1. Let $I \subset \mathbb{R}^N$ be a compact interval, let $f, g : I \to \mathbb{R}$ be continuous, and let $\epsilon > 0$. Show that there is a partition $P_{\epsilon}$ of $I$ such that, for each refinement $P$ of $P_{\epsilon}$, we have

$$\left| \int_I fg - \sum_{\nu} f(x_{\nu})g(y_{\nu})\mu(I_{\nu}) \right| < \epsilon,$$

where $(I_{\nu})_{\nu}$ is the subdivision of $I$ corresponding to $P$ and $x_{\nu}, y_{\nu} \in I_{\nu}$ are arbitrary.

2. Show that:

(a) if $D \subset \mathbb{R}^N$ has content, then so has $\overline{D}$ such that $\mu(D) = \mu(\overline{D})$;

(b) if $\emptyset \neq U \subset \mathbb{R}^N$ is open, $Z$ is a set of content zero with $Z \subset U$, and $\phi : U \to \mathbb{R}^N$ is a $C^1$-function, then $\phi(Z)$ has content zero.

3. Let $\emptyset \neq U \subset \mathbb{R}^N$ be open, let $K \subset U$ be compact with content, let $\phi \in C^1(U, \mathbb{R}^N)$, and suppose that there is $Z \subset K$ with content zero such that $\det J\phi(x) \neq 0$ for all $x \in K \setminus Z$. Show that $\phi(K)$ has content. (Hint: Show that $\partial\phi(K) \subset \phi(Z) \cup \phi(\partial K)$.)

4. We may identify $\mathbb{C}$ with $\mathbb{R}^2$. For $0 < \rho < R$, let

$$A_{R,\rho} := \{z \in \mathbb{C} : \rho \leq |z| \leq R\}.$$

Calculate $\int_{A_{R,\rho}} \frac{1}{z}.$

5. Let $\emptyset \neq K \subset \mathbb{R}^3$ be a compact body with content, and let $\mu : K \to \mathbb{R}$ be a continuous density. The Newton potential generated by $K$ at $x_0 \in \mathbb{R}^3 \setminus K$ is given (up to a factor) by

$$u(x_0) := \int_K \frac{\mu(x)}{\|x_0 - x\|} \, dx.$$

Suppose that $K = B_R[(0,0,0)]$ with $R > 0$, and that $\mu$ is rotation symmetric, i.e., there is a continuous function $\tilde{\mu} : [0, R] \to \mathbb{R}$ such $\mu(x) = \tilde{\mu}(\|x\|)$ for all $x \in K$. Show that

$$u(x_0) = \frac{1}{\|x_0\|} \int_K \mu,$$

for all $x_0 \in \mathbb{R}^3 \setminus K$. (Hint: First, argue that we can suppose that $x_0 = (0,0,\|x_0\|)$ without loss of generality; then use spherical coordinates.)

6*. Show that a slice of pizza of radius $r > 0$ and with angle $\alpha$ has the area $\frac{1}{2}r^2\alpha$.

Due Monday, February 4, 2019, at 10:00 a.m.; no late assignments.