1. Let 

\[ D := \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x^2 + y^2 \leq z \leq 1\} \]

Evaluate 

\[ \int_D \sqrt{x^2 + y^2}. \]

**Solution:** Passing to Cylindrical Coordinates, we see that 

\[ \int_D \sqrt{x^2 + y^2} = \int_{\{(r, \theta, z) : z \in [0, 1], r \in [0, \sqrt{z}], \theta \in [0, 2\pi]\}} \sqrt{r^2 (\cos \theta)^2 + r^2 (\sin \theta)^2} \, r \, dz = \int_0^1 \left( \int_0^{2\pi} \left( \int_0^{\sqrt{z}} \sqrt{r^2 (\cos \theta)^2 + r^2 (\sin \theta)^2} \, dr \right) \, d\theta \right) \, dz = \int_0^1 \left( \int_0^{2\pi} \frac{z^2}{3} \, d\theta \right) \, dz = 2\pi \int_0^1 \frac{z^2}{3} \, dz = \frac{4\pi}{15}. \]

2. Determine and classify all stationary points of 

\[ f : (-\pi, \pi) \times (-3, 4) \to \mathbb{R}, \quad (x, y) \mapsto (3 + 2 \cos x) \cos y. \]

If \( f \) attains a local minimum or maximum at one of its stationary points, evaluate it there.

**Solution:** The first order partial derivatives of \( f \) are computed as 

\[ \frac{\partial f}{\partial x} = -2(\sin x) \cos y \quad \text{and} \quad \frac{\partial f}{\partial y} = -(3 + 2 \cos x) \sin y. \]

Since \( 3 + 2 \cos x \neq 0 \) for all \( x \in \mathbb{R} \), a necessary and sufficient condition for \( \frac{\partial f}{\partial y}(x, y) = 0 \) is that \( \sin y = 0 \), i.e., \( y \in \pi \mathbb{Z} \). Since \( y \in (-3, 4) \), this means that \( y \in \{0, \pi\} \). Since \( \cos y \neq 0 \) for those \( y \), we require that \( \sin x = 0 \) in order to have \( \frac{\partial f}{\partial x}(x, y) = 0 \), i.e., \( x = 0 \) (because \( x \in \pi \mathbb{Z} \cap (-\pi, \pi) \)).

Hence, \((0, 0)\) and \((0, \pi)\) are the only stationary points of \( f \).
The next step is to compute Hess $f$. We have
\[
\frac{\partial^2 f}{\partial x^2} = -2(\cos x) \cos y, \quad \frac{\partial^2 f}{\partial y^2} = -(3 + 2 \cos x) \cos y,
\]
and
\[
\frac{\partial^2 f}{\partial x \partial y} = 2(\sin x)(\sin y),
\]
so that
\[
(Hess f)(0, 0) = \begin{bmatrix} -2 & 0 \\ 0 & -5 \end{bmatrix}
\]
and
\[
(Hess f)(0, \pi) = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}.
\]
Hence, $(Hess f)(0, 0)$ is negative definite, so that $f$ attains a local maximum at $(0, 0)$, namely $5$, whereas $(Hess f)(0, \pi)$ is positive definite, so that $f$ attains a local minimum at $(0, \pi)$, namely $-5$.

3. Let $I \subset \mathbb{R}^N$ and $J \subset \mathbb{R}^M$ be compact intervals, let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be continuous, and define
\[
f \otimes g : I \times J \to \mathbb{R}, \quad (x, y) \mapsto f(x)g(y).
\]
Then $f \otimes g$ is continuous and thus Riemann integrable. Show that
\[
\int_{I \times J} f \otimes g = \left( \int_I f \right) \left( \int_J g \right).
\]
Solution: By Fubini’s Theorem, we have
\[
\int_{I \times J} f \otimes g = \int_I \left( \int_J f(x)g(y) \, d\mu_M(y) \right) \, d\mu_N(x)
= \int_I \left( f(x) \int_J g(y) \, d\mu_M(y) \right) \, d\mu_N(x)
= \left( \int_I f \right) \left( \int_J g \right),
\]
which proves the claim.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by
\[
f(x, y) := \begin{cases} \frac{x^3y^3 - 1}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0), \\ 0, & \text{otherwise}. \end{cases}
\]
Check—and justify— whether or not $f$ is (a) partially differentiable, (b) continuous, (c) totally differentiable, (d) continuously partially differentiable, and (e) Riemann integrable on $[-1, 1] \times [-1, 1]$.

Solution:
(a) Clearly, $f$ is partially differentiable at every point of $\mathbb{R}^2 \setminus \{(0,0)\}$. Since 
\[
\frac{f(h,0) - f(0,0)}{h} = 0 = \frac{f(0,h) - f(0,0)}{h}
\]
for $h \neq 0$, it is clear that $f$ is partially differentiable at $(0,0)$ as well.

(b) Since 
\[
f\left(\frac{1}{n}, \frac{1}{n}\right) = e^{\frac{1}{n^2}} - 1 \xrightarrow{\frac{1}{n} \to 0} \frac{1}{2} \neq 0,
\]
$f$ is not continuous at $(0,0)$.

(c) Since total differentiability implies continuity, $f$ is not totally differentiable.

(d) Since continuously partially differentiable functions are totally differentiable, $f$

is not continuously partially differentiable.

(e) Clearly, $f$ is discontinuous only at $(0,0)$. It is therefore sufficient to show

that $f$ is bounded on $[-1,1] \times [-1,1]$. First note that, since $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$,

there is $C \geq 0$ such that $|e^h - 1| \leq C|h|$ for all $h \in [-1,1]$. For $(x,y) \in

([-1,1] \times [-1,1]) \setminus \{(0,0)\}$, we obtain

\[
|f(x,y)| = \frac{|e^{xy} - 1|}{x^2 + y^2} \leq C \frac{|xy|}{x^2 + y^2} = C \frac{\sqrt{x^2 y^2}}{x^2 + y^2} \leq C \frac{\frac{1}{2} x^2 + \frac{1}{2} y^2}{x^2 + y^2},
\]

by the inequality between geometric and arithmetic mean,

\[
= \frac{C}{2}.
\]

Consequently, $f$ is Riemann integrable on $[-1,1] \times [-1,1]$.

5. Let $\emptyset \neq K \subset \mathbb{R}^N$ be compact, and let $f : K \to \mathbb{R}^M$ be injective and continuous. Show that the inverse map

$f^{-1} : f(K) \to K, \quad f(x) \mapsto x$

is also continuous.

Solution: Let $x \in K$, and let $(x_n)_{n=1}^\infty$ be a sequence in $K$ such that $\lim_{n \to \infty} f(x_n) = f(x)$. We need to show that $\lim_{n \to \infty} x_n = x$. Assume that this is not true. Then there is $\epsilon > 0$ and a subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that $\|x_{n_k} - x\| \geq \epsilon_0$ for all $k \in \mathbb{N}$. Since $K$ is compact, we may suppose that $(x_{n_k})_{k=1}^\infty$ converges to
some $x' \in K$. Since $f$ is continuous, this means that $\lim_{k \to \infty} f(x_n_k) = f(x')$. Since $\lim_{n \to \infty} f(x_n) = f(x)$, this implies that $f(x) = f(x')$, and the injectivity of $f$ yields $x = x'$, so that $\lim_{k \to \infty} x_n_k = x$. This, however, contradicts that $\|x_n_k - x\| \geq \varepsilon_0$ for all $k \in \mathbb{N}$.

6. Let $K \subset \mathbb{R}^2$ be the triangle with vertices $(0,0)$, $(1,3)$, and $(0,3)$. Evaluate the line integral

$$\int_{\partial K} x^2 y^2 \, dx + 4xy^3 \, dy$$

where $\partial K$ is oriented counterclockwise.

**Solution**: Note that

$$K = \{(x,y) \in \mathbb{R}^2 : x \in [0,1], y \in [3x,3]\}.$$

Green’s Theorem then yields

$$\int_{\partial K} x^2 y^2 \, dx + 4xy^3 \, dy = \int_K \frac{\partial}{\partial x} 4xy^3 - \frac{\partial}{\partial y} x^2 y^2$$

$$= \int_K 4y^3 - 2x^2 y$$

$$= \int_0^1 \left( \int_{3x}^3 4y^3 - 2x^2 y \, dy \right) \, dx$$

$$= \int_0^1 y^4 - x^2 y^3 \bigg|_{3x}^3 \, dx$$

$$= \int_0^1 81 - 9x^2 - 72x^4 \, dx$$

$$= 81x - 3x^3 - \frac{72x^5}{5} \bigg|_0^1$$

$$= \frac{318}{5}.$$

7. Decide whether or not the set

$$\{ t \left( \cos \left( t^{-1} \right), \sin \left( t^{-1} \right) \right) : t \in (0, \pi) \} \cup \{(0,0)\}$$

is (a) open, (b) closed, (c) compact, or (d) connected in $\mathbb{R}^2$. Justify your answers.

**Solution**: Define

$$f : [0, \pi] \to \mathbb{R}^2, \quad t \mapsto \begin{cases} t \left( \cos \left( t^{-1} \right), \sin \left( t^{-1} \right) \right), & t \in (0, \pi] \\ 0, & t = 0. \end{cases}$$

Since

$$\lim_{t \to 0^+} t \cos \left( t^{-1} \right) = \lim_{t \to 0^+} t \sin \left( t^{-1} \right) = 0,$$

...
it follows that $f$ is continuous. Since $[0, \pi]$ is connected and compact and

$$S := \{ t (\cos (t^{-1}), \sin (t^{-1})) : t \in (0, \pi] \} \cup \{(0,0)\} = f([0, \pi]),$$

it follows that $S$ is connected and compact and thus—in particular—closed. Since $\emptyset \neq S \neq \mathbb{R}^2$, it is clear that $S$ is not open.

8. Let

$$D := \{(e^{xy}, e^{-x^2-z}) : (x, y, z) \in \mathbb{R}^3\}$$

Show that there is $(u, v) \in D$ such that

$$\ln(u + v) = \sqrt{13}.$$

Solution: Since

$$\mathbb{R}^3 \to \mathbb{R}^2, \quad (x, y, z) \mapsto (e^{xy}, e^{-x^2-z})$$

is continuous and since $\mathbb{R}^3$ is connected, $D$ must be connected.

Let

$$f : D \to \mathbb{R}, \quad (u, v) \mapsto \ln(u + v).$$

Then $f$ is continuous. Clearly $(1, 1) = (e^0, e^0) \in D$ and

$$f(1, 1) = \ln 2 < \ln e = 1 < \sqrt{13}$$

Let $(x, y, z) \in \mathbb{R}^3$ such that $xy > \sqrt{13}$. Then $(e^{xy}, e^{-x^2-z}) \in D$ such that

$$f(e^{xy}, e^{-x^2-z}) = \ln(e^{xy} + e^{-x^2-z}) > \ln e^{xy} = xy > \sqrt{13}.$$

The Intermediate Value Theorem then yields $(u, v) \in D$ with the desired property.