1. Let $U := \mathbb{R}^2 \setminus \{(0,0)\}$, and let

$$f : U \to \mathbb{R}^2, \quad (x,y) \mapsto \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right).$$

(a) Calculate $\det J_f(x,y)$ for all $(x,y) \in U$.

(b) Determine $f(U)$. Does it contain a non-empty open subset?

2. Is the following “theorem” true or not?

Let $\emptyset \neq U \subset \mathbb{R}^N$ be open, let $x_0 \in U$, and let $f \in C^1(U, \mathbb{R}^N)$ be such that $f(V)$ is open for each open neighborhood $V \subset U$ of $x_0$. Then $\det J_f(x_0) \neq 0$.

Give a proof or provide a counterexample.

3. Let $\emptyset \neq U \subset \mathbb{R}^N$ be open, and let $f \in C^1(U, \mathbb{R}^N)$ be such that $\det J_f(x) \neq 0$ for all $x \in U$.

(a) Show that

$$U \to \mathbb{R}, \quad x \mapsto \|f(x)\|$$

has no local maximum.

(b) Suppose that $U$ is bounded (so that $\overline{U}$ is compact) and that $f$ has a continuous extension $\tilde{f} : \overline{U} \to \mathbb{R}^N$. Show that the continuous map

$$\overline{U} \to \mathbb{R}, \quad x \mapsto \|\tilde{f}(x)\|$$

attains its maximum on $\partial U$.

4. Let

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad (x,y) \mapsto x^2 + y^2.$$

Show that, there is $\epsilon > 0$ and a $C^1$-function $\phi : (-\epsilon, \epsilon) \to \mathbb{R}$ with $\phi(0) = 1$ such that $y = \phi(x)$ solves the equation $f(x,y) = 1$ for all $x \in \mathbb{R}$ with $|x| < \epsilon$. Show without explicitly determining $\phi$ that

$$\phi'(x) = -\frac{x}{\phi(x)} \quad (x \in (-\epsilon, \epsilon)).$$
5. Show that there are $\epsilon > 0$, and $u, v, w \in C^1(B_\epsilon((1, 1)), \mathbb{R})$ such that $u(1, 1) = 1$, $v(1, 1) = 1$, and $w(1, 1) = -1$, and

$$u(x, y)^5 + x v(x, y)^2 - y + w(x, y) = 0,$$

$$v(x, y)^5 + y u(x, y) - x + w(x, y) = 0,$$

and

$$w(x, y)^4 + y^5 - x^4 = 1$$

for $(x, y) \in B_\epsilon((1, 1))$.

6*. Let $f \in C^1(\mathbb{R}^2, \mathbb{R})$ be such that

$$f^{-1}(\{0\}) = \{(x, y) \in \mathbb{R}^2 : (x - 1)^2 + y^2 = 1 \text{ or } (x + 1)^2 + y^2 = 1\}.$$

(a) Sketch $f^{-1}(\{0\})$.

(b) Show that

$$\frac{\partial f}{\partial y}(0, 0) = \frac{\partial f}{\partial x}(0, 0) = 0.$$

Due Monday, January 21, 2019, at 10:00 a.m.; no late assignments.

*** IMPORTANT !!!

1. The completed assignment has to be placed in the marked assignment box on the third floor of CAB.

2. You are allowed to collaborate on homework assignments—in fact, I encourage you to do so. Still, all students must hand in their own homework assignments.

3. All problems have equal weight.

4. Problems marked with an * are bonus problems: they allow you to earn extra marks on an assignment. On this assignment, for instance, you can thus get a mark of 120%.