1. Compute $\Delta f$ for

$$f: \mathbb{R}^3 \setminus \{(0,0,0)\} \to \mathbb{R}, \quad (x,y,z) \mapsto \frac{1}{\sqrt{x^2 + y^2 + z^2}}.$$ 

**Solution:** For $(x,y,z) \neq (0,0,0)$, we have

$$\frac{\partial f}{\partial x} = -\frac{x}{\sqrt{x^2 + y^2 + z^2}^3}, \quad \frac{\partial f}{\partial y} = -\frac{y}{\sqrt{x^2 + y^2 + z^2}^3},$$

and

$$\frac{\partial f}{\partial z} = -\frac{z}{\sqrt{x^2 + y^2 + z^2}^3},$$

so that

$$\frac{\partial^2 f}{\partial x^2} = -\frac{1}{\sqrt{x^2 + y^2 + z^2}^3} + \frac{3x^2}{\sqrt{x^2 + y^2 + z^2}^5},$$

$$\frac{\partial^2 f}{\partial y^2} = -\frac{1}{\sqrt{x^2 + y^2 + z^2}^3} + \frac{3y^2}{\sqrt{x^2 + y^2 + z^2}^5},$$

and

$$\frac{\partial^2 f}{\partial z^2} = -\frac{1}{\sqrt{x^2 + y^2 + z^2}^3} + \frac{3z^2}{\sqrt{x^2 + y^2 + z^2}^5}.$$ 

It follows that

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = -\frac{3}{\sqrt{x^2 + y^2 + z^2}^3} + \frac{3}{\sqrt{x^2 + y^2 + z^2}^5} x^2 + y^2 + z^2$$

$$= -\frac{3}{\sqrt{x^2 + y^2 + z^2}^3} + \frac{3}{\sqrt{x^2 + y^2 + z^2}^5} x^2 + y^2 + z^2$$

$$= 0.$$ 

2. Let $\varnothing \neq D \subset \mathbb{R}^N$, let $f: D \to \mathbb{R}^M$ be continuous, and let $(x_n)_{n=1}^\infty$ be a Cauchy sequence in $D$. Show that $(f(x_n))_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}^M$ if $D$ is closed or if $f$ is uniformly continuous.

Does this remain true without any additional requirements for $D$ or $f$?

**Solution:** Suppose that $D$ is closed. As $(x_n)_{n=1}^\infty$ is a Cauchy sequence, it converges to a limit $x_0 \in \mathbb{R}^N$ that—due to the closedness of $D$—must lie in $D$. As $f$ is continuous, it follows that $\lim_{n \to \infty} f(x_n) = f(x_0)$, so that $(f(x_n))_{n=1}^\infty$ is also a Cauchy sequence.
Suppose that $f$ is uniformly continuous. Let $\epsilon > 0$. By the uniform continuity of $f$, there is $\delta > 0$ such that $\|f(x) - f(y)\| < \epsilon$ for all $x, y \in D$ such that $\|x - y\| < \delta$. As $(x_n)_{n=1}^\infty$ is a Cauchy sequence, there is $n_0 \in \mathbb{N}$ such that $\|x_n - x_m\| < \delta$ for all $n, m \geq n_0$. From the choice of $\delta > 0$ it thus follows that $\|f(x_n) - f(x_m)\| < \epsilon$ for all $n, m \geq n_0$. Hence, $(f(x_n))_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}^M$.

Let $D = (0, 1]$, and let

$$f: D \to \mathbb{R}, \quad x \mapsto \frac{1}{x}.$$  

Then $D$ is not closed, and $f$ is continuous, but not uniformly continuous. For $n \in \mathbb{N}$, set $x_n := \frac{1}{n}$. Then $(x_n)_{n=1}^\infty$ is a Cauchy sequence in $D$, but as $f(x_n) = n$ for $n \in \mathbb{N}$, the sequence $(f(x_n))_{n=1}^\infty$ is definitely not a Cauchy sequence.

3. Show that:

(a) if $\mathcal{C}$ is a family of connected subsets of $\mathbb{R}^N$ such that $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$, then $\bigcup_{C \in \mathcal{C}} C$ is connected;

(b) if $C_1 \subset \mathbb{R}^{N_1}$ and $C_2 \subset \mathbb{R}^{N_2}$ are connected, then so is $C_1 \times C_2 \subset \mathbb{R}^{N_1+N_2}$ (Hint: Argue that we can suppose that $C_1$ and $C_2$ are not empty, and fix $x_2 \in C_2$; then apply (a) to $\mathcal{C} := \{(C_1 \times \{x_2\}) \cup (\{x_1\} \times C_2) : x_1 \in C_1\}$);

(c) if $C_1, C_2 \subset \mathbb{R}^N$ are connected, then so is $C_1 + C_2 \subset \mathbb{R}^N$.

Solution:

(a) Assume that there is a disconnection $\{U, V\}$ for $\bigcup_{C \in \mathcal{C}} C$. For any $C \in \mathcal{C}$, we then have $(U \cap C) \cup (V \cap C) = C$ and $(U \cap C) \cap (V \cap C) = \emptyset$, and as $C$ is connected, this means that $C \subset U$ or $C \subset V$. It follows that

$$\emptyset = \left(U \cap \bigcup_{C \in \mathcal{C}} C\right) \cap \left(V \cap \bigcup_{C \in \mathcal{C}} C\right) = \bigcap_{C \in \mathcal{C}} C \cap \bigcap_{C \in \mathcal{C}} C = \bigcap_{C \in \mathcal{C}} C,$$

which contradicts $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.

(b) Let $C_1 \subset \mathbb{R}^{N_1}$ and $C_2 \subset \mathbb{R}^{N_2}$ be connected. If $C_1 = \emptyset$ or $C_2 = \emptyset$, nothing needs to be shown. Hence, suppose that $C_1 \neq \emptyset \neq C_2$. Fix $x_2 \in C_2$. As $C_1 \times \{x_2\}$ is the image of $C_1$ under the continuous map

$$\mathbb{R}^{N_1} \to \mathbb{R}^{N_1+N_2}, \quad x \mapsto (x, x_2),$$

it follows that $C_1 \times \{x_2\}$ is connected. Analogously, one sees that $\{x_1\} \times C_2$ is connected for each $x_1 \in C_1$. As $(x_1, x_2) \in (C_1 \times \{x_2\}) \cap (\{x_1\} \times C_2)$, it follows that $(C_1 \times \{x_2\}) \cup (\{x_1\} \times C_2)$ is connected for each $x_1 \in C_1$. As

$$\emptyset \neq C_1 \times \{x_2\} \subset \bigcap_{x_1 \in C_1} ((C_1 \times \{x_2\}) \cup (\{x_1\} \times C_2)),$$
we conclude that

\[ C_1 \times C_2 = \bigcup_{x_1 \in C_1} ((C_1 \times \{x_2\}) \cup (\{x_1\} \times C_2)) \]

is connected.

(c) By (b), \( C_1 \times C_2 \) is connected. As

\[ f : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N, \quad (x, y) \mapsto x + y \]

is continuous, \( C_1 + C_2 = f(C_1 \times C_2) \) is connected as well.