1. Is there \( f \in PC_{2\pi}(\mathbb{R}) \) such that
\[
f(x) \sim \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \sin(nx)\
\]
2. Let \( f \in PC_{2\pi}(\mathbb{R}) \) be given by
\[
f : (-\pi, \pi] \to \mathbb{R}, \quad x \mapsto x.
\]
Determine the Fourier series of \( f \) and conclude that
\[
\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) = x
\]
for \( x \in (-\pi, \pi) \).
3. Let \( f \in PC_{2\pi}(\mathbb{R}) \) be given by
\[
f : (-\pi, \pi] \to \mathbb{R}, \quad x \mapsto |x|.
\]
Determine the Fourier series of \( f \) and argue that it converges to \( f \) uniformly on \( \mathbb{R} \).
Conclude that
\[
\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.
\]
4. Let \( f, F \in PC_{2\pi}(\mathbb{R}) \), let \( a_0, a_1, a_2, \ldots, b_1, b_2, \ldots \) be the Fourier coefficients of \( f \), and let \( A_0, A_1, A_2, \ldots, B_1, B_2, \ldots \) be the Fourier coefficients of \( F \). Show that
\[
\frac{1}{\pi} \int_{-\pi}^{\pi} f(t)F(t) \, dt = \frac{a_0 A_0}{2} + \sum_{n=1}^{\infty} (a_n A_n + b_n B_n).
\]
(Hint: Apply Parseval’s Identity to \( f + F \).)
5. Although the theory of power series was developed in class only for real variables, it all works perfectly well over \( \mathbb{C} \) as well. We can thus extend \( \exp, \sin, \) and \( \cos \) to \( \mathbb{C} \) by defining
\[
e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad \sin z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \quad \cos z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}
\]
for \( z \in \mathbb{C} \).
(a) Show that
\[ e^{iz} = \cos z + i \sin z \]
holds for all \( z \in \mathbb{C} \), and derive Euler’s Identity: \( e^{i\pi} + 1 = 0 \).

(b) For \( n \in \mathbb{Z} \), the \( n \)-th complex Fourier coefficient of \( f \in \mathcal{PC}_{2\pi}(\mathbb{R}) \) is defined as
\[ c_n := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-int} \, dt. \]
Express the complex Fourier coefficients of \( f \) in terms of its real Fourier coefficients \( a_0, a_1, a_2, \ldots \) and \( b_1, b_2, \ldots \).

6* Let \( a < b \), let \( f: [a, b] \to \mathbb{R} \) be continuous, let \( a \leq t_1 < t_2 < \cdots < t_n \leq b \), and let \( \epsilon > 0 \). Show that there is a polynomial \( p \) with \( |f(t) - p(t)| < \epsilon \) for \( t \in [a, b] \) and \( p(t_j) = f(t_j) \) for \( j = 1, \ldots, n \). (Hint: First, treat the case where \( f(t_1) = \cdots = f(t_n) = 0 \), then apply this to the auxiliary function \( [a, b] \ni t \mapsto f(t) - \sum_{k=1}^{n} f(t_k) \prod_{j=1}^{n} \frac{t-t_j}{t_k-t_j} \).)

Due Monday, April 8, 2019, at 10:00 a.m.; no late assignments.