MATH 216 (Fall 2021)
Introduction to Analysis

Midterm #2 Model Solutions

1. Let $(x_n)_{n=1}^\infty$ be a sequence in $\mathbb{R}$, let x be an accumulation point of $(x_n)_{n=1}^\infty$, and let $(\epsilon_k)_{k=1}^\infty$ be a sequence of strictly positive reals such that $\lim_{k\to\infty}\epsilon_k = 0$. Show that there is a subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that $|x_{n_k} - x| < \epsilon_k$ for all $k \in \mathbb{N}$.

   Solution: For $k \in \mathbb{N}$, set
$$\mathbb{N}_k := \{n \in \mathbb{N} : |x_n - x| < \epsilon_k\}.$$
   As x is an accumulation point of $(x_n)_{n=1}^\infty$, the set $\mathbb{N}_k$ is infinite for each $k \in \mathbb{N}$. Let $n_1 \in \mathbb{N}_1$. Suppose that $n_1 < n_2 < \cdots < n_k$ have already been chosen such that $|x_{n_j} - x| < \epsilon_j$ for $j = 1, \dots, k$. As $\mathbb{N}_{k+1}$ is infinite, there is $n_{k+1} \in \mathbb{N}_{k+1}$ such that $n_{k+1} > n_k$. Clearly, the resulting subsequence $(x_{n_k})_{k=1}^\infty$ has the desired property.

2. A set $U \subset \mathbb{R}$ is called *open* if its complement $\mathbb{R} \setminus U$ is closed. Show that $U \subset \mathbb{R}$ is open if and only if, for each $x \in U$, there is $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U$. Conclude that (a, b) is open for $a < b$.

   Solution: Suppose that U is open, and assume that there is $x \in U$ such that $(x - \epsilon, x + \epsilon) \not\subset U$ for each $\epsilon > 0$, i.e., for each $\epsilon > 0$, there is $x_\epsilon \in \mathbb{R} \setminus U$ such that $|x_\epsilon - x| < \epsilon$. In particular, for each $n \in \mathbb{N}$, there is $x_n \in \mathbb{R}\setminus U$ such that $|x_n - x| < \frac{1}{n}$, so that $x = \lim_{n\to\infty} x_n$. As $\mathbb{R}\setminus U$ is closed, this means that $x \in \mathbb{R}\setminus U$ as well, which is a contradiction.

   Conversely, suppose that, for each $x \in U$, there is $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subset U$. Let $(x_n)_{n=1}^\infty$ be a convergent sequence in $\mathbb{R}\setminus U$ with limit x. Assume that $x \notin \mathbb{R}\setminus U$. This means that $x \in U$. Choose $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subset U$. As $x = \lim_{n\to\infty} x_n$, there is $n_\epsilon \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ for all $n \geq n_\epsilon$, i.e.,
$$x_n \in (x - \epsilon, x + \epsilon) \subset U$$
   for those n. This is a contradiction.

   Let $a < b$, and let $x \in (a, b)$. Set $\epsilon := \min\{x-a, b-a\}$. It is clear that $(x-\epsilon, x+\epsilon) \subset (a, b)$. Consequently, (a, b) is open.

3. Show that that
$$\sum_{k=1}^n kx^k = x\frac{nx^{n+1} - (n+1)x^n + 1}{(1-x)^2}$$
   for all $n \in \mathbb{N}$, and conclude that
$$\sum_{k=1}^\infty kx^k = \frac{x}{(1-x)^2}$$

if $|x| < 1$.

Solution: For the first part, we use induction on n.

$n = 1$ (induction anchor): Clearly,

$$\sum_{k=1}^{1} kx^k = x = x\frac{x^2 - 2x + 1}{(1-x)^2}$$

holds.

$n \rightsquigarrow n+1$ (induction step): Let $n \in \mathbb{N}$ be such that

$$\sum_{k=1}^{n} kx^k = x\frac{nx^{n+1} - (n+1)x^n + 1}{(1-x)^2}.$$

It follows that

$$\begin{aligned}
&\sum_{k=1}^{n+1} kx^k \\
&= \sum_{k=1}^{n} kx^k + (n+1)x^{n+1} \\
&= x\frac{nx^{n+1} - (n+1)x^n + 1}{(1-x)^2} + (n+1)x^{n+1}, \qquad \text{by the induction hypothesis,} \\
&= x\left(\frac{nx^{n+1} - (n+1)x^n + 1}{(1-x)^2} + (n+1)x^n\right) \\
&= x\frac{nx^{n+1} - (n+1)x^n + 1 + (1-x)^2(n+1)x^n}{(1-x)^2} \\
&= x\frac{nx^{n+1} - (n+1)x^n + 1 + (x^2 - 2x + 1)(n+1)x^n}{(1-x)^2} \\
&= x\frac{nx^{n+1} - (n+1)x^n + 1 + (n+1)x^{n+2} - 2(n+1)x^{n+1} + (n+1)x^n}{(1-x)^2} \\
&= x\frac{(n+1)x^{n+2} - (n+2)x^{n+1} + 1}{(1-x)^2}.
\end{aligned}$$

This proves the claim.

From Problem 4 on Assignment #6, it follows that

$$\lim_{n\to\infty} nx^{n+1} = \lim_{n\to\infty} (n+1)x^n = 0$$

whenever $|x| < 1$. It follows immediately that

$$\sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2}$$

if $|x| < 1$.

4. Let $\varnothing \neq D \subset \mathbb{R}$ be not bounded above, and let $f\colon D \to \mathbb{R}$. Show that the following are equivalent for $y_0 \in \mathbb{R}$:

   (i) $\lim_{x\to\infty} f(x) = y_0$;

   (ii) for every $\epsilon > 0$, there is $R > 0$ such that $|f(x) - y_0| < \epsilon$ for all $x \in D$ with $x > R$.

   Solution: (i) $\Longrightarrow$ (ii): Assume that (ii) is wrong. Then there is $\epsilon_0 > 0$ such that, for every $R > 0$, there is $x_R \in D$ with $x_R > R$, but $|f(x_R) - y_0| \geq \epsilon_0$. In particular, for each $n \in \mathbb{N}$, there is $x_n \in D$ with $x_n > n$ and $|f(x_n) - y_0| \geq \epsilon_0$. This means that $\lim_{n\to\infty} x_n = \infty$ whereas $(f(x_n))_{n=1}^{\infty}$ does not converge to y_0.

   (ii) $\Longrightarrow$ (i): Let $\epsilon > 0$, and let $(x_n)_{n=1}^{\infty}$ be a sequence in D such that $x_n \to \infty$. Let $R > 0$ be as in (ii). Then there is $n_R \in \mathbb{N}$ such that $x_n > R$ for all $n \geq n_R$. It follows that $|f(x_n) - y_0| < \epsilon$ for all $n \geq n_R$, which means that $\lim_{n\to\infty} f(x_n) = y_0$. It follows that $\lim_{x\to\infty} f(x) = y_0$.