1. Show that the improper integral
\[ \int_0^\infty \frac{\cos x}{1+x} \, dx \]
exists, but does not converge absolutely.

*Solution:* Let \( R > 0 \). Then integration by parts yields
\[ \int_0^R \frac{\cos x}{1+x} \, dx = \frac{\sin R}{1+R} + \int_0^R \frac{\cos x}{(1+x)^2} \, dx. \]
Clearly,
\[ \lim_{R \to \infty} \frac{\sin R}{1+R} = \lim_{R \to \infty} \frac{\sin R}{1+R} = 0 \]
holds, and since \( \int_0^\infty \frac{1}{(1+x)^2} \, dx \) is easily seen to exist, the existence (and even absolute convergence) of \( \int_0^\infty \frac{\cos x}{(1+x)^2} \, dx \) follows from the comparison test. Hence, the integral \( \int_0^\infty \frac{\cos x}{1+x} \, dx \) does exist.

To see that it does not converge absolutely, note that, for \( n \in \mathbb{N} \), we have
\[ \int_0^{\frac{\pi}{2}+\pi+n\pi} \frac{|\cos x|}{1+x} \, dx \geq \int_0^{\frac{n\pi+\pi}{2}} \frac{|\cos x|}{1+x} \, dx \]
\[ = \sum_{k=1}^{n} \int_{\frac{2\pi}{2}+(k-1)\pi}^{\frac{2\pi}{2}+k\pi} \frac{|\cos x|}{1+x} \, dx \]
\[ \geq \sum_{k=1}^{n} \frac{1}{2\pi} \int_{\frac{\pi}{2}+(k-1)\pi}^{\frac{\pi}{2}+k\pi} |\cos x| \, dx \]
\[ \geq \sum_{k=1}^{n} \frac{1}{2\pi} \int_{\frac{\pi}{2}+(k-1)\pi}^{\frac{\pi}{2}+k\pi} |\cos x| \, dx \]
\[ = \sum_{k=1}^{n} \frac{2}{2\pi} \cdot \frac{\pi}{2}. \]
As \( \sum_{n=1}^\infty \frac{2}{2\pi+n\pi} \) diverges (limit comparison with the harmonic series), the improper integral cannot converge absolutely.

2. Let \( S \) be the unit sphere in \( \mathbb{R}^3 \). Compute the surface integral
\[ \int_S xz^2 \, dy \wedge dz + \frac{7}{x^2 + z^2 + 3} \, dz \wedge dx + x^3 y^3 \, dx \wedge dy. \]
Solution: Let $V$ be the closed unit ball in $\mathbb{R}^3$, so that $S = \partial V$.

Set

$$f(x, y, z) := \left( x z^2, \frac{7}{x^2 + z^2 + 3}, x^3 y^7 \right).$$

By Gauß' Theorem, we have

$$\int_S x z^2 \, dy \wedge dz + \frac{7}{x^2 + z^2 + 3} \, dz \wedge dx + x^3 y^7 \, dx \wedge dy = \int_S f \cdot n \, d\sigma = \int_V \text{div} \, f = \int_V z^2.$$

To compute the three-dimensional Riemann integral, we use spherical coordinates.

Set $K := [0, 1] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times [0, 2\pi]$, and let

$$\phi : K \to \mathbb{R}, \quad (r, \theta, \sigma) \mapsto (r \cos \theta \cos \sigma, r \cos \theta \sin \sigma, r \sin \theta),$$

so that

$$\int_V z^2 = \int_{\phi(K)} z^2$$

$$= \int_K (r \sin \theta)^2 | \det J_\phi |$$

$$= \int_K (r \sin \theta)^2 r^2 \cos \theta$$

$$= \int_K r^4 (\sin \theta)^2 (\cos \theta), \quad \text{because } \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

$$= \int_0^1 \left( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \int_0^{2\pi} r^4 (\sin \theta)^2 (\cos \theta) \, d\sigma \right) d\theta \right) dr$$

$$= 2\pi \int_0^1 \left( r^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin \theta)^2 (\cos \theta) \, d\theta \right) dr$$

$$= 2\pi \int_0^1 \left( r^4 \int_{-1}^{1} u^2 \, du \right) dr, \quad \text{substituting } u = \sin \theta,$$

$$= 2\pi \int_0^1 \left( r^4 \frac{u^3}{3} \bigg|_{-1}^{1} \right) dr$$

$$= \frac{4\pi}{3} \int_0^1 r^4 \, dr$$

$$= \frac{4\pi}{3} \cdot \frac{1}{5}$$

$$= \frac{4\pi}{15}.$$

3. Let $\emptyset \neq D \subset \mathbb{R}^N$, and let $f, f_1, f_2, \ldots : D \to \mathbb{R}$. The sequence $(f_n)_{n=1}^\infty$ is said to converge to $f$ locally uniformly on $D$ if, for each $x \in D$, there is a neighborhood $U$ of $x$ such that $(f_n)_{n=1}^\infty$ converges to $f$ uniformly on $U \cap D$. 


Let $\emptyset \neq K \subset \mathbb{R}^N$ be compact, and let $f, f_1, f_2, \ldots : K \to \mathbb{R}$ be such that $(f_n)_{n=1}^\infty$ converges to $f$ locally uniformly on $K$. Show that $(f_n)_{n=1}^\infty$ converges to $f$ uniformly on $K$.

**Solution:** For each $x \in K$, there is a neighborhood $U_x$ of $x$ such that $(f_n)_{n=1}^\infty$ converges to $f$ uniformly on $U_x \cap K$; we can suppose that $U_x$ is open. Let $\epsilon > 0$. Then, for each $x \in K$, there is $n_{\epsilon,x} \in \mathbb{N}$ such that $|f_n(y) - f(y)| < \epsilon$ for all $n \geq n_{\epsilon,x}$ and for all $y \in U_x \cap K$. Since $K$ is compact, there are $x_1, \ldots, x_m \in K$ such that $K \subset U_{x_1} \cup \cdots \cup U_{x_m}$.

Let $n_{\epsilon} := \max\{n_{\epsilon,x} : j = 1, \ldots, m\}$. For $n \geq n_{\epsilon}$, we thus have $|f_n(y) - f(x)| < \epsilon$ for all $n \geq n_{\epsilon}$ and for all $y \in K$.

4. Let $f \in \mathcal{PC}_{2\pi}(\mathbb{R})$ be given by

$$f(t) := \begin{cases} 
-1, & t \in (-\pi, 0], \\
2, & t \in (0, \pi]. 
\end{cases}$$

Determine the Fourier series of $f$. Does it converge to $f$ uniformly on $\mathbb{R}$?

**Solution:** We have

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \, dt = -1 + 2 = 1$$

and, for $n \in \mathbb{N}$,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) \, dt$$

$$= \frac{1}{\pi} \left( - \int_{-\pi}^{0} \cos(nt) \, dt + 2 \int_{0}^{\pi} \cos(nt) \, dt \right)$$

$$= \frac{1}{\pi} \int_{0}^{\pi} \cos(nt) \, dt$$

$$= \frac{1}{\pi} \left( \frac{\sin(nt)}{n} \bigg|_{t=\pi}^{t=0} \right)$$

$$= 0$$
and
\[
b_n = \frac{1}{\pi} f(t) \sin(nt) \, dt
= \frac{1}{\pi} \left( -\int_{-\pi}^{0} \sin(nt) \, dt + 2\int_{0}^{\pi} \sin(nt) \, dt \right)
= \frac{3}{\pi} \int_{0}^{\pi} \sin(nt) \, dt
= \frac{3}{\pi} \left( -\cos(nt) \bigg|_{t=0}^{t=\pi} \right)
= \frac{3}{n\pi} (-\cos(n\pi) + 1)
= \frac{3((-1)^{n-1} + 1)}{n\pi},
\]
so that
\[
f(x) \sim \frac{1}{2} + \sum_{n=1}^{\infty} \frac{3((-1)^{n-1} + 1)}{n\pi} \sin(nx).
\]

As \( f \) is discontinuous, it cannot be the uniform limit of its Fourier series.

5. Let \( (a_n)_{n=1}^{\infty} \) be a decreasing sequence of non-negative real numbers. Show with the help of the Cauchy criterion for infinite series that \( \lim_{n \to \infty} na_n = 0 \) if \( \sum_{n=1}^{\infty} a_n \) converges. Does the converse also hold?

**Solution:** Let \( \epsilon > 0 \). Since \( \sum_{n=1}^{\infty} a_n \) converges, there is \( n_1 \) such that \( \sum_{k=m}^{n} a_k < \frac{\epsilon}{2} \) for \( n \geq m \geq n_1 \). Since \( (a_n)_{n=1}^{\infty} \) is decreasing, we have
\[
\frac{\epsilon}{2} > \sum_{k=n_1}^{n} a_k \geq (n - n_1 + 1) a_n
\]
for \( n \geq n_1 \). Since \( \lim_{n \to \infty} a_n = 0 \), there is \( n_2 \) such that \( (n_1 - 1) a_n < \frac{\epsilon}{2} \) for \( n \geq n_2 \). Letting \( n_\epsilon := \max\{n_1, n_2\} \), we obtain that
\[
|na_n| \leq (n - n_1 + 1) a_n + (n_1 - 1) a_n \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]
for \( n \geq n_\epsilon \).

Let \( a_1 := 2019 \) and \( a_n := \frac{1}{n\log n} \) for \( n \geq 2 \). As seen in class, \( \sum_{n=1}^{\infty} a_n \) diverges by the Integral Comparison Test whereas \( \lim_{n \to \infty} na_n = 0 \).

6. Let \( p \) be a polynomial, and let \( \theta \in (-1, 1) \). Show that the series \( \sum_{n=1}^{\infty} p(n)\theta^n \) converges absolutely.

**Solution:** Let \( \nu \) be the degree of \( p \), i.e.,
\[
p(x) = a_\nu x^\nu + q(x),
\]
where \( a_\nu \neq 0 \), and \( q \) is a polynomial whose degree is strictly less than \( \nu \). We then have

\[
p(x + 1) = a_\nu x^\nu + \sum_{k=1}^{\nu} a_\nu \binom{\nu}{k} x^{\nu-k} + q(x + 1).
\]

Hence, we obtain

\[
\frac{p(n+1)}{p(n)} = \frac{a_\nu n^\nu + \sum_{k=1}^{\nu} a_\nu \binom{\nu}{k} n^{\nu-k} + q(n+1)}{a_\nu n^\nu + q(n)}
= \frac{a_\nu + \frac{1}{n^\nu} \sum_{k=1}^{\nu} a_\nu \binom{\nu}{k} n^{\nu-k} + \frac{q(n+1)}{n^\nu}}{a_\nu + \frac{q(n)}{n^\nu}} \to 1.
\]

(We may take the quotient safely because polynomials only have a finite number of zeros.)

Let \( a_n := p(n)\theta^n \). By the foregoing, we obtain that

\[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{p(n+1)}{p(n)} \right| |\theta| \to |\theta| < 1.
\]

Hence, the series converges by the limit ratio test.

7. Let \( U := \{(x,y) \in \mathbb{R}^2 : x > 0\} \), let \( f = (P,Q) : U \to \mathbb{R}^2 \) be given by

\[
P(x,y) = \frac{\arctan y}{x} \quad \text{and} \quad Q(x,y) = \frac{\log x}{1 + y^2}
\]

for \((x,y) \in U\), and let \( \gamma = \gamma_1 \oplus \gamma_2 \), where \( \gamma_1 = [(1,7), (13, \pi)] \) and \( \gamma_2 = [(13, \pi), (e, 1)] \). Evaluate

\[
\int_\gamma P\, dx + Q\, dy.
\]

**Solution:** First note that

\[
\frac{\partial P}{\partial y}(x,y) = \frac{1}{x(1 + y^2)} = \frac{\partial Q}{\partial x}(x,y)
\]

for \((x,y) \in U\). As \( U \) is convex, that means that \( f \) is conservative. It is easy to see that

\[
F : U \to \mathbb{R}, \quad (x,y) \mapsto (\log x)(\arctan y)
\]

is a potential function for \( f \). It follows that

\[
\int_\gamma P\, dx + Q\, dy = F(e, 1) - F(1, 7) = \frac{\pi}{4}.
\]
8. For the power series

\[
\sum_{n=1}^{\infty} \frac{x^n}{n}
\]

determine the following:

(a) the radius of convergence;
(b) the set of those \(x\) for which \((*)\) converges;
(c) the value of \((*)\) for all such \(x\).

**Solution:**

(a) Let \(x \neq 0\). Since

\[
\left| \frac{x^{n+1}}{n+1} \right| = \left| \frac{n}{x^n} \right| \rightarrow |x|,
\]

it follows from the ratio test that the series converges for \(|x| < 1\) and diverges for \(|x| > 1\). Hence, the radius of convergence is 1.

(b) In view of (a), only \(x = \pm 1\) still has to be checked. For \(x = 1\), we have the harmonic series and thus divergence, whereas for \(x = -1\), we have convergence by the alternating series test. Hence, \((*)\) converges for all \(x \in [-1, 1)\).

(c) Since power series can be differentiated term by term on the open interval of convergence, we have, for \(x \in (-1, 1)\), that

\[
\frac{d}{dx} \sum_{n=1}^{\infty} \frac{x^n}{n} = \sum_{n=1}^{\infty} \frac{d}{dx} \frac{x^n}{n} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.
\]

Since \(-\log(1-x)\) is an antiderivative of \(\frac{1}{1-x}\), this means that there is \(C \in \mathbb{R}\) such that

\[
\sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x) + C
\]

for \(x \in (-1, 1)\). Plugging in \(x = 0\) yields \(C = 0\) and thus

\[
\sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x)
\]

for \(x \in (-1, 1)\). By Abel’s Theorem, this identity still holds for \(x = -1\).

9. Let \(C\) be the counterclockwise oriented circle \(x^2 - 2x + y^2 = 0\). Evaluate the curve integral

\[
\int_C 2xy \, dx + (x+1)^2 \, dy.
\]
Solution: Let $D$ denote the disc bounded by $C$, so that, by Green’s Theorem,
\[
\int_C 2xy \, dx + (x + 1)^2 \, dy = \int_D \frac{\partial}{\partial x} (x + 1)^2 - \frac{\partial}{\partial y} 2xy \\
= \int_D 2(x + 1) - 2x \\
= 2\mu(D).
\]
Since
\[
x^2 - 2x + y^2 = 0 \iff (x - 1)^2 + y^2 = 1,
\]
$D$ has radius 1, so that $\mu(D) = \pi$. Consequently,
\[
\int_C 2xy \, dx + (x + 1)^2 \, dy = 2\pi.
\]
holds.

10. For $a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{\infty\}$ with $a < b$, let $f, g: [a, b) \to \mathbb{R}$ be such that such that:

(a) $f$ and $g$ are both Riemann integrable on $[a, c]$ for each $c \in [a, b)$;
(b) $\int_a^b f(x) \, dx$ converges absolutely;
(c) $g$ is bounded.

Show that $\int_a^b f(x)g(x) \, dx$ converges absolutely.

If $\int_a^b f(x) \, dx$ is only required to converge, does then $\int_a^b f(x)g(x) \, dx$ necessarily converge?

Solution: Let $C \geq 0$ be such that $|g(x)| \leq C$ for $x \in [a, b)$. Let $c \in [a, b)$, and note that
\[
\int_a^c |f(x)g(x)| \, dx \leq C \int_a^c |f(x)| \, dx \leq C \int_a^b |f(x)| \, dx.
\]
Consequently, $\int_a^b |f(x)| \, dx$ exists, i.e., $\int_a^b f(x) \, dx$ converges absolutely.

Let
\[
f: [0, \infty) \to \mathbb{R}, \quad x \mapsto \begin{cases} \frac{\sin x}{x}, & x > 0, \\ 1, & x = 0. \end{cases}
\]
In class, we saw that $\int_0^\infty f(x) \, dx$ converges. Define
\[
g: [0, \infty) \to \mathbb{R}, x \mapsto \begin{cases} 1, & x \in [2(n-1)\pi, (2n-1)\pi] \text{ for some } n \in \mathbb{N}, \\ -1, & x \in [2n-1, 2n) \text{ for some } n \in \mathbb{N}. \end{cases}
\]
Then $g$ is bounded, and $g|_{[0,c]}$ is Riemann integrable for all $c \in (0, \infty)$. Note that
\[
f(x)g(x) = \frac{\sin x}{x}
\]
for $x \in (0, \infty)$. In class, we had see that $\int_0^\infty \frac{\sin x}{x} \, dx$ does not exist. Hence, $\int_a^b f(x)g(x) \, dx$ does not converge.
11. Let $\emptyset \neq K \subset \mathbb{R}^N$ and $\emptyset \neq L \subset \mathbb{R}^M$ be compact, let $F: K \times L \to \mathbb{R}$, and let $(y_n)_{n=1}^\infty$ be a sequence in $L$. For $n \in \mathbb{N}$, let

$$f_n: K \to \mathbb{R}, \quad x \mapsto F(x, y_n).$$

Show that $(f_n)_{n=1}^\infty$ has a subsequence that converges uniformly on $K$.

**Solution:** As $L$ is compact, $(y_n)_{n=1}^\infty$ has a subsequence $(y_{n_k})_{k=1}^\infty$ that converges to some $y_0 \in L$. Define

$$f: K \to \mathbb{R}, \quad x \mapsto F(x, y_0).$$

We claim that $f_{n_k} \to f$ uniformly on $K$. To see this, let $\epsilon > 0$, and note that—due to the compactness of $K \times L$—$F$ is uniformly continuous. Hence, there is $\delta > 0$ such that

$$|F(x, y) - F(x', y')| < \epsilon$$

for all $(x, y), (x', y') \in K \times L$ such that $\|(x, y) - (x', y')\| < \delta$. Let $k_\epsilon > 0$ be such that $\|y_{n_k} - y_0\| < \delta$ for $k \geq k_\epsilon$. It follows that, for $k \geq k_\epsilon$ and $x \in K$, we have

$$\|(y_{n_k}, x) - (y_0, x)\| = \|y_{n_k} - y_0\| < \delta,$$

so that

$$|f_{n_k}(x) - f(x)| = |F(x, y_{n_k}) - F(x, y_0)| < \epsilon$$

holds. This proves the claim.

12. Determine whether or not the improper integral

$$\int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}} \, dx$$

exists, and evaluate it if possible.

**Solution:** Let $\theta \in (0, 1)$, and consider the Riemann integral

$$\int_{0}^{\theta} \frac{1}{\sqrt{1 - x^2}} \, dx.$$
Change of variables with $x = \sin u$ yields

$$
\int_0^\theta \frac{1}{\sqrt{1-x^2}} \, dx = \int_{\arcsin 0}^{\arcsin \theta} \frac{\cos u}{\sqrt{1 - (\sin u)^2}} \, du
$$

$$
= \int_0^{\arcsin \theta} \frac{\cos u}{\sqrt{1 - (\sin u)^2}} \, du
$$

$$
= \int_0^{\arcsin \theta} \frac{\cos u}{\sqrt{\cos^2 u}} \, du
$$

$$
= \int_0^{\arcsin \theta} \frac{\cos u}{\cos u} \, du
$$

$$
= \int_0^{\arcsin \theta} 1 \, du
$$

$$
= \arcsin \theta.
$$

It follows that

$$
\frac{\pi}{2} = \lim_{\theta \uparrow 1} \arcsin \theta = \lim_{\theta \uparrow 1} \int_0^\theta \frac{1}{\sqrt{1-x^2}} \, dx = \int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx.
$$

Analogously, one sees that

$$
\int_{-1}^0 \frac{1}{\sqrt{1-x^2}} \, dx = \frac{\pi}{2},
$$

so that

$$
\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \, dx = \int_{-1}^0 \frac{1}{\sqrt{1-x^2}} \, dx + \int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx = \pi.
$$