1. Is there \( f \in \mathcal{P}C_{2\pi}(\mathbb{R}) \) such that

\[
f(x) \sim \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \sin(nx)\?
\]

2. Let \( f \in \mathcal{P}C_{2\pi}(\mathbb{R}) \) be given by

\[
f: (-\pi, \pi] \to \mathbb{R}, \quad x \mapsto x.
\]

Determine the Fourier series of \( f \) and conclude that

\[
\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) = x
\]

for \( x \in (-\pi, \pi) \).

3. Let \( f \in \mathcal{P}C_{2\pi}(\mathbb{R}) \) be given by

\[
f: (-\pi, \pi] \to \mathbb{R}, \quad x \mapsto |x|.
\]

Determine the Fourier series of \( f \) and argue that it converges to \( f \) uniformly on \( \mathbb{R} \). Conclude that

\[
\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.
\]

4. Let \( f, F \in \mathcal{P}C_{2\pi}(\mathbb{R}) \), let \( a_0, a_1, a_2, \ldots, b_1, b_2, \ldots \) be the Fourier coefficients of \( f \), and let \( A_0, A_1, A_2, \ldots, B_1, B_2, \ldots \) be the Fourier coefficients of \( F \). Show that

\[
\frac{1}{\pi} \int_{-\pi}^{\pi} f(t)F(t) \, dt = \frac{a_0 A_0}{2} + \sum_{n=1}^{\infty} (a_n A_n + b_n B_n).
\]

(Hint: Apply Parseval’s Identity to \( f + F \).)
5. Although the theory of power series was developed in class only for real variables, it all works perfectly well over $\mathbb{C}$ as well. We can thus extend exp, sin, and cos to $\mathbb{C}$ by defining

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad \sin z := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}, \quad \cos z := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$$

for $z \in \mathbb{C}$.

(a) Show that

$$e^{iz} = \cos z + i \sin z$$

holds for all $z \in \mathbb{C}$, and derive Euler’s Identity: $e^{i\pi} + 1 = 0$.

(b) For $n \in \mathbb{Z}$, the $n$-th complex Fourier coefficient of $f \in \mathcal{PC}_{2\pi}(\mathbb{R})$ is defined as

$$c_n := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} \, dt.$$

Express the complex Fourier coefficients of $f$ in terms of its real Fourier coefficients $a_0, a_1, a_2, \ldots$ and $b_1, b_2, \ldots$.

6*. Let $a < b$, let $f : [a, b] \to \mathbb{R}$ be continuous, let $a \leq t_1 < t_2 < \cdots < t_n \leq b$, and let $\epsilon > 0$. Show that there is a polynomial $p$ with $|f(t) - p(t)| < \epsilon$ for $t \in [a, b]$ and $p(t_j) = f(t_j)$ for $j = 1, \ldots, n$. (Hint: First, treat the case where $f(t_1) = \cdots = f(t_n) = 0$, then apply this to the auxiliary function $[a, b] \ni t \mapsto f(t) - \sum_{k=1}^{n} f(t_k) \prod_{j=1}^{n} \frac{t-t_j}{t_k-t_j}$.)

Due Monday, April 12, at 10:00 a.m.; no late assignments.