquad (i \in \mathbb{I}).$$

Consequently, Θ is well defined and a contraction.

To see that Θ is an isometry, let $(T_i)_{i \in \mathbb{I}} \in \ell^\infty\text{-}\bigoplus_{i \in \mathbb{I}} \mathcal{B}(E_i, F_i)$, and let $\epsilon > 0$. For each $i \in \mathbb{I}$, there is $x_i \in \operatorname{Ball}(E_i)$ such that $\|T_i x_i\| \geq \|T_i\| - \epsilon$. Then $(x_i)_\mathcal{U} \in \operatorname{Ball}((E_i)_\mathcal{U})$ and

$$\begin{aligned} \|\Theta((T_i)_\mathcal{U})\| &\geq \|\Theta((T_i)_\mathcal{U})(x_i)_\mathcal{U}\| \\ &= \|(T_i x_i)_\mathcal{U}\| \\ &= \lim_{i \to \mathcal{U}} \|T_i x_i\| \\ &\geq \lim_{i \to \mathcal{U}} (\|T_i\| - \epsilon) \\ &= \|(T_i)_\mathcal{U}\| - \epsilon. \end{aligned}$$

As $\epsilon > 0$ was arbitrary, this means that $\|\Theta((T_i)_{\mathcal{U}})\| = \|(T_i)_{\mathcal{U}}\|$. $\square$

Corollary 4.2.19. *Let $(E_i)_{i\in\mathbb{I}}$ be a family of Banach spaces, and let $\mathcal{U}$ be an ultrafilter over $\mathbb{I}$. Then there is a canonical isometry from $(E_i^*)_{\mathcal{U}}$ into $(E_i)_{\mathcal{U}}^*$.*

Proof. Apply the previous proposition to the case where $F_i = \mathbb{F}$ for $i \in \mathbb{I}$, and note that $(\mathbb{F})_{\mathcal{U}} = \mathbb{F}$. $\square$

Proposition 4.2.20. *Let $(E_i)_{i\in\mathbb{I}}$ be a family of Banach spaces, and let $\mathcal{U}$ be an ultrafilter over $\mathbb{I}$ such that $(E_i)_{\mathcal{U}}$ is reflexive. Then $(E_i^*)_{\mathcal{U}} \cong (E_i)_{\mathcal{U}}^*$ holds canonically.*

Proof. Let $\Theta\colon (E_i^*)_{\mathcal{U}} \to (E_i)_{\mathcal{U}}^*$ denote the canonical isometry, i.e.,

$$\langle (x_i)_{\mathcal{U}}, \Theta((\phi_i)_{\mathcal{U}})\rangle = \lim_{i\to\mathcal{U}} \langle x_i, \phi_i\rangle \qquad ((\phi_i)_{\mathcal{U}} \in (E_i^*)_{\mathcal{U}},\ (x_i)_{\mathcal{U}} \in (E_i)_{\mathcal{U}}).$$

We claim that $\mathrm{Ball}(\Theta(E_i^*)_{\mathcal{U}})$ is $\sigma((E_i)_{\mathcal{U}}^*, (E_i)_{\mathcal{U}})$-dense in $\mathrm{Ball}((E_i)_{\mathcal{U}}^*)$. Assume otherwise. Then there is $\psi \in \mathrm{Ball}((E_i)_{\mathcal{U}}^*)$ with $\psi \notin \overline{\mathrm{Ball}(\Theta(E_i^*)_{\mathcal{U}})}^{\sigma((E_i)_{\mathcal{U}}^*, (E_i)_{\mathcal{U}})}$. By the Hahn–Banach Separation Theorem, there is $(x_i)_{\mathcal{U}} \in (E_i)_{\mathcal{U}}$ and $c \in \mathbb{R}$ such that

$$\sup_{(\phi_i)_{\mathcal{U}}\in\mathrm{Ball}((E_i^*)_{\mathcal{U}})} \lim_{i\to\mathcal{U}} \mathrm{Re}\,\langle x_i, \phi_i\rangle \le c < \mathrm{Re}\,\langle (x_i)_{\mathcal{U}}, \psi\rangle.$$

For $i \in \mathbb{I}$ choose $\phi_i \in \mathrm{Ball}(E_i^*)$ such that $\langle x_i, \phi_i\rangle = \|x_i\|$. It follows that

$$\|(x_i)_{\mathcal{U}}\| = \lim_{i\to\mathcal{U}} \|x_i\| = \lim_{i\to\mathcal{U}} \mathrm{Re}\,\langle x_i, \phi_i\rangle \le c < \mathrm{Re}\,\langle (x_i)_{\mathcal{U}}, \psi\rangle \le \|(x_i)_{\mathcal{U}}\|,$$

which is a contradiction.

Now, $(E_i)_{\mathcal{U}}$ is reflexive. Consequently, $\mathrm{Ball}(\Theta(E_i^*)_{\mathcal{U}})$ is weakly dense in $\mathrm{Ball}((E_i)_{\mathcal{U}}^*)$ and therefore norm dense, i.e., all of $\mathrm{Ball}((E_i)_{\mathcal{U}}^*)$. $\square$

For our final result on duality of ultraproducts, we need a theorem we won't prove as well as yet another definition.

Theorem 4.2.21 (James' Theorem)**.** *The following are equivalent for a Banach space E:*

(i) *E is reflexive;*

(ii) *for every $\phi \in E^*$, there is $x \in E$ with $\|x\| = 1$ such that $\langle x, \phi\rangle = \|\phi\|$.*

Definition 4.2.22. An ultrafilter $\mathcal{U}$ is called *countably incomplete* if there are $U_1, U_2, \ldots \in \mathcal{U}$ with $U_1 \supset U_2 \supset \cdots$ such that $\bigcap_{n=1}^{\infty} U_n = \varnothing$.

Theorem 4.2.23. *Let $(E_i)_{i\in\mathbb{I}}$ be a family of Banach spaces, and let $\mathcal{U}$ be a countably incomplete ultrafilter over $\mathbb{I}$. Then the following are equivalent:*

(i) *$(E_i)_{\mathcal{U}}$ is reflexive;*

(ii) $(E_i^*)_{\mathcal{U}} \cong (E_i)_{\mathcal{U}}^*$.

Proof. (i) $\Longrightarrow$ (ii) is clear in view of Proposition 4.2.20.

(ii) $\Longrightarrow$ (i): Let $\phi \in (E_i)_{\mathcal{U}}^*$, and let $(\phi_i)_{\mathcal{U}} \in (E_i^*)_{\mathcal{U}}$ be such that $\phi = \Theta((\phi_i)_{\mathcal{U}})$. Let $U_1 \supset U_2 \supset \cdots$ be sets in $\mathcal{U}$ such that $\bigcap_{n=1}^{\infty} U_n = \varnothing$. For each $i \in \mathbb{I}$, there is a unique $n_i \in \mathbb{N}$ such that $i \in U_{n_i} \setminus U_{n_i+1}$. Let $x \in E_i$ with $\|x_i\| = 1$, and $\langle x_i, \phi_i \rangle \geq \|\phi_i\| - \frac{1}{n_i}$. Set $x = (x_i)_{\mathcal{U}}$.

We claim that $\lim_{i \to \mathcal{U}} \frac{1}{n_i} = 0$. To see this, let $\epsilon > 0$, and note that

$$\left\{ i \in \mathbb{I} : \frac{1}{n_i} < \epsilon \right\} \supset U_{\lfloor \frac{1}{\epsilon} \rfloor + 1} \in \mathcal{U}.$$

It follows that

$$\langle x, \phi \rangle = \lim_{i \to \mathcal{U}} \langle x_i, \phi_i \rangle \geq \lim_{i \to \mathcal{U}} \|\phi_i\| - \frac{1}{n_i} = \|\phi\| \geq \langle x, \phi \rangle.$$

From James' Theorem, we conclude that $(E_i)_{\mathcal{U}}$ is reflexive. $\square$

4.3 Uniform Convexity

All Banach spaces in this section are over $\mathbb{R}$.

Definition 4.3.1. Let E be a Banach space. For each $\epsilon \in (0, 2]$, the *modulus of convexity* of E at ϵ is defined as

$$\delta_E(\epsilon) := \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in \mathrm{Ball}(E),\ \|x - y\| \geq \epsilon \right\}$$

Remark. It is straightforward to see that

$$\delta_E(\epsilon) = \inf\{\delta_F(\epsilon) : F \text{ is a a two-dimensional subspace of } E\}.$$

Proposition 4.3.2. *Let E be a Banach space, and let $\epsilon \in (0, 2]$. Then*

$$\delta_E(\epsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in E,\ \|x\| = \|y\| = 1,\ \|x - y\| = \epsilon \right\}.$$

Proof. Let $x, y \in \mathrm{Ball}(E)$ such that $\|x - y\| \geq \epsilon$, and consider

$$f : [0, 1] \to \mathbb{R}, \quad t \mapsto \|tx + (1-t)y - (ty + (1-t)x)\|.$$

Then $f(0) = f(1) = \|x - y\| \geq \epsilon$ whereas $f\left(\frac{1}{2}\right) = 0$. The Intermediate Value Theorem yields $t_0 \in [0, 1]$ such that $f(t_0) = \epsilon$. Set $x' := t_0 x + (1 - t_0)y$ and $y' := t_0 y + (1 - t_0)x$. It follows that $x', y' \in \mathrm{Ball}(E)$, $\frac{x'+y'}{2} = \frac{x+y}{2}$, and $\|x' - y'\| = \epsilon$. It is therefore enough to show that

$$\begin{aligned} &\sup\{\|x + y\| : x, y \in \mathrm{Ball}(E),\ \|x - y\| = \epsilon\} \\ &\qquad = \sup\{\|x + y\| : x, y \in E,\ \|x\| = \|y\| = 1,\ \|x - y\| = \epsilon\}. \end{aligned}$$

Without loss of generality, suppose that $\dim E \leq 2$. Let $x_0, y_0 \in \mathrm{Ball}(E)$ be such that

$$\|x_0 + y_0\| = \sup\{\|x + y\| : x, y \in \mathrm{Ball}(E),\ \|x - y\| = \epsilon\}.$$

Assume that $x_0, y_0 \in \mathrm{ball}(E)$. Set $\delta := \frac{1}{2}\min\{1 - \|x_0\|, 1 - \|y_0\|\}$, and let

$$\tilde{x}_0 := x_0 + \delta(x_0 + y_0) \qquad \text{and} \qquad \tilde{y}_0 := y_0 + \delta(x_0 + y_0).$$

It follows that $\tilde{x}_0, \tilde{y}_0 \in \mathrm{Ball}(E)$ and $\|\tilde{x}_0 - \tilde{y}_0\| = \|x_0 - y_0\| = \epsilon$. However,

$$\|\tilde{x}_0 + \tilde{y}_0\| = (1 + 2\delta)\|x_0 + y_0\| > \|x_0 + y_0\|,$$

which is a contradiction. It therefore follows that $\|x_0\| = 1$ or $\|y_0\| = 1$.

Assume that $x_0 \in \mathrm{ball}(E)$, so that necessarily $\|y_0\| = 1$. Let $\phi \in E^*$ be such that $\|\phi\| = 1$ and $\langle x_0 - y_0, \phi\rangle = \|x_0 - y_0\|$, and set

$$S := \{x \in E : \|x - y_0\| = \epsilon\},$$

so that

$$\sup\{\langle x - y_0, \phi\rangle : x \in S\} = \epsilon = \|x_0 - y_0\|.$$

Let $x \in \mathrm{Ball}(E) \cap S$. Then

$$\langle x, \phi\rangle - \langle y_0, \phi\rangle = \langle x - y_0, \phi\rangle \leq \|x - y_0\| \leq \|x_0 - y_0\| = \langle x_0, \phi\rangle - \langle y_0, \phi\rangle.$$

As $x_0 \in \mathrm{ball}(E)$, this means that ϕ has a local maximum at x_0 on S. A geometric inspection

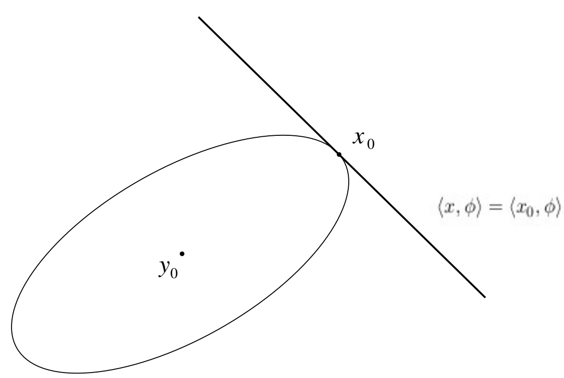

Figure 4.1: Geometric inspection of ϕ

yields that $\langle \cdot - y, \phi\rangle$ even has a global maximum at x_0 on S, i.e.,

$$\langle x_0 - y_0, \phi\rangle = \sup_{x \in S}\langle x - y_0, \phi\rangle = \|x_0 - y_0\|.$$

It follows that

$$\begin{aligned}
\|y_0\| &\leq \frac{1}{2}(\|y_0 - x_0\| + \|y_0 + x_0\|) \\
&= \frac{1}{2}(\langle x_0 - y_0, \phi\rangle + \langle x_0 + y_0, \phi\rangle) \\
&= \langle x_0, \phi\rangle \\
&\leq \|x_0\| \\
&< 1,
\end{aligned}$$

which is a contradiction.

Similarly, we deal with the case where $y_0 \in \operatorname{ball}(E)$. □

Remarks. 1. Let $x, y \in E$ be such that $\|x\| = \|y\| = 1$ and $\|x - y\| = \epsilon$. It follows that

$$\left\|\frac{x+y}{2}\right\| = \left\|x + \frac{y-x}{2}\right\| \geq \|x\| - \left\|\frac{y-x}{2}\right\| = 1 - \frac{\epsilon}{2},$$

so that

$$\delta_E(\epsilon) \leq \frac{\epsilon}{2}.$$

2. Let $\mathfrak{H}$ be a Hilbert space, and let $\epsilon \in (0, 2]$. For $\xi, \eta \in \mathfrak{H}$ with $\|\xi\| = \|\eta\| = 1$ and $\|\xi - \eta\| = \epsilon$, we obtain

$$\begin{aligned}
\left\|\frac{\xi+\eta}{2}\right\| &= \sqrt{\left\langle \frac{\xi+\eta}{2}, \frac{\xi+\eta}{2}\right\rangle} \\
&= \sqrt{\frac{1}{2}\|\xi\|^2 + \frac{1}{2}\|\eta\|^2 - \left\|\frac{\xi-\eta}{2}\right\|^2} \\
&= \sqrt{1 - \frac{\epsilon^2}{4}},
\end{aligned}$$

so that

$$\delta_{\mathfrak{H}}(\epsilon) = 1 - \sqrt{1 - \frac{\epsilon^2}{4}} > 0.$$

Definition 4.3.3. A Banach space E is called *uniformly convex* if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$.

Examples. 1. Every Hilbert space is uniformly convex.

2. ℓ^1_2 and ℓ^∞_2 are *not* uniformly convex.

ℓ^1_2: Let $x = (1, 0)$ and $y = (0, 1)$. Then

$$\|x - y\|_1 = 2 \qquad \text{and} \qquad \left\|\frac{x+y}{2}\right\|_1 = 1.$$

ℓ^∞_2: Let $x = (1, -1)$ and $y = (1, 1)$. Then

$$\|x - y\|_\infty = 2 \qquad \text{and} \qquad \left\|\frac{x+y}{2}\right\|_\infty = 1.$$

3. Every Banach space containing isometric copies of ℓ_2^1 and ℓ_2^∞ is not uniformly convex.

4. Let $(E_i)_{i\in\mathbb{I}}$ be a family of Banach spaces, let $\mathcal{U}$ be an ultrafilter over $\mathbb{I}$, and suppose that $\lim_{i\to\mathcal{U}} \delta_{E_i}(\epsilon) > 0$ for all $\epsilon \in (0,2]$. Then $(E_i)_{\mathcal{U}}$ is uniformly convex.

   Proof. Let $\epsilon \in (0,2]$. Let $(x_i)_{\mathcal{U}}, (y_i)_{\mathcal{U}}$ be such that

   $$\|(x_i)_{\mathcal{U}}\| = \lim_{i\to\mathcal{U}} \|x_i\| = 1, \qquad \|(y_i)_{\mathcal{U}}\| = \lim_{i\to\mathcal{U}} \|y_i\| = 1,$$
   $$\text{and} \qquad \|(x_i - y_i)_{\mathcal{U}}\| = \lim_{i\to\mathcal{U}} \|x_i - y_i\| = \epsilon.$$

   Replacing x_i with $\frac{x_i}{\|x_i\|}$ and y_i with $\frac{y_i}{\|y_i\|}$, we can suppose that $\|x_i\| = \|y_i\| = 1$ for all $i \in \mathbb{I}$. Set
   $$U := \left\{ i \in \mathbb{I} : \|x_i - y_i\| \geq \frac{\epsilon}{2} \right\}.$$
   Then $U \in \mathcal{U}$, and for all $i \in U$, we have

   $$1 - \left\| \frac{x_i + y_i}{2} \right\| \geq \delta_{E_i}\left(\frac{\epsilon}{2}\right),$$

   so that
   $$1 - \left\| \frac{(x_i)_{\mathcal{U}} + (y_i)_{\mathcal{U}}}{2} \right\| = \lim_{i\to\mathcal{U}} \left(1 - \left\| \frac{x_i + y_i}{2} \right\| \right) \geq \lim_{i\to\mathcal{U}} \delta_{E_i}\left(\frac{\epsilon}{2}\right) > 0.$$

   It follows that $\delta_{(E_i)_{\mathcal{U}}}(\epsilon) \geq \lim_{i\to\mathcal{U}} \delta_{E_i}\left(\frac{\epsilon}{2}\right) > 0$. □

   As a consequence, if E is uniformly convex, then so is $(E)_{\mathcal{U}}$ for every ultrafilter $\mathcal{U}$.

Definition 4.3.4. A Banach space E is called *superreflexive* if $(E)_{\mathcal{U}}$ is reflexive for every ultrafilter $\mathcal{U}$.

Theorem 4.3.5. *Let E be a uniformly convex Banach space. Then E is superreflexive.*

Proof. It is sufficient to show that E is reflexive.

Let $X \in E^{**}$, and suppose that $\|X\| = 1$. By Goldstine's Theorem, there is a net $(x_\alpha)_\alpha$ in $\operatorname{Ball}(E)$ such that $x_\alpha \overset{\sigma(E^{**},E^*)}{\longrightarrow} X$. It follows that $x_\alpha + x_\beta \overset{\sigma(E^{**},E^*)}{\longrightarrow} 2X$, and therefore $\|x_\alpha + x_\beta\| \to 2$.

Let $\epsilon \in (0,2]$, and choose α_ϵ such that

$$1 - \left\| \frac{x_\alpha + x_\beta}{2} \right\| < \delta_E(\epsilon)$$

for all indices α and β with $\alpha_\epsilon \preceq \alpha, \beta$. It follows that $\|x_\alpha - x_\beta\| < \epsilon$ for all such α and β, i.e., (x_α) is a Cauchy net in E and therefore convergent. Set $x := \lim_\alpha x_\alpha$. It follows that $x_\alpha \overset{\sigma(E^{**},E^*)}{\longrightarrow} x$ and therefore $X = x \in E$. □

Lemma 4.3.6. *Let $p \in (1, \infty)$, and let $\epsilon > 0$. Then there is $\tilde{\delta}(\epsilon) > 0$ such that*

$$\left| \frac{t+s}{2} \right|^p < \left(1 - \tilde{\delta}(\epsilon)\right) \frac{|t|^p + |s|^p}{2}$$

for all $s, t \in \mathbb{R}$ with $|s - t| \geq \epsilon \max\{|t|, |s|\}$.

Proof. Assume otherwise. Then, for each $n \in \mathbb{N}$, there are $s_n, t_n \in \mathbb{R}$ with

$$|s_n - t_n| \geq \epsilon \max\{|t_n|, |s_n|\} \qquad \text{and} \qquad \frac{|t_n|^p + |s_n|^p}{2}\left(1 - \frac{1}{n}\right) \leq \left| \frac{t_n + s_n}{2} \right|^p$$

Replacing s_n and t_n by $\frac{s_n}{\max\{|s_n|,|t_n|\}}$ and $\frac{t_n}{\max\{|s_n|,|t_n|\}}$, respectively, we can suppose without loss of generality that $|s_n|, |t_n| \leq 1$ for $n \in \mathbb{N}$. Let $(s_{n_k})_{k=1}^\infty$ and $(t_{n_k})_{k=1}^\infty$ be convergent subsequences of $(s_n)_{n=1}^\infty$ and $(t_n)_{n=1}^\infty$, respectively, and let $s := \lim_{k\to\infty} s_{n_k}$ and $t := \lim_{k\to\infty} t_{n_k}$. It follows that4

$$|s - t| \geq \epsilon \qquad \text{and} \qquad \frac{|t|^p + |s|^p}{2} \leq \left| \frac{t+s}{2} \right|^p.$$

Consider

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto |x|^p.$$

Then f is convex, so that

$$\left| \frac{t+s}{2} \right|^p = f\left(\frac{t}{2} + \frac{s}{2}\right) \leq \frac{1}{2} f(t) + \frac{1}{2} f(s) = \frac{|t|^p + |s|^p}{2}$$

and therefore

$$\frac{|t|^p + |s|^p}{2} = \left| \frac{t+s}{2} \right|^p.$$

Now, f is strictly convex because $p > 1$. It follows that $s = t$, which is a contradiction. $\square$

Theorem 4.3.7. *Let $p \in (1, \infty)$, and let $(X, \mathfrak{S}, \mu)$ be a measure space. Then $L^p(X, \mathfrak{S}, \mu)$ is uniformly convex.*

Proof. Let $\epsilon \in (0, 2]$, and let $\delta := \tilde{\delta}\left(\epsilon 4^{-\frac{1}{p}}\right)$ with the notation from Lemma 4.3.6.

Let $f, g \in L^p(X, \mathfrak{S}, \mu)$ with $\|f\|_p, \|g\|_p \leq 1$ and $\|f - g\|_p \geq \epsilon$. Set

$$Y := \{x \in X : \epsilon^p(|f(x)|^p + |g(x)|^p) \leq 4|f(x) - g(x)|^p\}.$$

It follows that

$$\int_{Y^c} |f - g|^p \leq \frac{\epsilon^p}{4} \int_{Y^c} |f|^p + |g|^p \leq \frac{\epsilon^p}{4} \int_X |f|^p + |g|^p \leq \frac{\epsilon^p}{2}.$$

As $\int_X |f - g|^p \geq \epsilon^p$, this yields $\int_Y |f - g|^p \geq \frac{\epsilon^p}{2}$ and therefore

$$\max\left\{ \int_Y |f|^p, \int_Y |g|^p \right\} \geq \frac{1}{2} \int_Y |f - g|^p \geq \frac{\epsilon^p}{2^{p+1}}.$$

We obtain

$$\begin{aligned}
\int_X \frac{|f|^p + |g|^p}{2} - \left|\frac{f+g}{2}\right|^p &\geq \int_Y \frac{|f|^p + |g|^p}{2} - \left|\frac{f+g}{2}\right|^p \\
&\geq \int_Y \delta \frac{|f|^p + |g|^p}{2}, \qquad \text{by Lemma 4.3.6,} \\
&\geq \frac{\delta}{2} \max\left\{\int_Y |f|^p, \int_Y |g|^p\right\} \\
&\geq \frac{\delta \epsilon^p}{2^{p+2}}.
\end{aligned}$$

We conclude that

$$\int_X \left|\frac{f+g}{2}\right|^p = \int_X \frac{|f|^p + |g|^p}{2} + \int_X \left(\left|\frac{f+g}{2}\right|^p - \frac{|f|^p + |g|^p}{2}\right) \leq \int_X \frac{|f|^p + |g|^p}{2} - \frac{\delta \epsilon^p}{2^{p+2}}.$$

and therefore

$$\left\|\frac{f+g}{2}\right\|_p \leq \left(1 - \delta \frac{\epsilon^p}{2^{p+2}}\right)^{\frac{1}{p}},$$

i.e.,

$$\delta_{L^p(X,\mathfrak{S},\mu)}(\epsilon) \geq 1 - \left(1 - \delta \frac{\epsilon^p}{2^{p+2}}\right)^{\frac{1}{p}} > 0,$$

which completes the proof. □

Proposition 4.3.8. *The following are equivalent for a Banach space E:*

(i) *E is uniformly convex;*

(ii) *if $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ are sequences in E with $(x_n)_{n=1}^\infty$ bounded and*

$$\lim_{n\to\infty} (2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2) = 0,$$

then $\lim_{n\to\infty} \|x_n - y_n\| = 0$*;*

(iii) *if $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ are sequences in* $\mathrm{Ball}(E)$ *with* $\lim_{n\to\infty} \|x_n + y_n\| = 2$*, then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Proof. (i) $\Longrightarrow$ (iii): Let $\epsilon \in (0, 2]$. Let $n_\epsilon \in \mathbb{N}$ be such that

$$1 - \left\|\frac{x_n + y_n}{2}\right\| < \delta_E(\epsilon) \qquad (n \geq n_\epsilon).$$

It follows that $\|x_n - y_n\| < \epsilon$ for all $n \geq n_\epsilon$.

(iii) $\Longrightarrow$ (i): Assume that there is $\epsilon \in (0, 2]$ such that $\delta_E(\epsilon) = 0$. For each $n \in \mathbb{N}$, there are are therefore $x_n, y_n \in \mathrm{Ball}(E)$ such that $1 - \left\|\frac{x_n+y_n}{2}\right\| < \frac{1}{n}$, i.e.,

$$2 \leq \|x_n + y_n\| + \frac{2}{n} \leq 2 + \frac{2}{n} \qquad (n \in \mathbb{N}),$$

but also

$$\|x_n - y_n\| \geq \epsilon.$$

It follows that $\lim_{n\to\infty} \|x_n + y_n\| = 2$ whereas $\|x_n - y_n\| \not\to 0$.

(ii) $\Longrightarrow$ (iii): As $\|x_n + y_n\| \to 2$, we have $\|x_n\| \to 1$ and $\|y_n\| \to 1$, so that

$$2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2 \to 0$$

and therefore $\|x_n - y_n\| \to 0$.

(iii) $\Longrightarrow$ (ii): We have

$$\begin{aligned} 2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2 &\geq 2\|x_n\|^2 + 2\|y_n\|^2 - (\|x_n\|^2 + \|y_n\|^2) \\ &= \|x_n\|^2 - 2\|x_n\|\|y_n\| + \|y_n\|^2 \\ &= (\|x_n\| - \|y_n\|)^2, \end{aligned}$$

so that

$$|\|x_n\| - \|y_n\|| \to 0.$$

Assume towards a contradiction that $\|x_n - y_n\| \not\to 0$. We can suppose without loss of generality that there is $\epsilon_0 > 0$ such that $\|x_n - y_n\| \geq \epsilon_0$ for all $n \in \mathbb{N}$. We can also suppose that $r := \lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| > 0$ exists. We obtain

$$0 = \lim_{n\to\infty} \left(2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2\right) = 4r^2 - \lim_{n\to\infty} \|x_n + y_n\|^2,$$

i.e., $\lim_{n\to\infty} \|x_n + y_n\| = 2r$ and therefore

$$\left\| \frac{x_n}{\|x_n\|} + \frac{y_n}{\|y_n\|} \right\| \to 2.$$

By (iii), this means

$$\left\| \frac{x_n}{\|x_n\|} - \frac{y_n}{\|y_n\|} \right\| \to 0$$

which entails $\|x_n - y_n\| \to 0$. $\square$

4.4 Differentiability of Norms

Definition 4.4.1. Let E be a Banach space, let $U \subset E$ be open, let $f\colon U \to \mathbb{R}$, and let $x_0 \in U$. We say that f is *Gâteaux differentiable* at $x_0 \in U$ if there is $\phi \in E^*$ such that

$$\lim_{\substack{t\to 0 \\ t\neq 0}} \frac{f(x_0 + th) - f(x_0)}{t} = \langle h, \phi \rangle$$

for every $h \in E$. We say that f is *Fréchet differentiable* at x_0 if this limit is uniform in $h \in E$ such that $\|h\| = 1$. We call ϕ the *Gâteaux (Fréchet) derivative* of of f at x_0 and set $f'(x_0) = \phi$. We say that f is *Gâteaux (Fréchet) differentiable* on U if it is so at each point of U.

Examples. 1. The norm

$$\|\cdot\|\colon E \to [0,\infty), \quad x \mapsto \|x\|$$

is not Gâteaux differentiable at 0 because

$$\frac{\|0+th\| - \|0\|}{t} = \frac{|t|}{t}\|h\|$$

for all $t \neq 0$ and $\|h\| = 1$.

2. $\|\cdot\|$ is Gâteaux (Fréchet) differentiable at each $x \in E\setminus\{0\}$ if and only if it is Gâteaux (Fréchet) differentiable at each $x \in E$ with $\|x\| = 1$.

3. Suppose that $\|\cdot\|$ is Gâteaux differentiable at $x \in E$ with $\|x\| = 1$. Let $h \in E$ with $\|h\| = 1$ and note that

$$\left|\frac{\|x+th\| - \|x\|}{t}\right| \leq \frac{\|th\|}{|t|} = 1,$$

so that $\|\|x\|'\| \leq 1$. On the other hand, we have for $0 < |t| < 1$ that

$$\frac{\|x+tx\| - \|x\|}{t} = \frac{(1+t)\|x\| - \|x\|}{t} = \frac{t\|x\|}{t} = \|x\| = 1.$$

It follows that

$$\langle x, \|x\|'\rangle = \lim_{\substack{t\to 0\\ t\neq 0}} \frac{\|x+tx\| - \|x\|}{t} = 1,$$

so that $\|\|x\|'\| = 1$.

Lemma 4.4.2. *Let E be a Banach space, let $U \subset E$ be open and convex, and let $f\colon U \to \mathbb{R}$ be convex and continuous at $x_0 \in U$. Then f is Gâteaux differentiable at x_0 if and only if*

$$\lim_{\substack{t\to 0\\ t\neq 0}} \frac{f(x_0+th) + f(x_0-th) - 2f(x_0)}{t} = 0$$

for all $h \in E$, and f is Fréchet differentiable at x_0 if this limit is uniform in $h \in E$ with $\|h\| = 1$.

Proof. We only prove the claim for the Gâteaux differentiability of f; the claim for Fréchet differentiability follows analogously.

Suppose that f is Gâteaux differentiable at x_0. Then

$$\begin{aligned}
\frac{f(x_0+th) + f(x_0-th) - 2f(x_0)}{t} &= \frac{f(x_0+th) - f(x_0)}{t} - \frac{f(x_0-th) - f(x_0)}{-t} \\
&\to \langle h, f'(x_0)\rangle - \langle h, f'(x_0)\rangle \\
&= 0.
\end{aligned}$$

For the converse define $\phi^+, \phi^- : E \to \mathbb{R}$ via

$$\phi^+(h) := \lim t \downarrow 0 \frac{f(x_0 + th) - f(x_0)}{t}$$

$$\text{and} \quad \phi^+(h) := \lim t \uparrow 0 \frac{f(x_0 + th) - f(x_0)}{t} \qquad (h \in E);$$

these limits exist because f is convex. It is easy to see that $\phi^+ \geq \phi^-$, ϕ^+ is subadditive, ϕ^- is superadditive, and both ϕ^+ and ϕ^- are positively homogeneous. As

$$\lim_{\substack{t \to 0 \\ t \neq 0}} \frac{f(x_0 + th) + f(x_0 - th) - 2f(x_0)}{t} = 0,$$

we have $\phi^+ = \phi^-$; in particular, $\phi := \phi^+ = \phi^-$ is linear. We need to show that ϕ is bounded. As f is continuous at x_0, there are $\delta, C > 0$ such that $f(y) \leq C$ for all $y \in x_0 + \delta \operatorname{Ball}(E)$, so that

$$\frac{f(x + th) - f(x)}{t} \leq \frac{f(x + \delta h) - f(x)}{\delta} \leq \frac{C - f(x)}{\delta} \qquad (h \in E, \ \|h\| = 1).$$

It follows that

$$\langle h, \phi(x_0) \rangle \leq \frac{C - f(x)}{\delta} \qquad (h \in E, \ \|h\| = 1),$$

which completes the proof. $\square$

Theorem 4.4.3. *Let E be a Banach space, and let $x \in E$ with $\|x\| = 1$. Then:*

(A) *The following are equivalent:*

  (i) *$\|\cdot\|$ is differentiable at x;*

  (ii) *for any two sequences $(\phi_n)_{n=1}^\infty$ and $(\psi_n)_{n=1}^\infty$ in the unit sphere of E^* with $\lim_{n\to\infty}\langle x, \phi_n \rangle = \lim_{n\to\infty}\langle x, \psi_n \rangle = 1$, we have $\lim_{n\to\infty} \|\phi_n - \psi_n\| = 0$;*

  (iii) *every sequence $(\phi_n)_{n=1}^\infty$ in the unit sphere of E^* with $\lim_{n\to\infty}\langle x, \phi_n \rangle = 1$ is norm convergent.*

(B) *The following are equivalent:*

  (i) *$\|\cdot\|$ is Gâteaux differentiable at x;*

  (ii) *for any two sequences $(\phi_n)_{n=1}^\infty$ and $(\psi_n)_{n=1}^\infty$ in the unit sphere of E^* with $\lim_{n\to\infty}\langle x, \phi_n \rangle = \lim_{n\to\infty}\langle x, \psi_n \rangle = 1$, we have $\phi_n - \psi_n \stackrel{\sigma(E^*,E)}{\longrightarrow} 0$;*

  (iii) *there is a unique ϕ in the unit sphere of E^* with $\langle x, \phi \rangle = 1$.*

Proof. (A) (i) $\Longrightarrow$ (ii): Let $\epsilon > 0$. Then there is $\delta > 0$ such that

$$\|x + h\| + \|x - h\| \leq 2 + \epsilon \|h\| \qquad (h \in \operatorname{Ball}_\delta(E)).$$

Let $(\phi_n)_{n=1}^{\infty}$ and $(\psi_n)_{n=1}^{\infty}$ be norm one sequences in E^* with

$$\lim_{n\to\infty} \langle x, \phi_n \rangle = \lim_{n\to\infty} \langle x, \psi_n \rangle = 1.$$

Choose $n_\epsilon \in \mathbb{N}$ such that

$$\max\{|\langle x, \phi_n \rangle - 1|, |\langle x, \psi_n \rangle - 1|\} < \epsilon\delta \qquad (n \geq n_\epsilon).$$

Let $h \in \mathrm{Ball}_\delta(E)$. It follows that

$$\begin{aligned}
\langle h, \phi_n - \psi_n \rangle &= \langle x + h, \phi_n \rangle + \langle x - h, \psi_n \rangle - \langle x, \phi_n \rangle - \langle x, \psi_n \rangle \\
&\leq \|x + h\| + \|x - h\| - \langle x, \phi_n \rangle - \langle x, \psi_n \rangle \\
&\leq 2 - \langle x, \phi_n \rangle - \langle x, \psi_n \rangle + \epsilon\|h\| \\
&\leq |1 - \langle x, \phi_n \rangle| + |1 - \langle x, \psi_n \rangle| + \epsilon\|h\| \\
&\leq 3\epsilon\delta \qquad (n \geq n_\epsilon).
\end{aligned}$$

We conclude that

$$\begin{aligned}
\|\phi_n - \psi_n\| &= \sup_{h \in \mathrm{Ball}(E)} \langle h, \phi_n - \psi_n \rangle \\
&= \sup_{h \in \mathrm{Ball}_\delta(E)} \frac{\langle h, \phi_n - \psi_n \rangle}{\delta} \\
&\leq 3\epsilon \qquad (n \geq n_\epsilon),
\end{aligned}$$

so that $\|\phi_n - \psi_n\| \to 0$.

(A) (ii) $\Longrightarrow$ (iii): Choose $\phi \in E^*$ with $\|\phi\| = 1$ such that $\langle x, \phi \rangle = \|x\|$, so that

$$\lim_{n\to\infty} \langle x, \phi_n \rangle = 1 = \langle x, \phi \rangle,$$

which implies $\|\phi_n - \phi\| \to 0$.

(A) (iii) $\Longrightarrow$ (ii): Let $(\phi_n)_{n=1}^{\infty}$ and $(\psi_n)_{n=1}^{\infty}$ be sequences in the unit sphere of E^* with $\lim_{n\to\infty}\langle x, \phi_n \rangle = \lim_{n\to\infty}\langle x, \psi_n \rangle = 1$. Define $(\tilde{\phi}_n)_{n=1}^{\infty}$ by letting

$$\tilde{\phi}_n := \begin{cases} \phi_{\frac{n}{2}}, & n \text{ even}, \\ \psi_{\frac{n+1}{2}}, & n \text{ odd}. \end{cases}$$

It follows that $\lim_{n\to\infty} \left\langle x, \tilde{\phi}_n \right\rangle = 1$. Therefore there is $\tilde{\phi} \in E^*$ with $\|\tilde{\phi}\| = 1$ with $\left\|\tilde{\phi}_n - \tilde{\phi}\right\| \to 0$, so that $\tilde{\phi} = \lim_{n\to\infty} \phi_n = \lim_{n\to\infty} \psi_n$ and therefore $\|\phi_n - \psi_n\| \to 0$.

(A) $\Longrightarrow$ (i): Assume that $\|\cdot\|$ *not* Fréchet differentiable at x. This means that there is an $\epsilon_0 > 0$ such that, for all $n \in \mathbb{N}$, there is $h_n \in E$ with $\|h_n\| \leq \frac{1}{n}$ with

$$\|x + h_n\| + \|x - h_n\| \geq 2 + \epsilon_0\|h_n\|.$$

For $n \in \mathbb{N}$, choose $\phi_n, \psi_n \in E^*$ with $\|\phi_n\| = \|\psi_n\| = 1$ and

$$\langle x + h_n, \phi \rangle = \|x + h_n\| \quad \text{and} \quad \langle x - h_n, \psi_n \rangle = \|x - h_n\| \qquad (n \in \mathbb{N}).$$

It follows that

$$\langle x, \phi_n \rangle = \langle x + h_n, \phi_n \rangle - \langle h_n, \phi_n \rangle = \|x + h_n\| - \langle h_n, \phi_n \rangle \to 1$$

and—similarly—$\langle x, \psi_n \rangle \to 1$. On the other hand, we have

$$\begin{aligned} \langle h_n, \phi_n - \psi_n \rangle &= \langle x + h_n, \phi_n \rangle + \langle x - h_n, \psi_n \rangle - \langle x, \phi_n + \psi_n \rangle \\ &\geq \|x + h_n\| + \|x - h_n\| - 2 \\ &\geq \epsilon_0 \|h_n\| \qquad (n \in \mathbb{N}), \end{aligned}$$

so that $\|\phi_n - \psi_n\| \geq \epsilon_0$ for all $n \in \mathbb{N}$, which is a contradiction.

(B) (i) $\Longleftrightarrow$ (ii) is proven as for (A).

(B) (iii) $\Longrightarrow$ (ii): Assume that (ii) is false. Then there are sequences $(\phi_n)_{n=1}^\infty$ and $(\psi_n)_{n=1}^\infty$ in the unit sphere of E^* as well as $y \in E$ and $\epsilon_0 > 0$ such that $\langle x, \phi_n \rangle \to 1$, $\langle x, \psi_n \rangle \to 1$, and $|\langle y, \phi_n \rangle - \langle y, \psi_n \rangle| \geq \epsilon_0$ for all $n \in \mathbb{N}$. Let ϕ and ψ be $\sigma(E^*, E)$-accumulation points of $(\phi_n)_{n=1}^\infty$ and $(\psi_n)_{n=1}^\infty$, respectively, so that $\langle x, \phi \rangle = \langle x, \psi \rangle = 1$, therefore $\|\phi\| = \|\psi\| = 1$, and consequently $\phi = \psi$ whereas $\langle y, \phi \rangle \neq \langle y, \psi \rangle$, which is a contradiction.

(B) (ii) $\Longrightarrow$ (iii): Let $\phi, \psi \in E^*$ be any norm one functionals such that $\langle x, \phi \rangle = \langle x, \psi \rangle = 1$. For $n \in \mathbb{N}$, set $\phi_n := \phi$ and $\psi_n := \psi$, so that—trivially—$\lim_{n\to\infty} \langle x, \phi_n \rangle = \lim_{n\to\infty} \langle x, \psi_n \rangle = 1$ and therefore $\phi_n - \psi_n \overset{\sigma(E^*,E)}{\longrightarrow} 0$, so that $\phi = \psi$. □

Corollary 4.4.4. *Let E be a Banach space such that the norm of E^* is differentiable on $E^* \setminus \{0\}$. Then E is reflexive.*

Proof. Let $\phi \in E^*$ with $\|\phi\| = 1$. For $n \in \mathbb{N}$, choose $x_n \in E$ with $\|x_n\| = 1$ such that $\langle x_n, \phi \rangle \to \|\phi\| = 1$. Then $(x_n)_{n=1}^\infty$ is norm convergent to some $x \in E$ with $\|x\| = 1$ and $\langle x, \phi \rangle = 1$. By Theorem 4.2.21, this means that E is reflexive. □

Definition 4.4.5. Let E be a Banach space, and let $\tau > 0$. Then the *modulus of smoothness* of E at τ is defined as

$$\rho_E(\tau) := \sup \left\{ \frac{\|x + \tau h\| + \|x - \tau h\| - 2}{2} : x \in E, \, \|x\| = \|h\| = 1 \right\}.$$

We call E *uniformly smooth* if $\lim_{\tau \downarrow 0} \frac{\rho_E(\tau)}{\tau} = 0$.

We record the following without proof:

Proposition 4.4.6. *The following are equivalent for a Banach space E:*

(i) *E is uniformly smooth;*

(ii) *the limit*

$$\lim_{t \to 0} \frac{\|x + th\| - \|x\|}{t}$$

exits uniformly in $x, h \in E$ with $\|x\| = \|h\| = 1$;

(iii) *$\| \cdot \|$ is Fréchet differentiable on $E \setminus \{0\}$, and*

$$\partial \operatorname{Ball}(E) \to E^*, \quad x \mapsto \|x\|'$$

is uniformly continuous.

Lemma 4.4.7. *Let E be a Banach space, and let $\tau > 0$. Then*

$$\rho_{E^*}(\tau) = \sup \left\{ \tau \frac{\epsilon}{2} - \delta_E(\epsilon) : \epsilon \in (0, 2] \right\}.$$

Proof. Let $\epsilon \in (0, 2]$, and let $\tau > 0$. Let $x, y \in E$ be with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$. Choose $\phi, \psi \in E^*$ with $\|\phi\| = \|\psi\| = 1$ such that

$$\langle x + y, \phi \rangle = \|x + y\| \qquad \text{and} \qquad \langle x - y, \phi \rangle = \|x - y\|.$$

It follows that

$$\begin{aligned} 2\, \rho_{E^*}(\tau) &\geq \|\phi + \tau\psi\| + \|\phi - \tau\psi\| - 2 \\ &\geq \langle x, \phi + \tau\psi \rangle + \langle x, \phi - \tau\psi, y \rangle - 2 \\ &= \langle x + y, \phi \rangle + \tau \, \langle x - y, \psi \rangle - 2 \\ &= \|x + y\| + \tau \|x - y\| - 2, \end{aligned}$$

so that

$$1 - \left\| \frac{x + y}{2} \right\| \geq \tau \frac{\epsilon}{2} - \rho_{E^*}(\tau).$$

and therefore

$$\delta_E(\epsilon) + \rho_{E^*}(\tau) \geq \tau \frac{\epsilon}{2}.$$

We conclude that

$$\rho_{E^*}(\tau) \geq \sup \left\{ \tau \frac{\epsilon}{2} - \delta_E(\epsilon) : \epsilon \in (0, 2] \right\}.$$

For the reversed inequality, let $\tau > 0$, and let $\phi, \psi \in E^*$ with $\|\phi\| = \|\psi\| = 1$. Let $\theta > 0$ and choose $x, y \in E$ with $\|x\| = \|y\| = 1$ such that

$$\langle x, \phi + \tau\psi \rangle \geq \|\phi + \tau\psi\| - \theta \qquad \text{and} \qquad \langle y, \phi - \tau\psi \rangle \geq \|\phi - \tau\psi\| - \theta.$$

We obtain

$$\begin{aligned}
\frac{1}{2}(\|\phi+\tau\psi\|+\|\phi-\tau\psi\|-2) &\le \frac{1}{2}(\langle x,\phi+\tau\psi\rangle+\langle y,\phi-\tau\psi\rangle-2)+\theta \\
&= \frac{1}{2}(\langle x+y,\phi\rangle-2)+\frac{\tau}{2}\langle x-y,\psi\rangle+\theta \\
&\le \left(\left\|\frac{x+y}{2}\right\|-1\right)+\frac{\tau}{2}\|x-y\|+\theta \\
&\le -\delta_E(\|x-y\|)+\frac{\tau}{2}\|x-y\|+\theta \\
&\le \sup\left\{\tau\frac{\epsilon}{2}-\delta_E(\epsilon):\epsilon\in(0,2]\right\}+\theta
\end{aligned}$$

and, all in all,

$$\rho_{E^*}(\tau)\le\sup\left\{\tau\frac{\epsilon}{2}-\delta_E(\epsilon):\epsilon\in(0,2]\right\},$$

which completes the proof. □

Theorem 4.4.8. *The following are equivalent for a Banach space E:*

(i) *E is uniformly convex;*

(ii) *E^* is uniformly smooth.*

Proof. (i) $\Longrightarrow$ (ii): Let $\epsilon\in(0,2]$. Then $\delta_E(\epsilon')\ge\delta_E(\epsilon)>0$ for each $\epsilon'\in[\epsilon,2]$.

Let $\tau\in(0,\delta_E(\epsilon)$. For all $\epsilon'\in[\epsilon,2]$, we have

$$\frac{\epsilon'}{2}-\frac{\delta_E(\epsilon')}{\tau}\le\frac{\epsilon'}{2}-\frac{\delta_E(\epsilon)}{\tau}\le\frac{\epsilon'}{2}-1\le 0$$

By Lemma 4.4.7, this means that

$$\begin{aligned}
\frac{\rho_{E^*}(\tau)}{\tau} &= \sup\left\{\frac{\epsilon'}{2}-\frac{\delta_E(\epsilon')}{\tau}:\epsilon'\in(0,2]\right\} \\
&= \sup\left\{\frac{\epsilon'}{2}-\frac{\delta_E(\epsilon')}{\tau}:\epsilon'\in(0,\epsilon)\right\} \\
&\le \frac{\epsilon}{2},
\end{aligned}$$

so that $\lim_{\tau\downarrow 0}\frac{\rho_{E^*}(\tau)}{\tau}=0$.

(ii) $\Longrightarrow$ (i): Assume towards a contradiction that E is not uniformly convex, i.e., there is $\epsilon_0\in(0,2]$ such that $\delta_E(\epsilon_0)=0$. Let $\tau>0$. Then

$$\rho_{E^*}(\tau)=\sup\left\{\tau\frac{\epsilon}{2}-\delta_E(\epsilon):\epsilon\in(0,2]\right\}\ge\frac{\tau\epsilon_0}{2}$$

and therefore

$$\frac{\rho_{E^*}(\tau)}{\tau}\ge\frac{\epsilon_0}{2}>0,$$

which contradicts $\lim_{\tau\downarrow 0}\frac{\rho_E(\tau)}{\tau}=0$. □

Example. Let $p\in(1,\infty)$ then $L^p(X,\mathfrak{S},\mu)$ is uniformly smooth for each measure space $(X,\mathfrak{S},\mu)$.

Chapter 5

Tensor Products of Banach spaces

5.1 The Algebraic Tensor Product

The tensor product $E_1 \otimes E_2$ of two linear spaces E_1 and E_2 is a universal linearizer: every bilinear map from $E_1 \times E_2$ factors uniquely through $E_1 \otimes E_2$.

We give the formal definition of *a*—we still have to justify that we may speak of *the*—tensor product of linear spaces for an arbitrary finite number of spaces:

Definition 5.1.1. Let $E_1, \ldots, E_n$ be linear spaces. A *tensor product* of $E_1, \ldots, E_n$ is a pair $(\mathcal{T}, \tau)$ where $\mathcal{T}$ is a linear space and $\tau \colon E_1 \times \cdots \times E_n \to \mathcal{T}$ is an n-linear map with the following (universal) property: for each linear space F, and for each n-linear map $V \colon E_1 \times \cdots \times E_n \to F$, there is a unique linear map $\tilde{V} \colon \mathcal{T} \to F$ such that $V = \tilde{V} \circ \tau$.

It is clear that a tensor product $(\mathcal{T}, \tau)$ of $E_1, \ldots, E_n$ is not unique: if $\mathcal{T}'$ is another linear space, and if $\theta \colon \mathcal{T}' \to \mathcal{T}$ is an isomorphism of linear spaces, then $(\mathcal{T}', \theta \circ \tau)$ is another tensor product of $E_1, \ldots, E_n$. We call tensor products arising from one another in this fashion *isomorphic*: Given two tensor products $(\mathcal{T}_1, \tau_1)$ and $(\mathcal{T}_2, \tau_2)$, an isomorphism of $(\mathcal{T}_1, \tau_1)$ and $(\mathcal{T}_2, \tau_2)$ is an isomorpism $\theta \colon \mathcal{T}_1 \to \mathcal{T}_2$ of linear spaces such that $\tau_2 = \theta \circ \tau_1$. The best we can thus hope for is uniqueness up to isomorphism, which is indeed what we have:

Proposition 5.1.2. *Let $E_1, \ldots, E_n$ be linear spaces, and let $(\mathcal{T}_1, \tau_1)$ and $(\mathcal{T}_2, \tau_2)$ be tensor products of $E_1, \ldots, E_n$. Then there is a unique isomorphism of $(\mathcal{T}_1, \tau_1)$ and $(\mathcal{T}_2, \tau_2)$.*

Proof. Abstract nonsense. $\square$

We may thus speak of *the* tensor product of n—in most situations: two—linear spaces. From now on, we shall also use the standard notation for tensor products: Given linear spaces $E_1, \ldots, E_n$ and their tensor product $(\mathcal{T}, \tau)$, we write $E_1 \otimes \cdots \otimes E_n$ for $\mathcal{T}$.

Furthermore, we define

$$x_1 \otimes \cdots \otimes x_n := \tau(x_1, \ldots, x_n) \qquad (x_1 \in E_1, \ldots, x_n \in E_n). \tag{5.1}$$

Elements of $E_1 \otimes \cdots \otimes E_n$ are called *tensors*, and elements of the form (5.1) are called *elementary tensors*. Not every tensor is an elementary tensor: the set of all elementary tensors is (except in trivial cases) not a linear space. Hence, the tensor product must contain at least all linear combinations of elementary tensors. As the following proposition shows, these are all there is:

Proposition 5.1.3. *Let $E_1, \ldots, E_n$ be linear spaces, and let $\boldsymbol{x} \in E_1 \otimes \cdots E_n$. Then there is $m \in \mathbb{N}$, and for each $j = 1, \ldots, n$ there are $x_j^{(1)}, \ldots, x_j^{(m)} \in E_j$ such that*

$$\boldsymbol{x} = \sum_{k=1}^{m} x_1^{(k)} \otimes \cdots \otimes x_n^{(k)}. \tag{5.2}$$

Proof. Let F be the set of all tensors of the form (5.2), and define

$$V \colon E_1 \times \cdots \times E_n \to F, \quad (x_1, \ldots, x_n) \mapsto x_1 \otimes \cdots \otimes x_n.$$

It is routinely seen that F is a linear space, and that V is n-linear. From the defining property of $E_1 \otimes \cdots \otimes E_n$, there is a unique linear map $\tilde{V} \colon E_1 \otimes \cdots \otimes E_n \to F$ such that

$$\tilde{V}(x_1 \otimes \cdots \otimes x_n) = V(x_1, \ldots, x_n) \qquad (x_1 \in E_1, \ldots, x_n \in E_n).$$

It follows that $\tilde{V}$ is the identity on F and—via a simple Hamel basis argument—on $E_1 \otimes \cdots \otimes E_n$. □

Remarks. 1. Let E be a linear space. Then $E \times \mathbb{F} \ni (x, \lambda) \mapsto \lambda x$ induces an isomorphism of $E \otimes \mathbb{F}$ and E.

2. Show that the tensor product is associative: if E_1, E_2, and E_3 are linear spaces, then

$$E_1 \otimes E_2 \otimes E_3 \cong E_1 \otimes (E_2 \otimes E_3) \cong (E_1 \otimes E_2) \otimes E_3$$

through canonical isomorphisms.

3. Let $E_1, F_1, \ldots, E_n, F_n$ be linear spaces and let $T_j \colon E_j \to F_j$ be linear for $j = 1, \ldots, n$. Then there is a unique linear map $T_1 \otimes \cdots \otimes T_n : E_1 \otimes \cdots \otimes E_n \to F_1 \otimes \cdots \otimes F_n$ such that

$$(T_1 \otimes \cdots \otimes T_n)(x_1 \otimes \cdots \otimes x_n) = T_1 x_1 \otimes \cdots \otimes T_n x_n \qquad (x_1 \in E_1, \ldots, x_n \in E_n).$$

So far we have dealt with tensor products without bothering to ask if they exist at all. Luckily, they do:

Theorem 5.1.4. *Let $E_1, \ldots, E_n$ be linear spaces. Then their tensor product $E_1 \otimes \cdots \otimes E_n$ exists.*

Proof. For convenience, we switch back to the notation of Definition 5.1.1.

Let $\tilde{\mathcal{T}}$ be the linear space of all maps from the set $E_1 \times \cdots \times E_n$ to $\mathbb{F}$ with finite support. For $(x_1, \ldots, x_n) \in E_1 \times \cdots \times E_n$ define $\delta_{(x_1,\ldots,x_n)}$, the *point mass at* $(x_1, \ldots, x_n)$, through

$$\delta_{(x_1,\ldots,x_n)}(y_1, \ldots, y_n) := \begin{cases} 1, & \text{if } (y_1, \ldots, y_n) = (x_1, \ldots, x_n), \\ 0, & \text{otherwise.} \end{cases}$$

Also define

$$\tilde{\tau} \colon E_1 \times \cdots \times E_n \to \tilde{\mathcal{T}}, \quad (x_1, \ldots, x_n) \to \delta_{(x_1,\ldots,x_n)}.$$

Note that $\tilde{\tau}$ is not an n-linear map: in order to "make it linear", we have to factor out a certain subspace. Let $\tilde{\mathcal{T}}_0$ be the subspace of $\tilde{\mathcal{T}}$ spanned by all elements of the form

$$\lambda\, \delta_{(x_1,\ldots,x_j,\ldots,x_n)} + \mu\, \delta_{(x_1,\ldots,y_j,\ldots,y_n)} - \delta_{(x_1,\ldots,\lambda x_j + \mu y_j,\ldots,x_n)}$$
$$(\lambda, \mu \in \mathbb{F},\, x_1, y_1 \in E_1, \ldots, x_j, y_j \in E_j, \ldots, x_n, y_n \in E_n) \quad (5.3)$$

where j ranges from 1 to n. Define $\mathcal{T} := \tilde{\mathcal{T}}/\tilde{\mathcal{T}}_0$, let $\pi \colon \tilde{\mathcal{T}} \to \mathcal{T}$ be the quotient map, and set $\tau := \pi \circ \tilde{\tau}$. From the definition of $\tilde{\mathcal{T}}_0$, it follows that τ is n-linear.

We claim that $(\mathcal{T}, \tau)$ is a tensor product. To prove this, let F be another linear space, and let $V \colon E_1 \times \cdots \times E_n \to F$ be an n-linear map. Define a linear map $\bar{V} \colon \tilde{\mathcal{T}} \to F$ through

$$\bar{V}(\delta_{(x_1,\ldots,x_n)}) = V(x_1, \ldots, x_n) \qquad (x_1 \in E_1, \ldots, x_n \in E_n).$$

Then V satisfies $V = \bar{V} \circ \tilde{\tau}$; since every element of $\tilde{\mathcal{T}}$ is a finite linear combination of point masses, V is uniquely determined by this property. Define

$$\tilde{V}(x + \tilde{\mathcal{T}}_0) := \bar{V}(x) \qquad (x \in \tilde{\mathcal{T}}). \quad (5.4)$$

It is clear that $V = \tilde{V} \circ \tau$—once we have established that $\tilde{V}$ is well defined. However, since $V = \bar{V} \circ \tilde{\tau}$, and since V is n-linear, it is easily, albeit tediously verified (nevertheless, do it) that $\bar{V}$ vanishes on tensors of the form (5.3) and thus on all of $\tilde{\mathcal{T}}_0$, which shows that $\tilde{V}$ is indeed well defined.

Finally, suppose that $\tilde{W} \colon \mathcal{T} \to F$ is another linear map satisfying $\phi = \tilde{W} \circ \tau$. Then $\tilde{W}$ and $\bar{V}$ are necessarily related in the same way as $\tilde{V}$ and $\bar{V}$ are in (5.4). Since $\bar{V}$, however, is uniquely determined through $V = \bar{V} \circ \tilde{\tau}$, this establishes $\tilde{V} = \tilde{W}$. □

The construction of $E_1 \otimes \cdots \otimes E_n$ is not very illuminating, and we won't come back to it anymore; actually, it is needed only to establish once and for all that tensor products of linear spaces exist. Whenever we work with tensor products, however, we shall either use their defining property or Proposition 5.1.3.

For use in the next section, we require the following lemma:

Lemma 5.1.5. *Let $m \in \mathbb{N}$, let $E_1, \ldots, E_n$ be linear spaces, and, for $j = 1, \ldots, n$, let $x_j^{(1)}, \ldots, x_j^{(m)} \in E_j$ be such that*

$$\sum_{k=1}^{m} x_1^{(k)} \otimes \cdots \otimes x_n^{(k)} = 0. \tag{5.5}$$

Then, if $x_n^{(1)}, \ldots, x_n^{(m)}$ are linearly independent, we have

$$x_1^{(k)} \otimes \cdots \otimes x_{n-1}^{(k)} = 0 \qquad (k = 1, \ldots, m).$$

Proof. Suppose that there is $k_0 \in \{1, \ldots, m\}$ such that $x_1^{(k_0)} \otimes \cdots \otimes x_{n-1}^{(k_0)} \neq 0$. Define a linear map $\phi \colon E_n \to \mathbb{F}$ via a Hamel basis argument such that

$$\left\langle x_n^{(k_0)}, \phi \right\rangle = 1 \qquad \text{and} \qquad \left\langle x_n^{(k)}, \phi \right\rangle = 0 \quad (k \neq k_0).$$

Then

$$E_1 \times \cdots \times E_{n-1} \times E_n \to E_1 \otimes \cdots \otimes E_{n-1}, \quad (x_1, \cdots, x_{n-1}, x_n) \mapsto \langle x_n, \phi \rangle (x_1 \otimes \cdots \otimes x_{n-1})$$

is an n-linear map and therefore induces a linear map $\psi : E_1 \otimes \cdots \otimes E_{n-1} \otimes E_n \to E_1 \otimes \cdots \otimes E_{n-1}$ such that

$$\left\langle \sum_{k=1}^{m} x_1^{(k)} \otimes \cdots \otimes x_n^{(k)}, \psi \right\rangle = x_1^{(k_0)} \otimes \cdots \otimes x_{n-1}^{(k_0)} \neq 0$$

contradicting (5.5). □

5.2 The Injective Tensor Product

Suppose that $E_1, \ldots, E_n$ are Banach spaces. Then their tensor product $E_1 \otimes \cdots \otimes E_n$ exists by Theorem 5.1.4. Except in rather trivial cases, however, there is no need for $E_1 \otimes \cdots \otimes E_n$ to be a Banach space.

The first problem is to define a norm on $E_1 \otimes \cdots \otimes E_n$. Once we have found a suitable norm, we can obtain a Banach space tensor product of $E_1, \ldots, E_n$ by completing $E_1 \otimes \cdots \otimes E_n$ with respect to this norm. Of course, one can define norms on $E_1 \otimes \cdots \otimes E_n$ as one pleases via elementary Hamel basis arguments. However, very few norms obtained in this fashion will yield a useful notion of a Banach space tensor product. In order to obtain Banach space tensor products we can work with, we have to require at least one property for the norm on $E_1 \otimes \cdots \otimes E_n$:

Definition 5.2.1. Let $E_1, \ldots, E_n$ be Banach spaces. A norm $\| \cdot \|$ on $E_1 \otimes \cdots \otimes E_n$ is called a *cross norm* if

$$\|x_1 \otimes \cdots \otimes x_n\| = \|x_1\| \cdots \|x_n\| \qquad (x_1 \in E_1, \ldots, x_n \in E_n).$$

Two questions arise naturally in connection with Definition 5.2.1:

1. Is there a cross norm on $E_1 \otimes \cdots \otimes E_n$?

2. Is there more than one cross norm on $E_1 \otimes \cdots \otimes E_n$?

We deal with the first question first. Let $E_1, \ldots, E_n$ be Banach spaces with dual spaces $E_1^*, \ldots, E_n^*$, and let $\phi_j \in E_j^*$ for $j = 1, \ldots, n$. As $\mathbb{F} \otimes \cdots \otimes \mathbb{F} \cong \mathbb{F}$, the linear map $\phi_1 \otimes \cdots \otimes \phi_n$ is a linear functional on $E_1 \otimes \cdots \otimes E_n$.

Definition 5.2.2. Let $E_1, \ldots, E_n$ be Banach spaces with dual spaces $E_1^*, \ldots, E_n^*$. Then we define for $\boldsymbol{x} \in E_1 \otimes \cdots \otimes E_n$

$$\|\boldsymbol{x}\|_\epsilon := \sup\{|\langle \boldsymbol{x}, \phi_1 \otimes \cdots \otimes \phi_n\rangle| : \phi_j \in \operatorname{Ball}(E_j^*) \text{ for } j = 1, \ldots, n\}.$$

We call $\|\cdot\|_\epsilon$ the *injective norm* on $E_1 \otimes \cdots \otimes E_n$.

Although we have just called $\|\cdot\|_\epsilon$ the injective norm, this is just a name tag: we don't know yet if it is a norm at all. It is not difficult, however, to show this (and even more):

Proposition 5.2.3. *Let $E_1, \ldots, E_n$ be Banach spaces. Then $\|\cdot\|_\epsilon$ is a cross norm on $E_1 \otimes \cdots \otimes E_n$.*

Proof. It is clear from the definition that

$$\|x_1 \otimes \cdots \otimes x_n\|_\epsilon \le \|x_1\| \cdots \|x_n\| \qquad (x_1 \in E_1, \ldots, x_n \in E_n).$$

This also implies, by Proposition 5.1.3, that the supremum in Definition 5.2.2 is always finite. Let $x_j \in E_j$ for $j = 1, \ldots, n$. By the Hahn–Banach Theorem, there are $\phi_j \in E_j^*$ with $\|\phi_j\| = 1$ and $\langle x_j, \phi_j\rangle = \|x_j\|$ for $j = 1, \ldots, n$, so that

$$\|x_1 \otimes \cdots \otimes x_n\|_\epsilon \ge |\langle x_1 \otimes \cdots \otimes x_n, \phi_1 \otimes \cdots \otimes \phi_n\rangle| = \|x_1\| \cdots \|x_n\|.$$

All that remains to be shown, is thus that $\|\cdot\|_\epsilon$ is a norm.

It is easy to see that $\|\cdot\|_\epsilon$ is a seminorm on $E_1 \otimes \cdots \otimes E_n$. What has to be shown is therefore that $\|\boldsymbol{x}\|_\epsilon = 0$ implies $\boldsymbol{x} = 0$ for all $\boldsymbol{x} \in E_1 \otimes \cdots \otimes E_n$. We proceed by induction on n. Let $\boldsymbol{x} \in E_1 \otimes \cdots \otimes E_n \setminus \{0\}$. By Proposition 5.1.3, there is $m \in \mathbb{N}$, and for each $j = 1, \ldots, n$ there are $x_j^{(1)}, \ldots, x_j^{(m)} \in E_j$ such that

$$\boldsymbol{x} = \sum_{k=1}^m x_1^{(k)} \otimes \cdots \otimes x_n^{(k)}.$$

We may suppose that $x_n^{(1)}, \ldots, x_n^{(m)}$ are linearly independent. By Lemma 5.1.5, this means that there is $k_0 \in \{1, \ldots, m\}$ such that $x_1^{(k_0)} \otimes \cdots \otimes x_{n-1}^{(k_0)} \neq 0$. By the induction hypothesis, we have

$$\left\|x_1^{(k_0)} \otimes \cdots \otimes x_{n-1}^{(k_0)}\right\|_\epsilon \neq 0.$$

Use the Hahn–Banach Theorem to find $\phi \in \mathrm{Ball}(E_n^*)$ such that

$$\left\langle x_n^{(k_0)}, \phi \right\rangle > 0 \qquad \text{and} \qquad \left\langle x_n^{(k)}, \phi \right\rangle = 0 \quad (k \neq k_0).$$

We conclude that

$$\|\boldsymbol{x}\|_\epsilon \geq \|(\mathrm{id} \otimes \phi)(\boldsymbol{x})\|_\epsilon = \left\langle x_n^{(k_0)}, \phi \right\rangle \left\| x_1^{(k_0)} \otimes \cdots \otimes x_{n-1}^{(k_0)} \right\|_\epsilon > 0,$$

which completes the proof. $\square$

Definition 5.2.4. Let $E_1, \ldots, E_n$ be Banach spaces. Then their *injective tensor product* $E_1 \check{\otimes} \cdots \check{\otimes} E_n$ is the completion of $E_1 \otimes \cdots \otimes E_n$ with respect to $\| \cdot \|_\epsilon$.

Remarks.

1. There is an associative law for the injective tensor product (with isometric isomorphisms).
2. Let $E_1, F_1, \ldots, E_n, F_n$ be Banach spaces, and let $T_j \in \mathcal{B}(E_j, F_j)$. Then $T_1 \otimes \cdots \otimes T_n$ is continuous with respect to the injective norms on $E_1 \otimes \cdots \otimes E_n$ and $F_1 \otimes \cdots \otimes F_n$, respectively, and satisfies

$$\|T_1 \otimes \cdots \otimes T_n\| = \|T_1\| \cdots \|T_n\|.$$

Towards the end of our discussion of the injective tensor product, we give a concrete description of the injective tensor product in a particular case.

Let Ω be a set, and let E be a linear space. For $f \in \mathbb{F}^\Omega$ and $x \in E$, we define $fx \in E^\Omega$ through

$$(fx)(\omega) := f(\omega)x \qquad (\omega \in \Omega).$$

Theorem 5.2.5. *Let Ω be a locally compact Hausdorff space, and let E be a Banach space. Then the bilinear map*

$$\mathcal{C}_0(\Omega) \times E \to \mathcal{C}_0(\Omega, E), \quad (f, x) \mapsto fx \tag{5.6}$$

induces an isometric isomorphism $\mathcal{C}_0(\Omega) \check{\otimes} E \cong \mathcal{C}_0(\Omega, E)$.

Proof. We only treat the case where Ω is compact. (The general case can be deduced from it by passing to the one-point-compactification of Ω.)

From the defining property of the algebraic tensor product $\mathcal{C}(\Omega) \otimes E$, it follows that (5.6) extends to a linear map from $\mathcal{C}(\Omega) \otimes E$ into $\mathcal{C}(\Omega, E)$.

Let $f_1, \ldots, f_n \in \mathcal{C}_0(\Omega)$ and $x_1, \ldots, x_n \in E$; then

$$\begin{aligned}
\left\| \sum_{j=1}^n f_n x_j \right\|_{\mathcal{C}_0(\Omega, E)} &= \sup \left\{ \left\| \sum_{j=1}^n f_j(\omega) x_j \right\| : \omega \in \Omega \right\} \\
&= \sup \left\{ \left| \sum_{j=1}^n f_j(\omega) \langle x_j, \phi \rangle \right| : \omega \in \Omega,\ \phi \in \mathrm{Ball}(E^*) \right\} \\
&= \sup \left\{ \left\| (\mathrm{id} \otimes \phi) \left(\sum_{j=1}^n f_j \otimes x_j \right) \right\| : \phi \in \mathrm{Ball}(E^*) \right\} \\
&= \left\| \sum_{j=1}^n f_j \otimes x_j \right\|_\epsilon .
\end{aligned}$$

Therefore, (5.6) is an isometry and thus injective with closed range. It remains to be shown that it has dense range as well.

Let $f \in \mathcal{C}(\Omega, E)$, and $\epsilon > 0$. Being the continuous image of a compact space, $K := f(\Omega) \subset E$ is compact. We may therefore find $x_1, \ldots, x_n \in E$ such that

$$K \subset \mathrm{ball}_\epsilon(x_1, E) \cup \cdots \cup \mathrm{ball}_\epsilon(x_n, E).$$

Let $U_j := f^{-1}(\mathrm{ball}_\epsilon(x_j, E))$ for $j = 1, \ldots, n$. Choose $f_1, \ldots, f_n \in \mathcal{C}(\Omega)$ with $f_1, \ldots, f_n \geq 0$ such that

$$f_1 + \cdots + f_n \equiv 1 \qquad \text{and} \qquad \mathrm{supp}(f_j) \subset U_j \quad (j = 1, \ldots, n).$$

For $\omega \in \Omega$, we then have

$$\|f(\omega) - (f_1 x_1 + \cdots + f_n x_n)(\omega)\| \leq \sum_{j=1}^n f_j(\omega) \|f(\omega) - x_j\|. \tag{5.7}$$

It easy to see that the right hand side of (5.7) is less than ϵ. This completes the proof. $\square$

Corollary 5.2.6. *Let Ω_1 and Ω_2 be locally compact Hausdorff spaces. Then there is a canonical isometric isomorphism $\mathcal{C}_0(\Omega_1) \check{\otimes} \mathcal{C}_0(\Omega_2) \cong \mathcal{C}_0(\Omega_1 \times \Omega_2)$.*

There is a connection between the injective tensor product and finite-rank operators between Banach spaces. For any two Banach spaces E and F, we denote by $\mathcal{F}(E, F)$ the bounded finite-rank operators from E to F.

Definition 5.2.7. Let E and F be Banach spaces, let $\phi_1, \ldots, \phi_n \in E^*$, and let $x_1, \ldots, x_n \in F$. Then $T := \sum_{j=1}^n \phi_j \odot x_j \in \mathcal{F}(E, F)$ is defined through

$$Tx = \sum_{j=1}^n \langle x, \phi_j \rangle x_j \qquad (x \in E).$$

It is clear that every finite rank operator arises in this fashion.

Proposition 5.2.8. *Let E and F be Banach spaces. Then the linear map*

$$E^* \otimes F \to \mathcal{B}(E,F), \quad \phi \otimes x \mapsto \phi \odot x$$

is an isometry with respect to the injective norm on $E^* \otimes F$ onto $\mathcal{F}(E,F)$.

Proof. For $\phi_1, \ldots, \phi_n \in E^*$ and $x_1, \ldots, x_n \in F$, let $T := \sum_{j=1}^n \phi_j \odot x_j$ and $\boldsymbol{x} := \sum_{j=1}^n \phi_j \otimes x_j$. Then

$$\begin{aligned}
\|T\| &= \|T^{**}\| \\
&= \sup\{\|T^{**}X\| : X \in \mathrm{Ball}(E^{**})\} \\
&= \sup\{|\langle \phi, T^{**}X\rangle| : \phi \in \mathrm{Ball}(F^*),\ X \in \mathrm{Ball}(E^{**})\} \\
&= \sup\left\{\left|\sum_{j=1}^n \langle \phi_j, X\rangle\langle x_j, \phi\rangle\right| : \phi \in \mathrm{Ball}(F^*),\ X \in \mathrm{Ball}(E^{**})\right\} \\
&= \sup\{|\langle \boldsymbol{x}, \phi \otimes X\rangle| : \phi \in \mathrm{Ball}(F^*),\ X \in \mathrm{Ball}(E^{**})\} \\
&= \|\boldsymbol{x}\|_\epsilon.
\end{aligned}$$

This completes the proof. $\square$

5.3 The Projective Tensor Product

The injective tensor product of Banach spaces has a major drawback: it is not characterized by the obvious functional analytic analogue of the definining (universal) property of the algebraic tensor product.

As we shall now see, there is a cross norm such that the corresponding tensor product of Banach spaces has the desired properties.

Definition 5.3.1. Let $E_1, \ldots, E_n$ be Banach spaces. Then we define, for $\boldsymbol{x} \in E_1 \otimes \cdots \otimes E_n$,

$$\|\boldsymbol{x}\|_\pi := \inf\left\{\sum_{k=1}^m \left\|x_1^{(k)}\right\| \cdots \left\|x_n^{(k)}\right\| : \boldsymbol{x} = \sum_{k=1}^m x_1^{(k)} \otimes \cdots \otimes x_n^{(k)}\right\}.$$

We call $\|\cdot\|_\pi$ the *projective norm* on $E_1 \otimes \cdots \otimes E_n$.

Proposition 5.3.2. *Let $E_1, \ldots, E_n$ be Banach spaces. Then $\|\cdot\|_\pi$ is a cross norm on $E_1 \otimes \cdots \otimes E_n$ such that*

$$\|\boldsymbol{x}\| \leq \|\boldsymbol{x}\|_\pi \qquad (\boldsymbol{x} \in E_1 \otimes \cdots \otimes E_n),$$

for any cross norm $\|\cdot\|$ on $E_1 \otimes \cdots \otimes E_n$.

Proof. It is immediate that $\|\cdot\|$ is a seminorm on $E_1 \otimes \cdots \otimes E_n$ satisfying

$$\|x_1 \otimes \cdots \otimes x_n\|_\pi \leq \|x_1\| \cdots \|x_n\| \qquad (x_1 \in E_1, \ldots, x_n \in E_n).$$

Let $\boldsymbol{x} \in E_1 \otimes \cdots \otimes E_n$. For $j = 1, \ldots, n$ choose $x_j^{(1)}, \ldots, x_j^{(m)} \in E_j$ such that $\boldsymbol{x} = \sum_{k=1}^m x_1^{(k)} \otimes \cdots \otimes x_n^{(k)}$. For any cross norm $\|\cdot\|$, we have

$$\|\boldsymbol{x}\| \leq \sum_{k=1}^m \left\|x_1^{(k)}\right\| \cdots \left\|x_n^{(k)}\right\|. \tag{5.8}$$

As $\|\boldsymbol{x}\|_\pi$ is the infimum over all possible expressions occurring as right hand sides of (5.8), we obtain $\|\boldsymbol{x}\| \leq \|\boldsymbol{x}\|_\pi$. In the special case where $\|\cdot\| = \|\cdot\|_\epsilon$, we obtain $\|\cdot\|_\epsilon \leq \|\cdot\|_\pi$, which establishes that $\|\cdot\|_\pi$ is a norm, and

$$\|x_1\| \cdots \|x_n\| = \|x_1 \otimes \cdots \otimes x_n\|_\epsilon \leq \|x_1 \otimes \cdots \otimes x_n\|_\pi \qquad (x_1 \in E_1, \ldots, x_n \in E_n),$$

so that $\|\cdot\|_\pi$ is a cross norm. $\square$

Definition 5.3.3. Let $E_1, \ldots, E_n$ be Banach spaces. Then their *projective tensor product* $E_1 \hat{\otimes} \cdots \hat{\otimes} E_n$ is the completion of $E_1 \otimes \cdots \otimes E_n$ with respect to $\|\cdot\|_\pi$.

Remarks. 1. The projective tensor product of Banach spaces satisfies the following universal property: Let $E_1, \ldots, E_n$ be Banach spaces. Then for every Banach space F, and for every bounded n-linear map $V : E_1 \times \cdots \times E_n \to F$, there is a unique $\tilde{V} \in \mathcal{B}(E_1 \hat{\otimes} \cdots \hat{\otimes} E_n; F)$ with $\|\tilde{V}\| = \|V\|$ such that

$$V(x_1, \ldots, x_n) = \tilde{V}(x_1 \otimes \cdots \otimes x_n) \qquad (x_1 \in E_1, \ldots, x_n \in E_n).$$

2. Let $E_1, \ldots, E_n$ be Banach spaces, and let $E_1 \tilde{\otimes} \cdots \tilde{\otimes} E_n$ be the completion of $E_1 \otimes \cdots \otimes E_n$ with respect to some cross norm. Then the identity on $E_1 \otimes \cdots \otimes E_n$ extends to a contraction from $E_1 \hat{\otimes} \cdots \hat{\otimes} E_n$ to $E_1 \tilde{\otimes} \cdots \tilde{\otimes} E_n$.

3. The projective tensor product is associative.

4. Let $E_1, F_1, \ldots, E_n, F_n$ be Banach spaces, and let $T_j \in \mathcal{B}(E_j, F_j)$. Then $T_1 \otimes \cdots \otimes T_n$ is continuous with respect to the projective norms on $E_1 \otimes \cdots \otimes E_n$ and $F_1 \otimes \cdots \otimes F_n$, respectively, and satisfies

$$\|T_1 \otimes \cdots \otimes T_n\| = \|T_1\| \cdots \|T_n\|.$$

There is a useful analogue of Proposition 5.1.3 for the projective tensor product:

Proposition 5.3.4. *Let $E_1, \ldots, E_n$ be Banach spaces, and let $\boldsymbol{x} \in E_1 \hat{\otimes} \cdots \hat{\otimes} E_n$. Then there are sequences $\left(x_j^{(k)}\right)_{k=1}^{\infty}$ in E_j for $j = 1, \ldots, n$ such that*

$$\sum_{k=1}^{\infty} \left\| x_1^{(k)} \right\| \cdots \left\| x_n^{(k)} \right\| < \infty \tag{5.9}$$

and

$$\boldsymbol{x} = \sum_{k=1}^{\infty} x_1^{(k)} \otimes \cdots \otimes x_n^{(k)} \tag{5.10}$$

Moreover, $\|\boldsymbol{x}\|_\pi$ is the infimum over all infinite series (5.9) *such that*(5.10) *is satisfied.*

Proof. For each $\boldsymbol{x} \in E_1 \hat{\otimes} \cdots \hat{\otimes} E_n$ having a series representation as in (5.10), let $\|\boldsymbol{x}\|_{\tilde{\pi}}$ be the infimum over all infinite series (5.9) such that (5.10) holds. It is easy to see that $\| \cdot \|_{\tilde{\pi}}|_{E_1 \otimes \cdots \otimes E_n}$ is a cross norm such that the resulting completion of $E_1 \otimes \cdots \otimes E_n$ enjoys the same universal property as the projective tensor product. It follows that $\| \cdot \|_{\tilde{\pi}} = \| \cdot \|_\pi$.

Let F be the subspace of $E_1 \hat{\otimes} \cdots \hat{\otimes} E_n$ consisting of all x having a series representation as in (5.10). It is not difficult to see that $(F, \| \cdot \|_{\tilde{\pi}})$ is a Banach space. It follows that $F = E_1 \hat{\otimes} \cdots \hat{\otimes} E_n$. $\square$

There is an analogue of Theorem 5.2.5 for the projective tensor product. For a measure space $(\Omega, \mathfrak{S}, \mu)$ and a Banach space E, let $L^1(\Omega, \mathfrak{S}, \mu; E)$ denote the space of all (equivalence classes of) μ-integrable functions on Ω with values in E.

Theorem 5.3.5. *Let $(\Omega, \mathfrak{S}, \mu)$ be a measure space, and let E be a Banach space. Then the bilinear map*

$$L^1(\Omega, \mathfrak{S}, \mu) \times E \to L^1(\Omega, \mathfrak{S}, \mu; E), \quad (f, x) \mapsto fx \tag{5.11}$$

induces an isometric isomorphism of $L^1(\Omega, \mathfrak{S}, \mu) \hat{\otimes} E$ and $L^1(\Omega, \mathfrak{S}, \mu; E)$.

Proof. It follows immediately from the universal property of the projective tensor product, that (5.11) induces a contraction from $L^1(\Omega, \mathfrak{S}, \mu) \hat{\otimes} E$ into $L^1(\Omega, \mathfrak{S}, \mu; E)$. From the definition of $L^1(\Omega, \mathfrak{S}, \mu; E)$, it is clear that this contraction has dense range; it remains to be shown that it is also an isometry.

For $f_1, \ldots, f_n \in L^1(\Omega, \mathfrak{S}, \mu)$ and $x_1, \ldots, x_n \in E$, let $f = \sum_{j=1}^n f_j x_j$ and $\boldsymbol{x} = \sum_{j=1}^n f_j \otimes x_j$. We claim that $\|f\|_1 = \|\boldsymbol{x}\|_\pi$. A simple density argument shows that we may confine ourselves to the case where $f_1, \ldots, f_n$ are step functions. Furthermore, we may suppose without loss of generality that there are mutually disjoint $\Omega_1, \ldots, \Omega_n \in \mathfrak{S}$ with $\mu(\Omega_j) < \infty$ for $j = 1, \ldots, n$ such that $f_j = \chi_{\Omega_j}$ for $j = 1, \ldots, n$. It follows that

$$\|f\|_1 = \int_\Omega \|f\| \, d\mu = \sum_{j=1}^n \|f_j x_j\|_1 = \sum_{j=1}^n \mu(\Omega_j) \|x_j\| = \sum_{j=1}^n \|f_j\|_1 \|x\| \geq \|\boldsymbol{x}\|_\pi,$$

which completes the proof. $\square$

We conclude with a result on the dual space of a projective tensor product:

Proposition 5.3.6. *Let E and F be Banach spaces. Then there is a unique isometric isomorphism between $\Theta \colon \mathcal{B}(E, F^*) \to (E \hat{\otimes} F)^*$ given by*

$$\langle x \otimes y, \Theta(T) \rangle := \langle y, Tx \rangle \qquad (x \in E,\ y \in F,\ T \in \mathcal{B}(E, F^*)).$$

Proof. It is clear that Θ is well defined and injective. A moment's thought reveals that it is also surjective and an isometry. $\square$

5.4 The Hilbert Space Tensor Product

Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be Hilbert spaces. When is $\mathfrak{H}_1 \hat{\otimes} \mathfrak{H}_2$ a Hilbert space (or $\mathfrak{H}_1 \check{\otimes} \mathfrak{H}_2$)? The answer is: hardly ever (even if we are willing to put up with merely topological, but not isometric isomorphism). If we want a tensor product of Hilbert spaces to again be a Hilbert space, we need a different construction.

Proposition 5.4.1. *Let $\mathfrak{H}_1, \ldots, \mathfrak{H}_n$ be Hilbert spaces. Then:*

(i) *there is a unique positive definite, sesquilinear form on $\mathfrak{H}_1 \otimes \cdots \otimes \mathfrak{H}_n$ such that*

$$\langle \xi_1 \otimes \cdots \otimes \xi_n, \eta_1 \otimes \cdots \otimes \eta_n \rangle := \langle \xi_1, \eta_1 \rangle \cdots \langle \xi_n, \eta_n \rangle \qquad (\xi_1, \eta_1 \in \mathfrak{H}_1, \ldots, \xi_n, \eta_n \in \mathfrak{H}_n);$$

(ii) *the norm on $\mathfrak{H}_1 \otimes \cdots \otimes \mathfrak{H}_n$ induced by the scalar product from* (i) *is a cross norm.*

Definition 5.4.2. Let $\mathfrak{H}_1, \ldots, \mathfrak{H}_n$ be Hilbert spaces, and let $\langle \cdot, \cdot \rangle$ be as in Proposition 5.4.1. Then the Hilbert space tensor product $\mathfrak{H}_1 \bar{\otimes} \cdots \bar{\otimes} \mathfrak{H}_n$ of $\mathfrak{H}_1, \ldots, \mathfrak{H}_n$ is the completion of $\mathfrak{H}_1 \otimes \cdots \otimes \mathfrak{H}_n$ with respect to the norm induced by $\langle \cdot, \cdot \rangle$.

Remark. Let $(\Omega, \mathfrak{S}, \mu)$ be a measure space, and let $\mathfrak{H}$ be a Hilbert space. Then the bilinear map

$$L^2(\Omega, \mathfrak{S}, \mu) \times \mathfrak{H} \to L^2(\Omega, \mathfrak{S}, \mu; \mathfrak{H}), \quad (f, \xi) \mapsto f\xi$$

induces an isometric isomorphism of $L^2(\Omega, \mathfrak{S}, \mu) \bar{\otimes} \mathfrak{H}$ and $L^2(\Omega, \mathfrak{S}, \mu; \mathfrak{H})$.