MATH 216 (Fall 2021)
Introduction to Analysis

Midterm #2 Practice Problems

1. Let $\varnothing \neq D \subset \mathbb{R}$, let $f : D \to \mathbb{R}$, and let $x_0 \in \overline{D}$. Show that the following are equivalent:

   (i) $\lim_{x \to x_0} f(x) = \infty$;

   (ii) for every $R \in \mathbb{R}$, there is $\delta > 0$ such that $f(x) > R$ for all $x \in D$ with $|x - x_0| < \delta$.

   Solution: (i) $\Longrightarrow$ (ii): Assume that (ii) is wrong. Then there is $R_0 \in \mathbb{R}$ such that, for every $\delta > 0$, there is $x_\delta \in D$ with $|x_\delta - x_0| < \delta$, but $f(x_\delta) \leq R_0$. In particular, for each $n \in \mathbb{N}$, there is $x_n \in D$ with $|x_n - x_0| < \frac{1}{n}$ and $f(x_n) \leq R_0$. This means that $\lim_{n\to\infty} x_n = x_0$ whereas $(f(x_n))_{n=1}^\infty$ is bounded above and therefore cannot diverge to ∞.

   (ii) $\Longrightarrow$ (i): Let $R \in \mathbb{R}$, and $(x_n)_{n=1}^\infty$ be a sequence in D such that $x_n \to x_0$. Let $\delta > 0$ be as in (ii). Then there is $n_R \in \mathbb{N}$ such that $|x_n - x_0| < \delta$ for all $n \geq n_R$. It follows that $f(x_n) > R$ for all $n \geq n_R$, which means that $\lim_{n\to\infty} f(x_n) = \infty$. It follows that $\lim_{x\to x_0} f(x) = \infty$.

2. Let p and q be polynomials, let ν be the degree of p, and let μ be the degree of q. Suppose that n_0 is such that $q(k) \neq 0$ for all $k \geq n_0$. Show that the series $\sum_{k=n_0}^\infty \frac{p(k)}{q(k)}$ converges if and only if $\mu - \nu \geq 2$. (*Hint*: Limit Comparison Test.)

   Solution: Let
   $$p(x) = \alpha_\nu x^\nu + \alpha_{\nu-1} x^{\nu-1} + \cdots + \alpha_1 x + \alpha_0$$
   and
   $$q(x) = \beta_\mu x^\mu + \beta_{\mu-1} x^{\mu-1} + \cdots + \beta_1 x + \beta_0,$$
   where $\alpha_\nu \neq 0 \neq \beta_\mu$. Without loss of generality, suppose that $\frac{\alpha_\nu}{\beta_\mu} > 0$ (otherwise, multiply p or q by -1).

   Let $a_n := \frac{p(k)}{q(k)}$, and let $b_n := \frac{1}{k^p}$, where $p := \mu - \nu$. Since
   $$\frac{a_n}{b_n} = \frac{\alpha_\nu k^\mu + \alpha_{\nu-1} k^{\mu-1} + \cdots + \alpha_n k^{p+1} + \alpha_0 k^p}{\beta_\mu k^\mu + \beta_{\mu-1} k^{\mu-1} + \cdots + \beta_1 n + \beta_0} \to \frac{\alpha_\nu}{\beta_\mu} > 0$$
   for all values of p, the Limit Comparison Test yields that the series $\sum_{k=1}^\infty a_k$ converges if and only if $\sum_{k=1}^\infty b_k$ converges. Since $\sum_{k=1}^\infty \frac{1}{k^p}$ converges for $p \geq 2$ and diverges for $p \leq 1$, this completes the proof.

3. Let $S \subset \mathbb{R}$. Show that

$$\overline{S} = \{x_0 \in \mathbb{R} : \text{for each } \epsilon > 0, \text{ there is } x \in S \text{ with } |x - x_0| < \epsilon\}.$$

Solution: Set

$$S_0 := \{x_0 \in \mathbb{R} : \text{for each } \epsilon > 0, \text{ there is } x \in S \text{ with } |x - x_0| < \epsilon\}.$$

We show that $S_0 = \overline{S}$ by showing that both $S_0 \subset \overline{S}$ and $\overline{S} \subset S_0$.

"$S_0 \subset \overline{S}$": Let $x_0 \in S_0$. Then for every $n \in \mathbb{N}$, there is $x_n \in S$ with $|x_n - x_0| < \frac{1}{n}$, so that $x_0 = \lim_{n\to\infty} x_n \in \overline{S}$.

"$\overline{S} \subset S_0$": Let $x_0 \in \overline{S}$, and let $\epsilon > 0$. By the definition of $\overline{S}$, there is a sequence $(x_n)_{n=1}^\infty$ such that $x_n \to x_0$. It follows that there is $n_\epsilon \in \mathbb{N}$ such that $|x_n - x_0| < \epsilon$ for all $n \geq n_\epsilon$. In particular, there is $x \in S$ with $|x - x_0| < \epsilon$, i.e., $x_0 \in S_0$.

4. Determine whether or not the following series diverge, converge, or converge absolutely:

(a) $\displaystyle\sum_{k=1}^\infty (-1)^{k^3+3k^2-7k+13} \frac{\sin k}{\cos k + k^2}$;

(b) $\displaystyle\sum_{\nu=1}^\infty \frac{1}{\nu^\nu}\binom{2\nu}{\nu}$;

(c) $\displaystyle\sum_{n=1}^\infty \frac{n^n}{n!}$.

Solution:

(a) For $k \in \mathbb{N}$, set

$$a_k := (-1)^{k^3+3k^2-7k+13} \frac{\sin k}{\cos k + k^2},$$

so that

$$|a_k| = \frac{|\sin k|}{|\cos k + k^2|}.$$

For $k \geq 2$, we have

$$|a_k| \leq \frac{1}{|\cos k + k^2|} \leq \frac{1}{k^2 - 1}$$

As $\sum_{k=2}^\infty \frac{1}{k^2-1} < \infty$, the Comparison Test yields that $\sum_{k=1}^\infty a_k$ converges absolutely.

(b) For $\nu \in \mathbb{N}$, set

$$a_\nu := \frac{1}{\nu^\nu}\binom{2\nu}{\nu},$$

so that $a_\nu \neq 0$. We have

$$\begin{aligned}
\frac{a_{\nu+1}}{a_\nu} &= \frac{1}{(\nu+1)^{\nu+1}} \frac{(2\nu+2)!}{((\nu+1)!)^2} \nu^\nu \frac{(\nu!)^2}{(2\nu!)} \\
&= \underbrace{\left(\frac{\nu}{\nu+1}\right)^\nu}_{\leq 1} \underbrace{\frac{1}{\nu+1}}_{\stackrel{\nu\to\infty}{\longrightarrow} 0} \underbrace{\frac{(2\nu+2)(2\nu+1)}{(\nu+1)^2}}_{\stackrel{\nu\to\infty}{\longrightarrow} 4} \\
&\stackrel{\nu\to\infty}{\longrightarrow} 0.
\end{aligned}$$

By the Limit Ratio Test, this means that $\sum_{\nu=1}^\infty a_\nu$ converges absolutely.

(c) For $n \in \mathbb{N}$, set

$$a_n := \frac{n^n}{n!},$$

so that

$$a_n = \frac{\overbrace{n \cdots n}^{n \text{ times}}}{n \cdot (n-1) \cdots 2 \cdot 1} = \frac{n}{n} \frac{n}{n-1} \cdots \frac{n}{2} \frac{n}{1} \geq 1.$$

In particular, $a_n \not\to 0$, so that $\sum_{n=1}^\infty a_n$ diverges.

5. Let $(a_n)_{n=1}^\infty$ be a sequence in $\mathbb{R}$ such that $\liminf_{n\to\infty} |a_n| = 0$, and let $(R_k)_{k=1}^\infty$ be a sequence of non-zero reals. Show that $(a_n)_{n=1}^\infty$ has a subsequence $(a_{n_k})_{k=1}^\infty$ such that $\sum_{k=1}^\infty R_k a_{n_k}$ converges absolutely.

Solution: From $\liminf_{n\to\infty} |a_n| = 0$ it is immediate that 0 is a cluster point of $(|a_n|)_{n=1}^\infty$. For $k \in \mathbb{N}$, set

$$\mathbb{N}_k := \left\{ n \in \mathbb{N} : |a_n| < \frac{1}{|R_k| 2^k} \right\}.$$

By Problem 4 on Assignment #5, we know that $\mathbb{N}_k$ is infinite for all $k \in \mathbb{N}$.

Choose $n_1 \in \mathbb{N}_1$. Then, suppose that $n_1 < \cdots < n_k$ with $n_j \in \mathbb{N}_j$ for $j = 1, \ldots, k$ have already been chosen. As $\mathbb{N}_{k+1}$ is infinite, there is $n_{k+1} \in \mathbb{N}_{k+1}$ such that $n_k < n_{k+1}$.

This way, we obtain a subsequence $(a_{n_k})_{k=1}^\infty$ of $(a_n)_{n=1}^\infty$ such that $|R_k a_{n_k}| < \frac{1}{2^k}$. As $\sum_{k=1}^\infty \frac{1}{2^k} < \infty$, the Comparison Test yields that $\sum_{k=1}^\infty R_k a_{n_k}$ converges absolutely.

6. Show the following:

(a) if $F_1, \ldots, F_m \subset \mathbb{R}$ are closed, then $F_1 \cup \cdots \cup F_m$ is closed;

(b) if $\mathcal{F} \subset \mathfrak{P}(\mathbb{R})$ is such that each $F \in \mathcal{F}$ is closed, then $\bigcap\{F : F \in \mathcal{F}\}$ is closed.

Solution:

(a) Let $(x_n)_{k=1}^{\infty}$ a convergent sequence in $F_1 \cup \cdots \cup F_m$. For $j = 1, \ldots, m$, let

$$\mathbb{N}_j := \{n \in \mathbb{N} : x_n \in F_j\}.$$

As $\mathbb{N} = \mathbb{N}_1 \cup \cdots \cup \mathbb{N}_m$ is infinite, there must be $j_0 \in \{1, \ldots, m\}$ such that $\mathbb{N}_{j_0}$ is infinite, i.e., there is a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ contained in F_{j_0}. As F_{j_0} is closed, it follows that

$$\lim_{n\to\infty} x_n = \lim_{k\to\infty} x_{n_k} \in F_{j_0} \subset F_1 \cup \cdots \cup F_m.$$

Therefore, $F_1 \cup \cdots \cup F_m$ is closed.

(b) Let $(x_n)_{k=1}^{\infty}$ be a convergent sequence in $\bigcap\{F : F \in \mathcal{F}\}$, i.e., $x_n \in F$ for all $n \in \mathbb{N}$ and all $F \in \mathcal{F}$. As each $F \in \mathcal{F}$ is closed, it is clear that $\lim_{n\to\infty} x_n \in F$ for each $F \in \mathcal{F}$, i.e., $\lim_{n\to\infty} x_n \in \bigcap\{F : F \in \mathcal{F}\}$. This means that $\bigcap\{F : F \in \mathcal{F}\}$ is closed.