Multicomponent wave separation using HOSVD/unimodal-ICA subspace method

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ABSTRACT

Multicomponent sensor arrays now are commonly used in seismic acquisition to record polarized waves. In this article, we use a three-mode model (polarization mode, distance mode, and temporal mode) to take into account the specific structure of signals that are recorded with these arrays, providing a data-structure-preserving processing. With the suggested model, we propose a multilinear decomposition named higher-order singular value decomposition and unimodal independent component analysis (HOSVD/unimodal ICA) to split the recorded three-mode data into two orthogonal subspaces: the signal and noise subspaces. This decomposition allows the separation and identification of polarized waves with infinite apparent horizontal propagation velocity. The HOSVD leads to a definition of a subspace method that is the counterpart of the well-known subspace method for matrices that is driven by singular value decomposition (SVD), a classic tool in monocomponent array processing.

INTRODUCTION

Multicomponent sensor arrays are used in many applications, such as seismic exploration and seismology, electromagnetism, communications, and acoustics (Nehorai and Paldi, 1994a; Nehorai and Paldi, 1994b; Anderson and Nehorai, 1996; Hawkes and Nehorai, 2001). Datasets that are recorded with these multicomponent arrays sometimes are called three-mode data (Le Bihan and Ginolhac, 2004) because they have a temporal mode (related to recording time on each sensor), a distance mode (related to the number of sensors), and a polarization mode (related to the vectorial type of each sensor). An intrinsic property of propagating waves is accessible using multicomponent sensors — the polarization.

Classically, multicomponent signal processing approaches are based on the use of long vectors. This consists of reorganizing the three-mode dataset into a matrix formulation. Consecutive lines (or columns) of the received three-mode data are concatenated into a large matrix (Nehorai and Paldi, 1994b). This data reorganization usually is performed to allow use of existing matrix algebra algorithms. Here, we propose a new type of processing to avoid modification of the original dataset structure and to carry the three modes through the processing.
The specificity of multicomponent sensors has been used to estimate the direction of arrival (Nehorai and Paldi, 1994b) and/or the polarization states of waves (Vidale, 1986). Some versions of classic array-processing algorithms, such as multiple-signal classification (MUSIC) (Marcos, 1998), have also been extended to the multicomponent case using matrix algebra tools (Wong and Zoltowski, 2000) (and so, using the long-vector approach).

These approaches are “destructive” in the sense that they do not take into account the original structure of the recording devices; consequently, the physical quantities that link the recorded signals might not be accessible after reorganization of the dataset. Also, long-vector techniques are “restrictive” in the sense that there exist several possible choices for the reorganization of a three-mode dataset into matrix format, and that arbitrarily choosing one is not optimal (Comon, 1994b). To fully use the polarization information directly in the processing requires a three-mode model of signals recorded on multicomponent arrays. This approach is more exhaustive because it simultaneously processes all possible long vectors that are built from the three-mode dataset.

In geophysical operations, the goal of signal processing is to separate and identify recorded waves to improve interpretation of the geophysical dataset. In this paper, we propose a subspace method that is based on the higher-order singular value decomposition HOSVD to separate the recorded three-mode data into a signal part (related to polarized waves with infinite apparent horizontal propagation velocity) and a noise part (related to other waves). For the three-mode data, the HOSVD gives a three-mode rank truncation, which is the counterpart of rank reduction in the matrix case (Freire and Ulrych, 1988; Golub and Van Loan, 1989). This is the backbone of the proposed three-mode subspace method. Other decompositions have also been used in signal processing, such as parallel factor analysis (PARAFAC) (Bro, 1998; Bro et al., 1999; Sidiropoulos et al., 2000) or Tucker decompositions (Tucker, 1966; Kroonenberg, 1983), but will not be considered. The HOSVD is preferred over other decompositions here because it provides an orthonormal basis, allowing an extension of the well-known SVD-based monocomponent subspace method to three-mode data case (Le Bihan and Ginolhac, 2004).

To enhance the results of the HOSVD-based subspace separation (e.g., when the signal-to-noise ratio is low or when orthogonality constraints imposed by the HOSVD are not well adapted to the waves recorded in the dataset), an independent component analysis (ICA) step can be performed on the temporal mode. This step is prompted by geophysical situations in which the recorded signals can be approximated as an instantaneous linear mixture of unknown waves that are supposed to be mutually independent. ICA allows blind estimation of these waves, i.e., \( \tau_{ci} \). After such preprocessing, aligned waves are said to have infinite apparent horizontal propagation velocity. Denoting \( s_i(t) \) as the aligned wave, equation 1 becomes

\[
y_{ci}(t) = m_{cip} s_i(t) + n_{cip}(t) = m_{cip} s_i(t - \tau_{ci}) + n_{cip}(t),
\]

where \( y_{ci}(t) = y(t + \tau_{ci}) \). It expresses products of three-mode arrays \( k = 1, 2, 3 \) in our case, with matrices along 3D space with the smallest possible rank, a velocity-correction (wave-alignment) operation often is applied to compensate the sensor-to-sensor delay for the dominant waveform, i.e., \( \tau_{ci} \). After such preprocessing, aligned waves are said to have infinite apparent horizontal propagation velocity. Denoting \( s_i(t) \) as the aligned wave, equation 1 becomes

\[
y_{ci}(t) = m_{cip} s_i(t) + n_{cip}(t),
\]

where \( y_{ci}(t) = y(t + \tau_{ci}), \tau_{ci} = \tau_{ci} - \tau_{ci}, \) and \( n_{cip}(t) = n_{cip}(t + \tau_{ci}). \)

In the following section, we consider this model and assume that the aligned wave \( s_i(t) \) is independent from the others \( s_j(t - \tau_{ci}) \) and from the noise \( n_{cip}(t) \).

**Multilinear model**

Supposing that \( N_i \) time samples are available, we rewrite the received signals into a three-mode format,

\[
\mathcal{Y} = \{ y_{ci}(t) \} \subseteq R^{N_i \times N_i \times N_i},
\]

where \( \mathcal{Y} \) is the three-mode data that are collected by the multicomponent array with \( N_i \) sensors, each one recording \( N_i \) directions of wavefields in 3D space. The different modes are the natural modes of the recorded signals; polarization, distance, and time.

To develop multilinear algorithms to process three-mode data, we introduce the \( k \)-mode product, denoted by \( \times_k \). It expresses products of three-mode arrays \( k = 1, 2, 3 \), in our case, with matrices along
each mode (De Lathauwer, 1997; Comon, 2000; De Lathauwer et al., 2000). The \( k \)-mode products can be computed for any three-mode array \( \mathbf{X} = \{x_{ijl}\} \in \mathbb{R}^{J \times I \times L} \), and for any matrix \( \mathbf{A} = \{a_{ik}\} \in \mathbb{R}^{I \times L} \) (with \( k = 1, 2, 3 \), or in case) as

\[
\begin{align*}
(\mathbf{X} \times_1 \mathbf{A})_{(ij)l} &= \sum x_{ijl}a_{ij1} \in \mathbb{R}^{J \times I \times L}, \\
(\mathbf{X} \times_2 \mathbf{A})_{(i)jl} &= \sum x_{ijl}a_{i1j} \in \mathbb{R}^{I \times J \times L}, \\
(\mathbf{X} \times_3 \mathbf{A})_{(i)lj} &= \sum x_{ijl}a_{i1j} \in \mathbb{R}^{I \times J \times L}.
\end{align*}
\]

(4) data by Meunier et al. (2001). Also, for dispersive waves, SVD processing can be used for wavefield discrimination and selection (Mars et al., 2004).

**Multilinear notation of SVD**

Using multilinear algebra notation, the SVD given in equation 6 can be rewritten as (De Lathauwer et al., 2000)

\[
\mathbf{Y}_c = \mathbf{A}_c \times_1 \mathbf{U}_c \times_2 \mathbf{V}_c.
\]

This \( k \)-mode product (see equation 4, with \( k = 1, 2 \)) includes left and right matrix products, which are the 1-mode and 2-mode cases, respectively. In this SVD reformulation, the matrix \( \mathbf{A}_c \) is multiplied on its first mode (lines) by the matrix \( \mathbf{U}_c \) and on its second mode (columns) by the matrix \( \mathbf{V}_c \). This multilinear notation is helpful for understanding the transition from matrices to three-mode arrays in the HOSVD definition.

**Subspace method using SVD**

The SVD of \( \mathbf{Y}_c \) leads to a global or partial estimation of the aligned waves, such as \( s_i(t) \), that are hidden in the mixture (Freire and Urych, 1988). Decomposition into a signal and a noise subspace is performed on each component in regard to singular values

\[
\mathbf{Y}_c = \mathbf{Y}_c^{\text{signal}} + \mathbf{Y}_c^{\text{noise}} = \sum_{j=1}^{p} \delta_{j} \mathbf{u}_{ij} \mathbf{v}_{cj}^T + \sum_{j=p+1}^{R_c} \delta_{j} \mathbf{u}_{ij} \mathbf{v}_{cj}^T.
\]

The first \( p \) left and right singular vectors of \( \mathbf{Y}_c \) form a basis that represents the highest sensor-to-sensor correlated waves such as \( s_i(t) \) on the \( c^a \) component, and the highest-amplitude sensor-to-sensor uncorrelated waves. Associated singular values are linked with the waveform’s magnitude contribution in the original dataset. Noise is spread across the whole set of singular values.

Using multilinear notation, the signal subspace also can be written as

\[
\mathbf{Y}_c^{\text{signal}} = \mathbf{Y}_c^{\text{signal}} \times_1 \mathbf{P}_c \times_2 \mathbf{P}_c^T.
\]

Figure 1 presents a schematic representation of the three possible \( k \)-mode products of a three-mode array \( \mathbf{X} \in \mathbb{R}^{J \times I \times L} \), with matrices \( \mathbf{A} = \{a_{ik}\} \in \mathbb{R}^{I \times L} \).
with the projectors

$$
\mathcal{P}_{U_c} = U_c^p U_c^{pT} \quad \text{and} \quad \mathcal{P}_{V_c} = V_c^p V_c^{pT},
$$

where $U_c = [u_c, \ldots, u_c]$ and $V_c = [v_c, \ldots, v_c]$ are the matrices that contain the first $p$ singular vectors. This formulation will be very useful for understanding the expression of the signal subspace for the three-mode data (see equation 17).

The noise subspace on each component then is obtained by subtracting the signal subspace from the initial data, i.e., $Y_{\text{noise}} = Y - Y_{\text{signal}}$.

The determination of $p$, the number of basis vectors giving the best (in the mean-square sense) signal part, is made by finding an abrupt change of slope in the decreasing curve of singular values for each component. In practice, velocity compensation is performed as preprocessing to ensure that $p$ never exceeds 2, which justifies the simplified model given in equation 2.

The SVD subspace method will be used as a component-wise subspace separation method in comparison with HOSVD and HOSVD/unimodal-ICA subspace methods.

HOSVD AND RELATED SUBSPACE METHOD

Here, we extend the SVD to three-mode arrays, and list some of its properties. We also present the way that this decomposition gives information about the so-called $(r_c, r_x, r_y)$-rank of three-mode arrays and about the three-mode subspace method.

**Definition**

HOSVD is a multilinear decomposition that was introduced by De Lathauwer (1997) and De Lathauwer et al. (2000). For three-mode data $Y \in R^{N_i \times N_x \times N_t}$, this decomposition can be written as

$$
Y = \mathcal{C} \times_1 U_{(c)} \times_2 U_{(x)} \times_3 U_{(t)},
$$

with $U_{(i)} = \{u_{(i)}[r_{(i)}], \ldots, u_{(i)}[p_{(i)}]\} \in R^{N_i \times p_{(i)}}$ being orthogonal matrices that contain the singular vectors $u_{(i)} \in R^{N_i}$ of $Y$ in the three modes $(i = c,x,t)$, and with $x_{(i)}$ being the $k$-mode products (see equation 4). Figure 2 is a schematic representation of HOSVD.

**Computation**

For a three-mode dataset $Y \in R^{N_i \times N_x \times N_t}$, the unfolding matrices $Y_{(i)}$ are defined as $(i = c, x, t)$:

$$
Y_{(i)} \in R^{N_i \times N_x N_t}.
$$

These unfolding matrices are obtained by stacking subarrays in large matrices. There are three possible ways to build such an unfolding matrix in the three-mode-array case. Figure 3 presents a schematic representation of these unfolding matrices.

In equation 12, each matrix $U_{(i)} (i = c, x, t)$ is the left matrix given by the SVD of each unfolding matrix $Y_{(i)}$,

$$
Y_{(i)} = U_{(i)} \Delta_{(i)} V_{(i)}^T.
$$

The matrices $U_{(i)}$ define orthonormal bases in the three modes of the vector space $R^{N_i \times N_x \times N_t}$. In the rest of the paper, the three sets of singular values given by the matrices $\Delta_{(i)}$ may also be referred to as three-mode singular values.

The core array $C$ (see Figure 2) is the equivalent of the diagonal matrix in the matrix SVD, except that it is not hyperdiagonal. Instead, it is imposed to be all-orthogonal, and the Frobenius norms of the subarrays (obtained by fixing one index of the core array) are ordered. One can see that there is no strict equivalent to singular values for matrices in the case of three-mode arrays. In spite of this, the ordered norms of the subarrays of $C$ can be seen as the equivalent of the singular values; moreover, the energy of the three-mode dataset $Y$ is concentrated at the $(1,1,1)$ corner of the core array $C$. It also is important to note that the Frobenius norms of the subarrays correspond to three-mode singular values $\Delta_{(i)}$. That is, $\|\mathcal{C}(1\ldots, 1)\| = \Delta_{(i)}(1,1)$, $\|\mathcal{C}(2\ldots, \cdot)\| = \Delta_{(i)}(2,1)$, ..., $\|\mathcal{C}(\cdot, \cdot, 1)\| = \Delta_{(i)}(1,1)$, and so on.

The core array $C$ is given by

$$
C = \mathcal{Y} \times_1 (U_{(c)}^T \times_2 U_{(x)}^T \times_3 U_{(t)}^T).
$$

It is calculated after estimation of the $U_{(i)}$, using equation 14. The interpretation of $C$ is given through the three-mode rank, corresponding to the non-null three-mode singular values.

**Rank of a three-mode array**

There are two main definitions for the rank of a three-mode array: the rank and the three-mode rank.
The rank is the minimal number of three-mode arrays of rank 1 that yields the array by linear combination. Obtaining the rank of a three-mode array is not easy because this rank can be higher than the highest dimension of the array (Comon, 1994b; De Lathauwer, 1997; Bro, 1998; Comon, 2000).

In this paper, we will not consider the rank, but we will focus on the three-mode rank, which is a generalization of the matrix case: it is a triplet \((r, r, r)\) that is composed of the ranks \(r_i\) of the three unfolding matrices \(Y(i)\) \((i = c, x, t)\),

\[ r = \text{rank}(Y(i)) = \text{dim}(\Delta(i)). \tag{16} \]

In the rest of the paper, the three-mode rank of a three-mode array may also be referred to as the \((r, r, r)\)-rank.

### Subspace method using HOSVD

To decompose a three-mode dataset \(Y\) of size \(N_x \times N_c \times N_t\) and of \((r, r, r)\)-rank, we define two subspaces.

The signal subspace \(Y_{\text{signal}}\) is built by using the first \(p_i \leq r_i\) singular vectors in the first mode, \(p_i \leq r\) in the second, and \(p_i \leq r\) in the third,

\[ Y_{\text{signal}} = Y_1 P_{p_1} Y_2 P_{p_2} Y_3 P_{p_3}, \tag{17} \]

where the projectors \(P_{p_i}\) are given by

\[ P_{p_i} = U_{p_i} U_{p_i}^T, \tag{18} \]

where \(U_{p_i} = [u_{(i)}] \ldots [u_{(i)}]\) are the matrices that contain the first \(p_i\) singular vectors \((i = c, x, t)\).

The noise subspace \(Y_{\text{noise}}\) is obtained by subtracting the \(Y_{\text{signal}}\) from the original three-mode data, i.e., \(Y_{\text{noise}} = Y - Y_{\text{signal}}\).

These two subspaces are orthogonal in at least one mode (because of the nonhyperfidiagonality of the core array \(\mathcal{C}\)). The orthogonality along different modes is obtained thanks to the orthogonality between the subspaces given by the unfolding matrices.

In the same way as for the subspace method for scalar-sensor arrays, the values \(p_c, p_x, p_t\) are chosen by finding abrupt changes of slope in the curves of three-mode singular values. For some critical cases in which no visible change of slope can be found in a set of singular values, the number of singular values kept for the signal subspace construction is chosen arbitrarily. It is chosen to be as small as possible, but this can leave uncertainty in validity of the result (Le Bihan and Ginolhac, 2004).

We have introduced a way to decompose a three-mode dataset into signal and noise subspaces by using the HOSVD technique. This algorithm, as applied on a three-mode dataset \(Y\), is summarized in Table 1.

We now propose a second step in the subspace method that uses an ICA procedure to improve wave separation.

### HOSVD/UNIMODAL ICA AND RELATED SUBSPACE METHOD

The SVD of \(Y(i)\) in equation 14 provides two orthogonal matrices, \(U_{p_i} \in R^{N_i \times r_i}\) and \(V_{p_i} \in R^{N_i \times r_i}\), that are made up of the left and right singular vectors \(u_{(i)}\) and \(v_{(i)}\). The \(u_{(i)}\) are the *estimated waves* because they give the time evolution of the received signals by the three-mode array (Vrabie et al., 2004). The \(v_{(i)}\) are called *propagation vectors* because they give the amplitude of these waves with respect to the unfolding matrix \(Y(i)\).

The estimated waves are orthogonal, and therefore are statistically independent at the second order. The propagation vectors also are orthogonal by construction (because they are singular vectors). In practical terms, there is no physical reason why the propagation vectors should be orthogonal. Imposing this orthogonality, the estimated waves \(u_{(i)}\) are forced to be a mixture of recorded waves; moreover, the signal subspace given by equation 17 is obtained using only the information contained in the left matrices \(U_{p_i}\) of the unfolding matrices \(Y(i)\). So, the signal subspace approximation is sensitive to an accurate extraction of the waves \(u_{(i)}\).

To enhance the subspace separation result, it is possible to impose a stronger criterion for the estimated waves, i.e., that they be fourth-order statistically independent, and consequently to relax the orthogonality condition for the propagation vectors. This can be done using ICA.

ICA is a blind technique that allows recovery, from a given set of vectors, of the statistically independent vectors that lead by linear combination to the original set of vectors (Cardoso and Souloumiac, 1993; Comon, 1994a; De Lathauwer, 1997). It is used in blind source separation (BSS) to recover independent sources (modeled as vectors) from a set of recordings that contain linear combinations of these sources. Using ICA, the crosscumulants of any order of independent sources must equal zero. As usual, the third-order cumulants are discarded because they generally are null or low. We use fourth-order cumulants, which have been proved to be sufficient for instantaneous mixtures.

A two-stage algorithm that consists of a prewhitening step and a high-order step can solve ICA. The prewhitening step is carried out by the SVD directly on the raw data \(Y(i)\). The estimated waves in \(U_{p_i}\) are an instantaneous linear mixture of recorded waves if the nonaligned waves in \(Y(i)\) are contained in a subspace of dimension \(R - 1\) that is smaller than \(r\) (Vrabie et al., 2004). Assuming this, only the first \(R\) estimated waves \([u_{(i)}] \ldots [u_{(i)}] = U_{p_i}\) are taken into account. Assuming that the recorded waves are mutually statistically independent, the high-order step consists of finding a rotation matrix \(B \in R^{R\times R}\) for which the components of

\[ U_{p_i} B = \bar{U}_{p_i} = [\bar{u}_{(i)}] \ldots [\bar{u}_{(i)}] \tag{19} \]

are independent at the fourth order.

There are different ways to find this rotation matrix: joint approximate diagonalization of eigenmatrices (JADE) (Cardoso and Sou-

| Table 1. HOSVD subspace methods steps for a three-mode dataset \(Y\): |
| --- | |
| 1) Construct unfolding matrices \(Y(i), Y(c), Y(t)\), as in equation 13. |
| 2) Compute matrices \(U_{p_i}, U_{p_c}, U_{p_t}\) using SVD of unfolding matrices, as in equation 14. |
| 3) Determine the \((r, r, r)-\)rank given by the triplet of ranks of the unfolding matrices, as in equation 16. |
| 4) Choose values \(p_c, p_x, p_t\) by finding abrupt changes of slope in the curves of three-mode singular values. |
| 5) Compute projectors \(P_{p_i}\) using equation 18. |
| 6) Estimate signal subspace \(Y_{\text{signal}}\) as \((p_c, p_x, p_t)-\)rank truncation of \(Y\) using equation 17. |
| 7) Compute noise subspace \(Y_{\text{noise}}\) by subtracting signal subspace from the original three-mode data. |
loumiac, 1993), maximal diagonality (MD) (Comon, 1994a), simultaneous third-order tensor diagonalization (STOTD) (De Lathauwer, 1997). The JADE algorithm is used here, but we believe that similar results would be obtained using other algorithms.

After using ICA, we get the matrix \(\mathbf{U}_0 = [\mathbf{U}_0^1; \mathbf{U}_0^2; \mathbf{U}_0^3; \mathbf{U}_0^4]\), in which the first \(R\) columns are statistically independent at the fourth order and the last \(N_t - R\) are unchanged from the matrix \(\mathbf{U}_0\). The HOSVD/unimodal ICA then is

\[ \mathbf{Y} = \mathbf{C} \times_1 \mathbf{U}_0^1 \times_2 \mathbf{U}_0^2 \times_3 \mathbf{U}_0^3, \]

where

\[ \mathbf{C} = \mathbf{Y} \times_1 \mathbf{U}_0^1 \times_2 \mathbf{U}_0^2 \times_3 \mathbf{U}_0^3, \]

Table 2. HOSVD/unimodal-ICA subspace method steps for a three-mode dataset \(\mathbf{Y}\).

1) Construct unfolding matrices \(\mathbf{Y}_{(i)}, \mathbf{Y}_{(a)}, \mathbf{Y}_{(o)}\), and \(\mathbf{V}_{(o)}\), as in equation 13.
2) Compute matrices \(\mathbf{U}_{(i)}, \mathbf{U}_{(a)}, \text{ and } \mathbf{U}_{(o)}\) using SVD of unfolding matrices, as in equation 14.
3) Determine \(R\) in the set of singular values in the third (temporal) mode.
4) Choose rotation matrix \(\mathbf{B}\) using JADE method.
5) Compute estimated waves basis \(\tilde{\mathbf{C}}\), as in equation 19.
6) Carry out permutation between the vectors of \(\mathbf{U}_{(i)}, \mathbf{U}_{(a)}, \text{ and } \mathbf{U}_{(o)}\) for ordering the Frobenius norms of the subarrays of \(\tilde{\mathbf{C}}\) obtained using equation 21.
7) Choose values \(p_x, p_y, \text{ and } \tilde{p}\), by finding abrupt changes of slope in the curves of modified three-mode singular values.
8) Compute projectors \(\mathcal{P}_{\mathbf{Y}_{(i)}}\) and \(\mathcal{P}_{\mathbf{Y}_{(a)}}\) using equation 18, and \(\mathcal{P}_{\mathbf{Y}_{(o)}}\) using equation 22.
9) Estimate signal subspace \(\tilde{\mathbf{Y}}\) using equation 23.
10) Compute noise subspace \(\tilde{\mathbf{Y}}\) by subtracting the signal subspace from the original three-mode data.

\[
\mathbf{Y} = \mathbf{C} \times_1 \mathbf{U}_0^1 \times_2 \mathbf{U}_0^2 \times_3 \mathbf{U}_0^3, \tag{20}
\]

Here, unimodal means that ICA is performed only on the temporal mode. Note that a permutation between the vectors of \(\mathbf{U}_{(i)}, \mathbf{U}_{(a)}, \text{ and } \mathbf{U}_{(o)}\) must be performed to order the Frobenius norms of the subarrays (obtained by fixing one index) of the new core array \(\tilde{\mathbf{C}}\).

Hence, we keep the same decomposition structure as in equations 12 and 15; the only difference here is that we have modified the orthogonality constraint into a fourth-order-independence constraint for the first \(R\) estimated waves on the third mode. Note that the \((r_x, r_y, r_z)\)-rank of the three-mode dataset \(\mathbf{Y}\) is unchanged.

Subspace method using HOSVD/unimodal ICA

We compute the projector on the third (temporal) mode by using

\[
\tilde{\mathbf{Y}} = \mathbf{Y} \times_1 \mathbf{P}_{\mathbf{Y}_{(i)}} \times_2 \mathbf{P}_{\mathbf{Y}_{(a)}} \times_3 \mathbf{P}_{\mathbf{Y}_{(o)}}, \tag{23}
\]

and the noise subspace \(\tilde{\mathbf{Y}}\) is obtained by subtracting the signal subspace from the original three-mode data.

In practice, \(R\) becomes a parameter. It is chosen to completely describe the aligned wave \(s_i(t)\) by the first \(R\) estimated waves. As in the HOSVD case, the values \(p_x, p_y, \text{ and } \tilde{p}\), are chosen by finding abrupt changes of slope in the curves of modified three-mode singular values that were obtained after the ICA step. Note that in the HOSVD/unimodal-ICA method, the rank for the signal subspace in the third mode \(\tilde{p}_z\) is not necessarily equal to the rank \(p_t\) that is obtained using only HOSVD.

Table 2 summarizes the three-mode HOSVD/unimodal-ICA algorithm as applied to a three-mode dataset \(\mathbf{Y}\).

Next, we present applications to simulated and real data to compare the behavior of the HOSVD/unimodal-ICA method with that of component-wise SVD and HOSVD subspace methods.

Simulation

The simulated data represent a 2C multicomponent array that is composed of \(N_t = 18\) sensors, each of which is made up of \(N_x = 2\) components (hence, records two directions in the 3D space): the first one related to a geophone \(\mathbf{Z}\), and the second one related to a hydrophone \(\mathbf{Hy}\). The recording time is 512 ms, corresponding to \(N_t = 256\) time samples. The distance between sensor positions is 5 m. This dataset, shown in Figure 4, is a three-mode dataset \(\mathbf{Y} \in \mathbb{R}^{2 \times 18 \times 256}\) that has polarization, distance, and time modes. The representation of the \(\mathbf{Z}\) component was scaled by 5 to obtain the same amplitude range.

To simulate a realistic case, this synthetic dataset \(\mathbf{Y}\) was obtained by addition between an original signal subspace \(\mathbf{S}\) (Figure 5) that is

![Figure 4. Simulated three-mode dataset \(\mathbf{Y}\): (a) \(\mathbf{Z}\) and (b) \(\mathbf{Hy}\) components.](image-url)
made up of several wavefronts that have infinite apparent horizontal propagation velocity, and consequently are associated with the wave $s(t)$ (see equation 2), and an original noise subspace $N$ (Figure 6) that is obtained from a geophysical acquisition after the subtraction of aligned waves. For the original signal subspace $S$, the wave amplitudes vary along the array, which simulates attenuation along the distance mode. The relation between $Z$ and $H_y$ (which is a linear relation here) can be assimilated to wave polarization (polarization mode) in the sense that it consists of phase and amplitude relations between two seismic sections. The signal-to-noise ratio (SNR) of this dataset is low (SNR = $-5.4$ dB). The SNR definition used here is

$$\text{SNR} = 20 \log_{10} \frac{\|S\|}{\|N\|}$$

(24)

where $\|\|$ is the three-mode-array Frobenius norm $^3$, $S$ and $N$ the original signal and noise subspaces.

Normalization to unit variance of each trace for each component was done before applying the described subspace methods. This ensures that even weak-amplitude recorded waves are well represented within the input data. After computation of the signal subspaces, a denormalization was applied to find the original signal subspace.

**Results using SVD subspace method**

First, we tested the SVD subspace method. From the original dataset $\mathcal{Y}$, we constructed two seismic sections, $\mathcal{Y}_1$ and $\mathcal{Y}_2$, which represent, respectively, the $Z$ and $H_y$ components of the mixture (see equation 5). The subspace method given by equation 10 (or equivalently by equation 9) was used, keeping only one singular vector (one singular value) for each seismic section. This choice was made by finding an abrupt change of slope after the first singular value in the relative singular values for each seismic section. The signal subspace components $\mathcal{Y}_1^{\text{signal}}$ and $\mathcal{Y}_2^{\text{signal}}$ that were obtained are presented in Figure 7a and b.

When SVD is used, waveforms are not well recovered, compared to the original signal components (see Figure 5). Moreover, no distinction between the different wavefronts is possible by looking at Figure 7. Furthermore, arrival time cannot be estimated using this technique. Low signal level is a strong handicap for a component-wise process.

**Results using HOSVD subspace method**

Components of the signal subspace $\mathcal{Y}^{\text{signal}}$ that was obtained with the HOSVD method, outlined in Table 1, are presented in Figure 8. In this case, the number of singular vectors kept are one each for the

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$^3$For any three-mode array $\chi = \{x_{ijk}\} \in \mathbb{R}^{I \times J \times K}$, its Frobenius norm is $\|\chi\| = \sqrt{\sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} x_{ijk}^2}$. 

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**Figure 5.** Original signal subspace $S$ of $\mathcal{Y}$: (a) $Z$ and (b) $H_y$ components.

**Figure 6.** Original noise subspace $N$ of $\mathcal{Y}$: (a) $Z$ and (b) $H_y$ components.

**Figure 7.** Signal subspace estimated with the SVD subspace method: (a) $Z$ and (b) $H_y$ components.
polarization, distance, and time modes, giving a (1,1,1)-rank truncation for the signal subspace $\mathcal{Y}^{\text{signal}}$ (see equation 17). This choice was made by finding an abrupt change of slope after the first singular value in all three sets of relative singular values shown in Figure 9a. Note from this figure that the three-mode dataset $\mathcal{Y}$ has a (2, 18, 35)-rank because the 36th singular value on the temporal mode vanishes.

As we can see, there remain some oscillations between the different aligned waves in $\mathcal{Z}$ and $\mathcal{H}_y$ of the signal subspace, which could induce detection errors.

**Results using HOSVD/unimodal-ICA subspace method**

An ICA step is required in this case to obtain a better signal separation and to cancel parasitic oscillations.

In Figure 10, the waves recovered in the signal subspace $\tilde{\mathcal{Y}}^{\text{signal}}$ obtained with the HOSVD/unimodal-ICA technique (outlined in Table 2) are very close to the original signal components. Here, the ICA method was applied on the first $R = 6$ estimated waves shown in Figure 11a. These waves describe the aligned waves of the original signal subspace $\mathcal{S}$. Figure 11b shows the estimated waves $\{\tilde{\mathbf{u}}_{i01}, \ldots, \tilde{\mathbf{u}}_{i61}\}$ after the ICA step. As we can see, the first one $\tilde{\mathbf{u}}_{i01}$ describes more precisely the aligned wave of the original subspace $\mathcal{S}$ than does the first estimated wave $\hat{\mathbf{u}}_{i01}$ before the ICA step (Figure 11a), so that the estimation of signal subspace is more accurate. That means that our proposed procedure delivers better results.

The number of singular vectors kept are one each for the polarization, distance, and time modes ($\tilde{p}_i = 1$). Figure 9b shows the relative singular values on the three modes in HOSVD/unimodal-ICA case after the permutation step. This figure justifies the choice of a (1,1,1)-rank truncation for the signal subspace because of the abrupt changes of slope in the singular values on these modes. Note that the three-mode rank of the dataset $\mathcal{Y}$ is unchanged, i.e., that $\mathcal{Y}$ has (2, 18, 35)-rank.

**Results comparison**

To compare these results qualitatively, in Figure 12 we present the stacks on the $\mathcal{Z}$ and $\mathcal{H}_y$ components for the original signal subspace

![Figure 8](image1.png)

**Figure 8.** Signal subspace estimated with the HOSVD subspace method: (a) $\mathcal{Z}$ and (b) $\mathcal{H}_y$ components.

![Figure 9](image2.png)

**Figure 9.** Relative three-mode singular values using (a) HOSVD and (b) HOSVD/unimodal-ICA methods.

![Figure 10](image3.png)

**Figure 10.** Signal subspace estimated with the HOSVD/unimodal-ICA subspace method: (a) $\mathcal{Z}$ and (b) $\mathcal{H}_y$ components.

![Figure 11](image4.png)

**Figure 11.** The first six estimated waves obtained (a) before and (b) after ICA.
and for the estimated signal subspaces obtained with SVD, HOSVD, and HOSVD/unimodal-ICA methods, respectively. To provide a good reference, we also present a simple stack on the initial three-mode dataset \( \mathcal{Y} \). We conclude from these stacks that the estimated waves given by the HOSVD/unimodal-ICA subspace method are the closest to the original signal components.

To quantitatively compare the three subspace methods used, an error criterion is used. Here, we use the relative mean-square error \( \varepsilon_{\text{Met}} \) between the original signal subspace \( \mathcal{S} \) and the estimated signal subspace \( \mathcal{Y}_{\text{Met}} \) given by the method \( \text{Met} \) (with \( \text{Met} = \text{SVD, HOSVD, HOSVD/unimodal-ICA} \)), defined as

\[
\varepsilon_{\text{Met}} = \frac{\| \mathcal{S} - \mathcal{Y}_{\text{Met}} \|_F^2}{\| \mathcal{S} \|_F^2},
\]  

where \( \| \cdot \|_F \) is the three-mode-array Frobenius norm that is defined in the footnote at the beginning of this section.

For the SVD method, the signal subspace \( \mathcal{Y}_{\text{SVD}} \) was obtained by concatenating the two estimated signal subspaces \( \mathcal{Y}_{\text{signal}}^{1} \) and \( \mathcal{Y}_{\text{signal}}^{2} \) that are shown in Figure 7 into a three-mode dataset \( \mathcal{Y}_{\text{SVD}} \). Using the HOSVD method, \( \mathcal{Y}_{\text{signal}}^{\text{HOSVD}} \) is the signal subspace \( \mathcal{Y}_{\text{signal}} \) that is shown in Figure 8. For the HOSVD/unimodal-ICA method, the signal subspace \( \mathcal{Y}_{\text{signal}}^{\text{HOSVD/unimodal-ICA}} \) is the signal subspace \( \mathcal{Y}_{\text{signal}} \) that is shown in Figure 10.

In Table 3, we list the relative errors obtained with the three presented subspace methods. We use the HOSVD/unimodal-ICA subspace method to obtain the lowest error. It is 3.75 times smaller than the error given by HOSVD, and seven times smaller than the error given by component-wise SVD. The ICA step enhances the wave separation results, and it minimizes the error on the estimated signal subspace.

### APPLICATION TO REAL DATA

The described subspace methods also were applied to real vertical-seismic-profile (VSP) geophysical data. This 3C dataset is composed of \( N_x = 42 \) sensors, each of which is made up of \( N_c = 3 \) geophones that record in three directions in the 3D space: \( X, Y, \) and \( Z \). The recording time is 880 ms, corresponding to \( N_t = 220 \) time samples. The depth sampling is 10 m. Figure 13 shows the dataset obtained after the preprocessing step (velocity correction based on direct downgoing wave). The representation of the \( Z \) component was scaled by 8 to obtain the same amplitude range.

To qualitatively compare the results obtained by the three subspace methods, in Figure 14 we present the stacks for the \( X, Y, \) and \( Z \) components on the initial three-mode dataset \( \mathcal{Y} \) and on the estimated signal subspaces given by the SVD, HOSVD, and HOSVD/unimodal-ICA methods, respectively. As in the simulation case, normalization of each trace for each component was done before applying the described subspace methods, and denormalization of signal subspaces was done after their estimation.

For the SVD subspace method, we keep only one singular vector for each seismic section constructed from the original dataset \( \mathcal{Y} \), in regard to an abrupt change of slope after the first singular value.

While using the HOSVD subspace method (Table 1), a (2,1,1)-rank truncation is used to define the signal subspace. The choice of

### Table 3. Errors associated with the three subspace methods.

<table>
<thead>
<tr>
<th>Method</th>
<th>SVD</th>
<th>HOSVD</th>
<th>HOSVD/unimodal-ICA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error ( \varepsilon_{\text{Met}} )</td>
<td>51.98%</td>
<td>27.43%</td>
<td>7.31%</td>
</tr>
</tbody>
</table>

![Figure 12. Stacks on the two components for 1 original signal subspace \( \mathcal{S} \); 2 initial three-mode dataset \( \mathcal{Y} \); and 3 SVD-, 4 HOSVD-, and 5 HOSVD/unimodal-ICA-estimated signal subspaces, respectively.](image1)

![Figure 13. Real VSP geophysical dataset \( \mathcal{Y} \): (a) \( X \), (b) \( Y \), and (c) \( Z \) components.](image2)
two singular vectors for the polarization mode is prompted here by the elliptical polarization of the aligned downgoing wave. Note that for a linear polarization, only one singular vector on the polarization mode is needed. For the other two modes, the choice is made by finding an abrupt change of slope after the first singular value.

The ICA step was applied here on the first \( R = 8 \) estimated waves \( \tilde{u}_{ij} \) shown in Figure 15a. As suggested, \( R \) becomes a parameter while using real data. But anyway, the estimated waves \( \tilde{u}_{ij} \) shown in Figure 15b are more realistic (shorter wavelet and no side lobes) than those obtained without ICA. This step enhances the wave-separation results, implying a minimization of the error on the estimated signal subspace.

The signal subspace using the HOSVD/unimodal-ICA subspace method (Table 2) was obtained by a (2,1,1)-rank truncation. This choice was prompted by the elliptical polarization for the first mode and the abrupt change of slope after the first singular value for the other two modes.

The results shown in Figure 14 suggest that the 3D subspace methods are more robust to noise than are the component-wise techniques because they exploit the relationship between components directly in the process. Also, the fourth-order-independence constraint of the estimated waves enhances the wave separation results and minimizes the error on the estimated signal subspace.

These results emphasize the potential of the HOSVD/unimodal-ICA subspace method for vector-sensor-array signal processing.

We can note here that when applying normal-moveout (NMO) correction, frequency distortion (stretching) can occur. Classically, to avoid severe damage, a mute zone is applied on the stretched zone. The three-mode subspace decompositions presented here can manage any remaining distortion by taking more singular values for the third (temporal) mode to express the signal subspace.

**CONCLUSION**

We have proposed a new subspace method for multicomponent seismic-dataset processing. Our technique is based on a three-mode data model for signals that are recorded on vector-sensor arrays. This model takes into account an intrinsic physical quantity of waves — the polarization — while simultaneously processing the whole dataset.

Compared to component-wise techniques that are based on matrix algebra, the proposed technique has been shown to be more robust to noise because it exploits the relationship between components directly in the process. Multilinear tools are needed, though, when processing vector-sensor signals, and HOSVD has been used here. This decomposition allows the projection of the dataset onto two multilinear subspaces: the signal and noise subspaces.

A unimodal-ICA step also has been added to the subspace separation technique. It is performed on the temporal mode of the dataset, and it leads to a fourth-order-independence constraint of the estimated waves. This additional step in the process enhances the wave-separation results and minimizes the error on the estimated signal subspace. Note that this step is not necessary, but it can enhance the SNR and improve estimation of waves.

The proposed technique has been applied to synthetic and real data. As shown, the proposed approach fits better physically with the data structure of signals that are recorded on vector-sensor arrays, and it should be preferred to the classic matrix and long-vector approaches.
REFERENCES


