**MATH 216** (Fall 2021)
Introduction to Analysis

***Midterm #1 Model Solutions***

1. Let $(x_n)_{n=1}^\infty$ be a convergent sequence of non-negative reals with limit $x$. Show that $\left(\sqrt{x_n}\right)_{n=1}^\infty$ is convergent with
$$\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}.$$
(*Hint*: $\left|\sqrt{x}-\sqrt{y}\right|^2 \leq \left|\sqrt{x}-\sqrt{y}\right| \left|\sqrt{x}+\sqrt{y}\right| = |x-y|$ for all $x, y \geq 0$.)
*Solution*: Let $\epsilon > 0$. As $\lim_{n\to\infty} x_n = x$, there is $n_\epsilon \in \mathbb{N}$ such that $|x_n - x| < \epsilon^2$ for all $n \geq n_\epsilon$. From the inequality in the hint, we obtain for those $n$ that
$$\left|\sqrt{x_n}-\sqrt{x}\right|^2 \leq \left|\sqrt{x_n}-\sqrt{x}\right| \left|\sqrt{x_n}+\sqrt{x}\right| = |x_n - x| < \epsilon^2$$
and therefore $\left|\sqrt{x_n}-\sqrt{x}\right| < \epsilon$.

2. Let
$$S := \left\{ n((-1)^n + 1) + \frac{1}{n} : n \in \mathbb{N} \right\}.$$
Determine $\inf S$ and $\sup S$.
*Solution*: First note that, for even $n \in \mathbb{N}$, we have
$$n((-1)^n + 1) + \frac{1}{n} = 2n + \frac{1}{n} \geq 2n.$$
It follows that $S$ is not bounded above, so that $\sup S = \infty$.
For odd $n \in \mathbb{N}$, we have
$$n((-1)^n + 1) + \frac{1}{n} = \frac{1}{n} \geq 0.$$
It follows that 0 is a lower bound for $S$, so that $\inf S \geq 0$. Assume towards a contradiction that $\inf S > 0$. By the Archimedian Property of $\mathbb{R}$, there is then $n \in \mathbb{N}$, which we can suppose to be odd, such that
$$\inf S > \frac{1}{n} = n((-1)^n + 1) + \frac{1}{n}.$$
This is a contradiction.

3. Show that
$$\prod_{k=1}^{n-1} \left(1 + \frac{1}{k}\right)^k = \frac{n^n}{n!}$$
for all $n \in \mathbb{N}$ such that $n \geq 2$.

*Solution*: $n = 2$ (*induction anchor*): In this case,

$$\prod_{k=1}^{1} \left(1 + \frac{1}{k}\right)^k = \left(1 + \frac{1}{1}\right)^1 = 2 = \frac{4}{2} = \frac{2^2}{2!}$$

holds.

$n \rightsquigarrow n+1$ (*induction step*): Suppose that

$$\prod_{k=1}^{n-1} \left(1 + \frac{1}{k}\right)^k = \frac{n^n}{n!}$$

holds. Then it follows that

$$\begin{aligned}
\prod_{k=1}^{n} \left(1 + \frac{1}{k}\right)^k &= \left(\prod_{k=1}^{n-1} \left(1 + \frac{1}{k}\right)^k\right) \left(1 + \frac{1}{n}\right)^n \\
&= \frac{n^n}{n!} \left(1 + \frac{1}{n}\right)^n, \qquad \text{by the induction hypothesis,} \\
&= \frac{n^n}{n!} \left(\frac{n+1}{n}\right)^n \\
&= \frac{n^n}{n!} \frac{(n+1)^n}{n^n} \\
&= \frac{n^n}{n!} \frac{(n+1)^{n+1}}{n^n} \frac{1}{n+1} \\
&= \frac{(n+1)^{n+1}}{n!(n+1)} \\
&= \frac{(n+1)^{n+1}}{(n+1)!}
\end{aligned}$$

4. Does the sequence $\left(\sqrt{n^2+n} - n\right)_{n=1}^{\infty}$ converge? If so, determine its limit.

   *Solution*: By the third binomial formula, we have

$$\sqrt{n^2+n} - n = \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+\frac{1}{n}}+1}$$

   for $n \in \mathbb{N}$. By Problem 1, we know that $\lim_{n\to\infty} \sqrt{1+\frac{1}{n}} = 1$. Then limit laws that yield that $\left(\sqrt{n^2+n} - n\right)_{n=1}^{\infty}$ converges to $\frac{1}{2}$.