MATH 216 (Fall 2021)
Introduction to Analysis

Final Practice Problem

1. Let $a < b$, let $f : [a, b] \to \mathbb{R}$ be Riemann integrable, and let $g : [a, b] \to \mathbb{R}$ be such that $\{x \in [a, b] : f(x) \neq g(x)\}$ is finite. Show that g is Riemann integrable and $\int_a^b g(x)\,dx = \int_a^b f(x)\,dx$. Proceed as follows:

   - first suppose, that f is a step function and show that g is also a step function and that $\int_a^b g(x)\,dx = \int_a^b f(x)\,dx$;
   - the use the definition of the Riemann integral to prove the general case.

   Solution: Suppose that f is a step function. Let $a = x_0 < x_1 < \cdots < x_n = b$ be such that f is constant on (x_{j-1}, x_j) for $j = 1, \ldots, n$. Choose $a = y_0 < y_1 < \cdots < y_m = b$ such that

$$\{y_0, y_1, \ldots, y_m\} \supset \{x_0, x_1, \ldots, x_n\} \cup \{x \in [a, b] : f(x) \neq g(x)\}.$$

   Then both f and g are constant—and equal—on (y_{j-1}, y_j) for $j = 1, \ldots, m$; in particular, g is also a step function. Let $\xi_j \in (y_{j-1}, y_j)$ for $j = 1, \ldots, m$. We obtain

$$\int_a^b f(x)\,dx = \sum_{j=1}^m f(\xi_j)(y_j - y_{j-1}) = \sum_{j=1}^m g(\xi_j)(y_j - y_{j-1}) = \int_a^b g(x)\,dx.$$

   Now, let f be a general Riemann integrable function. Let ϕ be a step function with $\phi \leq f$. Define

$$\tilde{\phi}\colon [a, b] \to \mathbb{R}, \quad x \mapsto \begin{cases} \phi(x), & g(x) = f(x), \\ g(x), & f(x) \neq g(x) \end{cases}$$

   Then $\tilde{\phi} \leq g$ such that $\left\{x \in [a, b] : \phi(x) \neq \tilde{\phi}(x)\right\}$ is finite. By the foregoing, $\tilde{\phi}$ is a step function such that $\int_a^b \tilde{\phi}(x)\,dx = \int_a^b \phi(x)\,dx$. It follows that

$$\begin{aligned}\underline{\int_a^b} f(x)\,dx &= \sup\left\{\int_a^b \phi(x)\,dx : \phi\colon [a, b] \to \mathbb{R} \text{ is a step function with } \phi \leq f\right\} \\ &\leq \sup\left\{\int_a^b \phi(x)\,dx : \phi\colon [a, b] \to \mathbb{R} \text{ is a step function with } \phi \leq g\right\} \\ &= \underline{\int_a^b} g(x)\,dx\end{aligned}$$

   Interchanging the rôles of f and g in this argument, we obtain that $\underline{\int_a^b} g(x)\,dx \leq \underline{\int_a^b} f(x)\,dx$ as well, so that, in fact, $\underline{\int_a^b} g(x)\,dx = \underline{\int_a^b} f(x)\,dx$. Similarly, we see that

$\overline{\int_a^b} g(x)\,dx = \overline{\int_a^b} f(x)\,dx$. As f is Riemann integrable, we obtain

$$\underline{\int_a^b} g(x)\,dx = \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \overline{\int_a^b} g(x)\,dx,$$

so that g is Riemann integrable and $\int_a^b g(x)\,dx = \int_a^b f(x)\,dx$.

2. Show that

$$\sum_{k=1}^{n} k(k!) = (n+1)! - 1$$

for all $n \in \mathbb{N}$.

Solution: We use induction on n.

$n = 1$: In this case, both the left and the right hand side of the equation equal 1.

$n \rightsquigarrow n+1$: Let $n \in \mathbb{N}$ be such that

$$\sum_{k=1}^{n} k(k!) = (n+1)! - 1$$

(*induction hypothesis*). It then follows that

$$\begin{aligned}
\sum_{k=1}^{n+1} k(k!) &= \sum_{k=1}^{n} k(k!) + (n+1)(n+1)! \\
&= (n+1)! - 1 + (n+1)(n+1)!, \\
&\qquad\qquad \text{by the induction hypothesis,} \\
&= (n+1)!(1+n+1) - 1 \\
&= (n+1)!(n+2) - 1 \\
&= (n+2)! - 1
\end{aligned}$$

which completes the proof.

3. Let $\theta > 0$, and define the sequence $(x_n)_{n=1}^{\infty}$ inductively through

$$x_1 := \sqrt{\theta} \qquad \text{and} \qquad x_{n+1} = \sqrt{\theta + x_n} \quad \text{for } n \in \mathbb{N}.$$

Show that $(x_n)_{n=1}^{\infty}$ increases and is bounded above by $1 + \sqrt{\theta}$ (and therefore converges), and compute its limit.

Solution: We use induction to prove that

$$x_n \le x_{n+1} \le 1 + \sqrt{\theta}$$

for all $n \in \mathbb{N}$.

$n = 1$: Clearly,

$$x_1 = \sqrt{\theta} \leq \underbrace{\sqrt{\theta + \sqrt{\theta}}}_{=x_2} = \sqrt{\sqrt{\theta}\left(1+\sqrt{\theta}\right)} \leq \sqrt{\left(1+\sqrt{\theta}\right)\left(1+\sqrt{\theta}\right)} = 1 + \sqrt{\theta}$$

holds.

$n \rightsquigarrow n+1$: Let $n \in \mathbb{N}$ be such that

$$x_n \leq x_{n+1} \leq 1 + \sqrt{\theta}$$

is true. It follows that

$$\begin{aligned} x_{n+1} = \sqrt{\theta + x_n} &\leq \underbrace{\sqrt{\theta + x_{n+1}}}_{=x_{n+2}} \\ &\leq \sqrt{\theta + 1 + \sqrt{\theta}} \leq \sqrt{\theta + 2\sqrt{\theta} + 1} = \sqrt{\left(1+\sqrt{\theta}\right)^2} = 1 + \sqrt{\theta}. \end{aligned}$$

Let x denote the limit of $(x_n)_{n=1}^\infty$. It then follows that

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{\theta + x_n} = \sqrt{\theta + x},$$

so that $x^2 = x + \theta$. Solving the quadratic equation for x yields $x = \frac{1}{2} \pm \sqrt{\theta + \frac{1}{4}}$. As, $\frac{1}{2} - \sqrt{\theta + \frac{1}{4}} < 0 \leq x$, it follows that $x = \frac{1}{2} + \sqrt{\theta + \frac{1}{4}}$.

4. Is the function

$$f\colon [0,1] \to \mathbb{R}, \quad x \mapsto \begin{cases} x, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}, \end{cases}$$

Riemann integrable? If so, evaluate its integral.

Solution: Let $0 = x_0 < x_1 < \cdots < x_n = 1$ and let $\phi, \psi\colon [0,1] \to \mathbb{R}$ be such that ϕ and ψ are constant on (x_{j-1}, x_j) for $j = 1, \ldots, n$ and $\phi \leq f \leq \psi$.

For each $j = 1, \ldots, n$, there is $r_j \in (x_{j-1}, x_j) \setminus \mathbb{Q}$. As $f(r_j) = 0 \geq \phi(r_j)$ and since ϕ is constant on (x_{j-1}, x_j), this means that $\phi(x) \leq 0$ and therefore

$$\int_0^1 \phi(x)\, dx = \sum_{j=1}^n \phi(r_j)(x_j - x_{j-1}) \leq 0.$$

Since $f \geq 0$, this implies that $\int_{*\,0}^1 f(x)\, dx = 0$.

On the other hand, we have $\psi(q) \geq f(q) = q$ for all $x \in [0,1] \cap \mathbb{Q}$ and therefore

$$\psi(x) \geq \sup\{q : q \in (x_{j-1}, x_j) \cap \mathbb{Q}\} = x_j$$

for all $j = 1, \dots, n$ and $x \in (x_{j-1}, x_j)$. It follows that

$$\int_0^1 \psi(x)\,dx \geq \sum_{j=1}^n x_j(x_{j-1} - x_j) > \sum_{j=1}^n \frac{x_{j-1} + x_j}{2}(x_{j-1} - x_j) = \frac{1}{2}\sum_{j=1}^n (x_{j-1}^2 - x_j^2) = \frac{1}{2}$$

and thus $\overset{*}{\int_0^1} f(x)\,dx \geq \frac{1}{2}$.

Consequently, f is not Riemann integrable.

5. Determine whether or not the following series converge, converge absolutely, or diverge:

   (a) $\displaystyle\sum_{k=1}^\infty \frac{k!}{(2k)!}$;

   (b) $\displaystyle\sum_{n=1}^\infty \frac{\cos((n-1)\pi) + \sqrt{n}}{n}$;

   (c) $\displaystyle\sum_{\nu=2}^\infty \frac{(-1)^{\nu-1}}{\nu(\log\nu)^p}$ where $p > 0$.

   (*Hint for* (c): The answer may depend on the value of p.)

   Solution:

   (a) For $k \in \mathbb{N}$, set $a_k := \frac{k!}{(2k)!}$. It follows that

   $$\frac{a_{k+1}}{a_k} = \frac{(k+1)!}{(2k+2)!}\,\frac{(2k)!}{k!} = \frac{k+1}{(2k+1)(2k+2)} \to 0,$$

   so that $\sum_{k=1}^\infty a_k$ converges absolutely by the Limit Ratio Test.

   (b) For $n \in \mathbb{N}$, set $a_n := \frac{\cos((n-1)\pi)+\sqrt{n}}{n}$, and note that

   $$a_n = \frac{(-1)^{n-1}}{n} + \frac{1}{\sqrt{n}}.$$

   By the Alternating Series Test, $\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n}$ converges. So, the convergence of $\sum_{n=1}^\infty a_n$ would imply the convergence of $\sum_{n=1}^\infty \frac{1}{\sqrt{n}}$, which diverges. Therefore, $\sum_{n=1}^\infty a_n$ must diverge.

   (c) For $\nu \in \mathbb{N}$, $\nu \geq 2$, set $a_\nu := \frac{1}{\nu(\log\nu)^p}$. It is clear that $(a_\nu)_{\nu=2}^\infty$ is a decreasing sequence converging to zero. The Alternating Series Test therefore yields the convergence of $\sum_{\nu=2}^\infty (-1)^{\nu-1} a_\nu$.

   The answer to the question of whether or not $\sum_{\nu=2}^\infty (-1)^{\nu-1} a_\nu$ converges absolutely, i.e., if $\sum_{\nu=2}^\infty a_\nu < \infty$ or $\sum_{\nu=2}^\infty a_\nu = \infty$, depends on the value of p.

We use Cauchy's Compression Theorem to tackle this question. For $\nu \in \mathbb{N}$, we have

$$2^\nu a_{2^\nu} = \frac{2^\nu}{2^\nu (\log 2^\nu)^p} = \frac{1}{(\log 2^\nu)^p} = \frac{1}{(\log 2)^p \nu^p}.$$

As

$$\sum_{\nu=1}^{\infty} \frac{1}{(\log 2)^p \nu^p} < \infty \quad \Longleftrightarrow \quad p > 1,$$

it follows that $\sum_{\nu=2}^{\infty} (-1)^{\nu-1} a_\nu$ converges absolutely if and only if $p > 1$.

6. Let $r \in \mathbb{R}$, and let $f\colon (0,\infty) \to \mathbb{R}$ be a differentiable function such that $f(1) = 1$ and

$$x\, f'(x) = r\, f(x)$$

all $x > 0$. Show that

$$f(x) = x^r$$

for $x > 0$. (*Hint*: Consider the function

$$g\colon (0,\infty) \to \mathbb{R}, \quad x \mapsto \frac{f(x)}{x^r}$$

and differentiate it. What do you notice?)

Solution: Let g be as in the hint. Then g is clearly differentiable. Differentiating, we obtain

$$\begin{aligned} g'(x) &= \frac{f'(x)x^r - f(x)rx^{r-1}}{x^{2r}} \\ &= \frac{f'(x)x^r - xf'(x)x^{r-1}}{x^{2r}} \\ &= \frac{f'(x)x^r - f'(x)x^r}{x^{2r}} \\ &= 0 \end{aligned}$$

for $x > 0$. It follows that g is constant, and as $h(1) = 1$, we obtain that $g(x) = 1$ and, consequently,

$$f(x) = x^r$$

for all $x > 0$.

7. Let $a < b$, and let $f, g\colon [a,b] \to \mathbb{R}$ be continuous and differentiable on (a,b) such that $f(a) = f(b) = 0$. Show that there is $\xi \in (a,b)$ such that

$$f'(\xi) + f(\xi)g'(\xi) = 0.$$

(*Hint*: Apply Rolle's Theorem to the function $[a,b] \ni x \mapsto f(x)\exp(g(x))$.)

Solution: Let the function in the hint be denoted by h. Then, clearly, h is continuous and differentiable on (a,b) and satisfies $h(a) = h(b) = 0$. By Rolle's Theorem, there is $\xi \in (a,b)$ such that

$$0 = h'(\xi) = f'(\xi)\exp(g(\xi)) + f(\xi)g'(\xi)\exp(g(\xi)) = (f'(\xi) + f(\xi)g'(\xi))\exp(g(\xi)).$$

As $\exp(g(\xi)) \neq 0$, this means that $f'(\xi) + f(\xi)g'(\xi) = 0$.

8. Let $a < b$, and let $f, g \colon [a,b] \to \mathbb{R}$ be continuously differentiable such that $f(a) \leq g(a)$ and $f' \leq g'$. Show that $f \leq g$.

   Solution: For $x \in [a,b]$, define

$$\tilde{f}(x) := \int_a^x f'(t)\,dt \qquad \text{and} \qquad \tilde{g}(x) := \int_a^x g'(t)\,dt.$$

   Then $\tilde{f}$ and $\tilde{g}$ are antiderivatives of f' and g', respectively, and since $f' \leq g'$, we have $\tilde{f} \leq \tilde{g}$. As $\tilde{f}(a) = \tilde{g}(a) = 0$, it follows that

$$f = \tilde{f} + f(a) \leq \tilde{g} + g(a) = g.$$

9. Define $f \colon \mathbb{R} \to \mathbb{R}$ by letting

$$f(x) := \begin{cases} 0, & x \notin \mathbb{Q}, \\ \frac{1}{q}, & x = \frac{p}{q} \neq 0 \text{ with } p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \text{ coprime}, \\ 1, & x = 0. \end{cases}$$

   Show that f is discontinuous at every $x \in \mathbb{Q}$, but continuous at every $x \in \mathbb{R} \setminus \mathbb{Q}$.

   Solution: Let $x \in \mathbb{Q}$, so that $f(x) \neq 0$. For each $n \in \mathbb{N}$, there is $r_n \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < r_n < x + \frac{1}{n}$. It follows that $\lim_{n\to\infty} r_n = x$, but $\lim_{n\to\infty} f(r_n) = 0 \neq f(x)$.

   Let $x \in \mathbb{R} \setminus \mathbb{Q}$, and let $\epsilon > 0$. We need to find $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $x \in \mathbb{R}$ with $|x - y| < \delta$. There are only finitely many $q \in \mathbb{N}$ with $\frac{1}{q} \geq \epsilon$. Consequently,

$$S := \left\{ \frac{p}{q} : p \in \mathbb{Z} \setminus \{0\} \text{ and } q \in \mathbb{N} \text{ coprime}, \ \frac{1}{q} \geq \epsilon, \ \left| x - \frac{p}{q} \right| < |x| \right\} \cup \{0\}$$

   is finite. Set

$$\delta := \min\{|x - s| : s \in S\},$$

   so that $\delta > 0$. Let $y \in \mathbb{R}$ be such that $|x - y| < \delta$. If $y \notin \mathbb{Q}$, then $f(y) = 0$, so that trivially $|f(x) - f(y)| = 0 < \epsilon$. Suppose that $y \in \mathbb{Q}$. It follows from the definition of δ that $y \notin S$; in particular, $y \neq 0$. Let $p \in \mathbb{Z} \setminus \{0\}$ and $q \in \mathbb{N}$ by coprime such that $y = \frac{p}{q}$. As $y \notin S$, this means that

$$\left| x - \frac{p}{q} \right| \geq |x| \qquad \text{or} \qquad \frac{1}{q} < \epsilon.$$

By the definition of δ, it is clear that $|x| \geq \delta$, and since $|x - y| < \delta$, the first case cannot occur. It follows that $f(y) = \frac{1}{q} < \epsilon$, which proves the continuity of f at x.

10. Let $a < b$, and let f be the function defined in the previous problem. Show that f is Riemann integrable on $[a, b]$ with $\int_a^b f(x)\,dx = 0$.

    Solution: Let $\epsilon > 0$, and set

$$S_\epsilon := \left\{ \frac{p}{q} : p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \text{ coprime}, \ \frac{1}{q} \geq \frac{\epsilon}{2(b-a)}, \ \frac{p}{q} \in [a, b] \right\}$$

    As in the solution to the previous problem, we see that S_ϵ is finite. From the definition of f, it follows that

$$S_\epsilon = \left\{ x \in [a, b] : f(x) \geq \frac{\epsilon}{2(b-a)} \right\}.$$

    Define step functions $\phi, \psi \colon [a, b] \to \mathbb{R}$ by setting $\phi :\equiv 0$ and defining

$$\psi(x) := \begin{cases} f(x), & x \in S_\epsilon, \\ \frac{\epsilon}{2(b-a)}, & \text{otherwise.} \end{cases}$$

    It is obvious that $\phi \leq f \leq \psi$ and that

$$\int_a^b \psi(x)\,dx - \int_a^b \phi(x)\,dx = \int_a^b \psi(x)\,dx = \frac{\epsilon}{2(b-a)}(b-a) = \frac{\epsilon}{2} < \epsilon.$$

    As $\epsilon > 0$ was arbitrary, this means that f is Riemann integrable.

    Finally, note that

$$0 \leq \int_a^b f(x)\,dx \leq \int_a^b \psi(x)\,dx < \epsilon.$$

    Again, as $\epsilon > 0$ was arbitrary, this means that $\int_a^b f(x)\,dx = 0$.

11. Let $a < b$, and let $f, g : [a, b] \to \mathbb{R}$ be continuous such that $f(a) \leq g(a)$ and $f(b) \geq g(b)$. Show that there is $x_0 \in [a, b]$ such that $f(x_0) = g(x_0)$.

    Solution: Define

$$h\colon [a, b] \to \mathbb{R}, \quad x \mapsto f(x) - g(x).$$

    Then h is continuous with $h(a) \leq 0$ and $h(b) \geq 0$. The Intermediate Value Theorem yields $x_0 \in [a, b]$ such that $h(x_0) = 0$, i.e., $f(x_0) = g(x_0)$.

12. Let $a < b$, and let $f : [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b). Show that:

    (a) if $f'(x) \geq 0$ for all $x \in (a, b)$, then f is increasing, and if $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing;

(b) if f is increasing, then $f'(x) \geq 0$ for all $x \in (a, b)$.

Give an example of a strictly increasing function f that is differentiable on (a, b) such that there is $\xi \in (a, b)$ with $f'(\xi) = 0$. (*Hint for* (a): Mean Value Theorem.)

Solution:

(a) Let $x, y \in [a, b]$ be such that $x < y$. By the Mean Value Theorem, there is $\xi \in (x, y)$ such that

$$\frac{f(x) - f(y)}{x - y} = f'(\xi) \geq 0,$$

so that $f(x) - f(y) = f'(\xi)(x - y) \leq 0$, i.e., $f(x) \leq f(y)$. If $f'(x) > 0$ for all $x \in (a, b)$, then the same argument shows that $f(x) < f(y)$ for all $x, y \in [a, b]$ with $x < y$.

(b) Let $x \in (a, b)$, and let $h > 0$ be such that $x + h \in [a, b]$. It follows that

$$\frac{f(x + h) - f(x)}{h} \geq 0.$$

Therefore, we have

$$f'(x) = \lim_{\substack{h \to 0 \\ h \neq 0 \\ x+h \in [a,b]}} \frac{f(x + h) - f(x)}{h} = \lim_{\substack{h \to 0 \\ h > 0 \\ x+h \in [a,b]}} \frac{f(x + h) - f(x)}{h} \geq 0.$$

For the example, let $a = -1$, $b = 1$, and define

$$f \colon [-1, 1] \to \mathbb{R}, \quad x \mapsto x^3.$$

Clearly, f is strictly increasing, but $f'(0) = 0$.