1. Let $\Phi$ be a surface in $\mathbb{R}^3$ with parameter domain $K \subset \mathbb{R}^2$, let $\gamma : [a, b] \to K$ be a $C^1$-curve, and let $\alpha := \Phi \circ \gamma$. Show that $\alpha'(t)$ is orthogonal to $N(\gamma(t))$ for each $t \in [a, b]$.

Interpret this in geometric terms.

2. Let $\Phi$ and $\Psi$ be $C^2$-surfaces with parameter domain $K$, which is a normal region, such that $\Phi|_{\partial K} = \Psi|_{\partial K}$, and let $f : V \to \mathbb{R}^3$ be continuously differentiable where $V \subset \mathbb{R}^3$ is open and contains $\{\Phi\} \cup \{\Psi\}$. Show that

$$\int_{\Phi} \text{curl} f \cdot n \, d\sigma = \int_{\Psi} \text{curl} f \cdot n \, d\sigma.$$

3. Let $S$ be the surface of the ball centered at $(0,0,0)$ with radius $r > 0$. Compute

$$\int_{S} x^3 \, dy \wedge dz + y^3 \, dz \wedge dx + z^3 \, dx \wedge dy.$$

4. Let $V$ be a normal domain with boundary $S$ such that $N \neq 0$ on $S$ throughout, and let $f$ and $g$ be $\mathbb{R}$-valued $C^2$-functions on an open set containing $V$.

(a) Prove Green’s First Formula:

$$\int_{V} (\nabla f) \cdot (\nabla g) + \int_{V} f \Delta g = \int_{S} f D_n g \, d\sigma.$$

(b) Prove Green’s Second Formula:

$$\int_{V} (f \Delta g - g \Delta f) = \int_{S} (f D_n g - g D_n f) \, d\sigma.$$

(Hint for (a): Apply Gauß’ Theorem to the vector field $f \nabla g$.)

5. Let $\emptyset \neq U \subset \mathbb{R}^3$ be open, and suppose that $f \in C^2(U, \mathbb{R})$ is harmonic, i.e., satisfies $\Delta f = 0$. Let $V \subset U$, $S$ and $n$ be as in the previous problem. Show that

$$\int_{S} D_n f \, d\sigma = 0 \quad \text{and} \quad \int_{S} f D_n f \, d\sigma = \int_{V} \|\nabla f\|^2.$$

6*. Let $a, b > 0$. Use Green’s Theorem to compute the area of the ellipse

$$E := \left\{ (x,y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}.$$

Due Monday, March 8, at 10:00 a.m.; no late assignments.