1. Let $I$ be a compact interval, and let $f = (f_1, \ldots, f_M): I \to \mathbb{R}^M$. Show that $f$ is Riemann integrable if and only if $f_j: I \to \mathbb{R}$ is Riemann integrable for each $j = 1, \ldots, M$ and that, in this case,

$$\int_I f = \left(\int_I f_1, \ldots, \int_I f_M\right)$$

holds.

2. Let $I \subset \mathbb{R}^N$ be a compact interval, and let $f: I \to \mathbb{R}^M$ be Riemann integrable. Show that $f$ is bounded.

3. Let $\emptyset \neq D \subset \mathbb{R}^N$ be bounded, and let $f, g: D \to \mathbb{R}$ be Riemann-integrable. Show that $fg: D \to \mathbb{R}$ is Riemann-integrable. Do we necessarily have

$$\int_D fg = \left(\int_D f\right)\left(\int_D g\right)?$$

(Hint: First, treat the case where $f = g$ and then the general case by observing that $fg = \frac{1}{2}((f + g)^2 - f^2 - g^2).$)

4. Let $\emptyset \neq D \subset \mathbb{R}^N$ have content zero, and let $f: D \to \mathbb{R}^M$ be bounded. Show that $f$ is Riemann-integrable on $D$ such that

$$\int_D f = 0.$$

5. Let $\emptyset \neq U \subset \mathbb{R}^N$ be open with content, and let $f: U \to [0, \infty)$ be bounded and continuous such that $\int_U f = 0$. Show that $f \equiv 0$ on $U$.

6*. The function

$$f: [0, 1] \times [0, 1] \to \mathbb{R}, \quad (x, y) \mapsto xy$$

is continuous and thus Riemann integrable. Evaluate $\int_{[0, 1] \times [0, 1]} f$ using only the definition of the Riemann integral, i.e., in particular, without use of Fubini’s Theorem.

Due Monday, November 19, 2018, at 10:00 a.m.; no late assignments.