MATH 216 (Fall 2021)
Introduction to Analysis

Final Model Solutions

1. Is the function

$$f\colon [0,1] \to \mathbb{R}, \quad x \mapsto \begin{cases} x^2, & x \in \mathbb{Q}, \\ 1, & x \notin \mathbb{Q}, \end{cases}$$

Riemann integrable? If so, evaluate its Riemann integral.

Solution: Let $\psi\colon [0,1] \to \mathbb{R}$ be a step function with $\psi \geq f$, and let $0 = x_0 < x_1 < \cdots < x_n = 1$ be such that ψ is constant on (x_{j-1}, x_j) for $j = 1, \ldots, n$. For each $j = 1, \ldots, n$, choose $\xi_j \in (x_{j-1}, x_j) \setminus \mathbb{Q}$. It follows that

$$\int_0^1 \psi(x)\, dx = \sum_{j=1}^n \psi(\xi_j)(x_j - x_{j-1}) \geq \sum_{j=1}^n x_j - x_{j-1} = 1$$

and therefore

$$\overline{\int_0^1} f(x)\, dx \geq 1.$$

Let $\phi : [0,1] \to \mathbb{R}$ be a step function with $\phi \leq f$, and let $0 = x_0 < x_1 < \cdots < x_n = 1$ be such that ϕ is constant on (x_{j-1}, x_j) for $j = 1, \ldots, n$. Note that, for all $j = 1, \ldots, n$ and $q \in (x_{j-1}, x_j) \cap \mathbb{Q}$, we have $\phi(q) \leq q^2$. As ϕ and $[0,1] \ni x \mapsto x^2$ are continuous on (x_{j-1}, x_j) for each $j = 1, \ldots, n$, this means that $\phi(x) \leq x^2$ for all $x \in \bigcup_{j=1}^n (x_{j-1}, x_j)$. Define

$$\tilde{\phi}\colon [0,1] \to \mathbb{R}, \quad \begin{cases} x^2, & x \in \{x_0, x_1, \ldots, x_n\}, \\ \phi(x), & x \in \bigcup_{j=1}^n (x_{j-1}, x_j). \end{cases}$$

It follows that $\tilde{\phi}$ is a step function with $\tilde{\phi}(x) \leq x^2$ for all $x \in [0,1]$ and $\int_0^1 \tilde{\phi}(x)\, dx = \int_0^1 \phi(x)\, dx$. We thus obtain

$$\int_0^1 \phi(x)\, dx \leq \int_0^1 x^2 = \frac{1}{3}$$

and therefore

$$\underline{\int_0^1} f(x) \leq \frac{1}{3}.$$

Consequently, f is not Riemann integrable.

2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be such that there is $C \geq 0$ with

$$|f(x) - f(y)| \leq C(x-y)^2$$

for all $x, y \in \mathbb{R}$. Show that f is constant.

Solution: It is clear that

$$\left|\frac{f(x)-f(y)}{x-y}\right| \leq C|x-y|$$

for all $x, y \in \mathbb{R}$ with $x \neq y$. It follows that

$$\lim_{\substack{x\to x_0\\ x\neq x_0}}\left|\frac{f(x)-f(x_0)}{x-x_0}\right| \leq \lim_{\substack{x\to x_0\\ x\neq x_0}} C|x-x_0| = 0$$

for all $x_0 \in \mathbb{R}$, so that $f' \equiv 0$. This means that f has to be constant.

3. Determine whether or not the following series converge, converge absolutely, or diverge:

   (a) $\displaystyle\sum_{n=1}^{\infty} \binom{2n}{n}\Big/\binom{3n}{n}$;

   (b) $\displaystyle\sum_{\nu=1}^{\infty} \frac{(-1)^{\nu}+2}{(-1)^{\nu-1}\nu}$;

   (c) $\sum_{m=1}^{\infty} \frac{\sin\left(\frac{\pi}{2}+m\pi\right)}{\sqrt{m}}$.

   Solution:

   (a) For $n \in \mathbb{N}$, set

   $$a_n := \binom{2n}{n}\Big/\binom{3n}{n} = \frac{(2n)!}{(n!)^2}\frac{n!(2n)!}{(3n)!} = \frac{(2n)!}{n!}\frac{(2n)!}{(3n)!}$$

   It follows that

   $$\begin{aligned}\frac{|a_{n+1}|}{|a_n|} &= \frac{(2n+2)!}{(n+1)!}\frac{(2n+2)!}{(3n+3)!}\frac{n!}{(2n)!}\frac{(3n)!}{(2n)!}\\ &= \frac{(2n+2)(2n+1)}{n+1}\frac{(2n+2)(2n+1)}{(3n+3)(3n+2)(3n+1)} \to \frac{16}{27} < 1,\end{aligned}$$

   so that $\sum_{n=1}^{\infty} a_n$ converges absolutely by the Limit Ratio Test.

   (b) For $\nu \in \mathbb{N}$, set $a_\nu := \frac{(-1)^\nu+2}{(-1)^{\nu-1}\nu}$. It follows that

   $$a_\nu = -\frac{1}{\nu} + (-1)^{\nu-1}\frac{2}{\nu}$$

   for all $\nu \in \mathbb{N}$. We claim that $\sum_{\nu=1}^{\infty} a_\nu$ diverges. Assume towards a contradiction that $\sum_{\nu=1}^{\infty} a_\nu$ converges. By the Alternating Series Test, $\sum_{\nu=1}^{\infty}(-1)^{\nu-1}\frac{2}{\nu}$ converges. As

   $$a_\nu - (-1)^{\nu-1}\frac{2}{\nu} = -\frac{1}{\nu}$$

   for $\nu \in \mathbb{N}$, this would mean that $\sum_{\nu=1}^{\infty} \frac{1}{\nu}$ converges, which is clearly false.

(c) First, note that

$$\sin\left(\frac{\pi}{2} + m\pi\right) = (-1)^m$$

for all $m \in \mathbb{N}$. It follows from the Alternating Series Test that $\sum_{m=1}^{\infty} \frac{\sin\left(\frac{\pi}{2}+m\pi\right)}{\sqrt{m}}$ converges. As $\sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} = \infty$, it is clear that the series does not converge absolutely.

4. Let $f, g \colon \mathbb{R} \to \mathbb{R}$ be differentiable functions such that

$$f(0) = 0, \qquad g(0) = 1, \qquad f' = g, \qquad \text{and} \qquad g' = -f.$$

Show that

$$f(x) = \sin x \qquad \text{and} \qquad g(x) = \cos x$$

for $x \in \mathbb{R}$. (*Hint*: Consider the function

$$h \colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto (f(x) - \sin x)^2 + (g(x) - \cos x)^2,$$

and differentiate it.)

Solution: Let h be as in the hint. Then h is clearly differentiable. Differentiating, we obtain

$$\begin{aligned} h'(x) &= 2(f(x) - \sin x)(f'(x) - \cos x) + 2(g(x) - \cos x)(g'(x) + \sin x) \\ &= 2(f(x) - \sin x)(g(x) - \cos x) + 2(g(x) - \cos x)(-f(x) + \sin x) \\ &= 2(f(x) - \sin x)(g(x) - \cos x) - 2(g(x) - \cos x)(f(x) - \sin x) \\ &= 0 \end{aligned}$$

for $x \in \mathbb{R}$. It follows that h is constant, and as $h(0) = 0$, we obtain that $h(x) = 0$ and, consequently,

$$f(x) = \sin x \qquad \text{and} \qquad g(x) = \cos x$$

for all $x \in \mathbb{R}$.

5. Show that

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

for all $n \in \mathbb{N}$.

Solution: Use induction on n.

$n = 1$ (*induction anchor*): In this case

$$\sum_{k=1}^{1} k^3 = 1 = \frac{1^2 \cdot 2^2}{4}$$

$n \rightsquigarrow n+1$ (*induction step*): Let $n \in \mathbb{N}$ be such that

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

holds (*induction hypothesis*). Then

$$\begin{aligned}
\sum_{k=1}^{n+1} k^3 &= \sum_{k=1}^{n} k^3 + (n+1)^3 \\
&= \frac{n^2(n+1)^2}{4} + (n+1)^3, \qquad \text{by the induction hypothesis,} \\
&= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} \\
&= \frac{(n+1)^2(n^2+4n+4)}{4} \\
&= \frac{(n+1)^2(n+2)^2}{4},
\end{aligned}$$

which proves the claim.

6. Let $a < b$, and let $f, g\colon [a,b] \to \mathbb{R}$ be continuous and differentiable on (a,b). Show that there is $\xi \in (a,b)$ such that

$$f'(\xi)(g(b) - g(a)) = g'(\xi)(f(b) - f(a)).$$

(*Hint*: Apply Rolle's Theorem to a suitable auxiliary function.)

Solution: Define

$$h\colon [a,b] \to \mathbb{R}, \quad x \mapsto f(x)(g(b) - g(a)) - g(x)(f(b) - f(a)).$$

Then, clearly, h is continuous and differentiable on (a,b). It follows that

$$h(a) = f(a)(g(b) - g(a)) - g(a)(f(b) - f(a)) = f(a)g(b) - g(a)f(b)$$

and

$$h(b) = f(b)(g(b) - g(a)) - g(b)(f(b) - f(a)) = -f(b)g(a) + g(b)f(a) = h(a).$$

By Rolle's Theorem, there is $\xi \in (a,b)$ such that

$$0 = h'(\xi) = f'(\xi)(g(b) - g(a)) - g'(\xi)(f(b) - f(a)).$$

7. Let $f : [0,2] \to \mathbb{R}$ be continuous such that $f(0) = f(2)$. Show that there are $x, y \in [0,2]$ with $|x - y| = 1$ and $f(x) = f(y)$. (*Hint*: Apply the Intermediate Value Theorem to the auxiliary function $[0,1] \ni x \mapsto f(x+1) - f(x)$.)

Solution: Define

$$g\colon [0,1] \to \mathbb{R}, \quad x \mapsto f(x+1) - f(x).$$

Then g is continuous with

$$g(0) = f(1) - f(0) = f(1) - f(2) = -(f(2) - f(1)) = -g(1).$$

This means that $g(0)$ and $g(1)$ have opposite signs, so that, by the Intermediate Value Theorem, there is $\xi \in [0,1]$ with $g(\xi) = 0$. Obviously, $x = \xi + 1$ and $y = \xi$ have the required properties.

8. Let $a < b$, let $f\colon [a,b] \to \mathbb{R}$ be continuously differentiable, and let

$$F\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto \int_a^b f(t)\sin(xt)\,dt.$$

Show that $\lim_{x\to\infty} F(x) = \lim_{x\to-\infty} F(x) = 0$. (*Hint*: Integration by Parts.)

Solution: Fix $x \in \mathbb{R} \setminus \{0\}$, and let

$$g\colon [a,b] \to \mathbb{R}, \quad t \mapsto -\frac{\cos(xt)}{x},$$

so that

$$g'(t) = \sin(xt)$$

for $t \in [a,b]$. Integration by Parts then yields

$$F(x) = \int_a^b f(t)g'(t) = -f(t)\frac{\cos(xt)}{x}\bigg|_{t=a}^{t=b} + \frac{1}{x}\int_a^b f'(t)\cos(xt)\,dt,$$

so that

$$(*) \qquad |F(x)| \le \frac{|f(b)|}{|x|} + \frac{|f(a)|}{|x|} + \frac{1}{|x|}\int_a^b |f'(t)|\,dt.$$

If $x \to \infty$ or $x \to -\infty$, then the right hand side of $(*)$ tends to zero as, consequently, does the left hand side.