MATH 216 (Fall 2021)
Introduction to Analysis

Midterm #1 Practice Problems

1. Let
$$S := \left\{ (-1)^n \left(1 - \frac{1}{n}\right) : n \in \mathbb{N} \right\}.$$
Determine $\sup S$ and $\inf S$.

Solution: As
$$\left| (-1)^n \left(1 - \frac{1}{n}\right) \right| = 1 - \frac{1}{n} \leq 1$$
for $n \in \mathbb{N}$, it is clear that $\sup S \leq 1$ and $\inf S \geq -1$.

Assume that $\sup S < 1$. Then, by the Archimedean Property, there is $n \in \mathbb{N}$ such that $\frac{1}{n} < 2(1 - \sup S)$ and therefore
$$\sup S < 1 - \frac{1}{2n} = (-1)^{2n} \left(1 - \frac{1}{2n}\right),$$
which is a contradiction. It follows that $\sup S = 1$.

Similarly, one shows that $\inf S = -1$.

2. Let $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ be convergent sequences in $\mathbb{R}$ such that $x_n < y_n$ for $n \in \mathbb{N}$. Does this entail that $\lim_{n\to\infty} x_n < \lim_{n\to\infty} y_n$?

Solution: Of course, not.

For $n \in \mathbb{N}$, set
$$x_n := 0 \qquad \text{and} \qquad y_n := \frac{1}{n},$$
so that $x_n < y_n$ for all $n \in \mathbb{N}$, but $\lim_{n\to\infty} x_n = 0 = \lim_{n\to\infty} y_n$.

3. Let p be a non-zero polynomial. Show that $\lim_{n\to\infty} \frac{p(n+1)}{p(n)} = 1$;

Solution: We can suppose that p is not constant. Then there are $c_k, \ldots, c_1, c_0 \in \mathbb{R}$ with $k \in \mathbb{N}$ and $c_k \neq 0$ such that
$$p(x) := c_k x^k + \cdots + c_1 x + c_0.$$

It follows that
$$p(x+1) := c_k \sum_{j=0}^{k} \binom{k}{j} x^j + \text{terms of degree} \leq k - 1,$$
i.e., there are $d_{k-1}, \ldots, d_0 \in \mathbb{R}$ such that
$$p(x+1) := c_k x^k + d_{k-1} x^{k-1} + \cdots + d_1 x + d_0.$$

We conclude that, for $n \in \mathbb{N}$ with $p(n) \neq 0$,

$$\begin{aligned}\frac{p(n+1)}{p(n)} &= \frac{c_k n^k + d_{k-1} n^{k-1} + \cdots + d_0}{c_k n^k + c_{k-1} n^{k-1} + \cdots + c_0} \\ &= \frac{c_k + \frac{d_{k-1}}{n} + \cdots + \frac{d_0}{n^k}}{c_k + \frac{c_{k-1}}{n} + \cdots + \frac{c_0}{n^k}} \\ &\overset{n\to\infty}{\longrightarrow} \frac{c_k}{c_k} \\ &= 1.\end{aligned}$$

4. In class, we have used the symbol $\sum$ for finite sums. There is a similar notation for finite products. Let $m, n \in \mathbb{N}$ be such that $m \leq n$, and let $a_m, a_{m+1}, \ldots, a_n \in \mathbb{R}$. Then we define

$$\prod_{k=m}^{n} a_k := a_m a_{m+1} \cdots a_n.$$

Show that

$$\prod_{k=2}^{n} \left(1 - \frac{1}{k^2}\right) = \frac{n+1}{2n}$$

for all $n \geq 2$.

Solution: We use induction on n.

$n = 2$: In this case, we have

$$\prod_{k=2}^{2} \left(1 - \frac{1}{k^2}\right) = \frac{3}{4} = \frac{2+1}{2 \cdot 2}.$$

$n \rightsquigarrow n+1$: Let $n \geq 2$ be such that

$$\prod_{k=2}^{n} \left(1 - \frac{1}{k^2}\right) = \frac{n+1}{2n}$$

(*induction hypothesis*). We obtain

$$\begin{aligned}\prod_{k=2}^{n+1} \left(1 - \frac{1}{k^2}\right) &= \left(\prod_{k=2}^{n} \left(1 - \frac{1}{k^2}\right)\right)\left(1 - \frac{1}{(n+1)^2}\right) \\ &= \frac{n+1}{2n}\left(1 - \frac{1}{(n+1)^2}\right), \qquad \text{by the induction hypothesis,} \\ &= \frac{n+1}{2n}\frac{n^2+2n}{(n+1)^2} \\ &= \frac{n+2}{2(n+1)},\end{aligned}$$

which proves the claim.

5. Let $(x_n)_{n=1}^\infty$ be a convergent sequence in $\mathbb{R}$ with limit x. Show that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_k = x.$$

(This problem is way too difficult for an actual exam question, but it's a lovely—and important!—result.)

Solution: Let $\epsilon > 0$. As $\lim_{n\to\infty} x_n = x$, there is $n_1 \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\epsilon}{2}$$

for $n \geq n_1$. Choose $n_2 \in \mathbb{N}$ such that

$$\frac{1}{n_2}\left|\sum_{k=1}^{n_1-1}(x_k - x)\right| < \frac{\epsilon}{2}.$$

Set $n_\epsilon := \max\{n_1, n_2\}$. For $n \geq n_\epsilon$, we then have

$$\begin{aligned}
\left|\frac{1}{n}\sum_{k=1}^{n} x_k - x\right| &= \left|\frac{1}{n}\sum_{k=1}^{n}(x_k - x)\right| \\
&\leq \frac{1}{n}\left|\sum_{k=1}^{n_1-1}(x_k - x)\right| + \frac{1}{n}\sum_{k=n_1}^{n}|x_k - x| \\
&\leq \underbrace{\frac{1}{n_2}\left|\sum_{k=1}^{n_1-1}(x_k - x)\right|}_{<\frac{\epsilon}{2}} + \frac{1}{n}\sum_{k=n_1}^{n}|x_k - x| \\
&< \frac{\epsilon}{2} + \underbrace{\frac{n+1-n_1}{n}}_{\leq 1}\max_{k=n_1,\ldots,n}\underbrace{|x_k - x|}_{<\frac{\epsilon}{2}} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

This proves the claim.