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The Theory of Magnetotelluric Methods When the Source Field Is Considered

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Abstract. The theory of the relationship between the tangential components of **E** and **H** for geomagnetic fluctuations over a stratified earth is extended to take account of the distribution of the ionospheric inducing field. It is shown that Cagniard's simple formulas on which magnetotelluric methods are generally based need modification to take account of the dimensions of this field. This is so even when the inducing field is much more extensive than the region under consideration and when the depth of the probe is quite moderate. It is further shown that, for deep probing, magnetotelluric methods can be satisfactorily applied only if an analysis of the field over a region having dimensions comparable with those of the inducing field is first made. The relation between these methods and the earlier methods of determining the conductivity distribution from analyses of the components of the surface magnetic field is discussed. The evaluation of the amplitude and phase relations of **E** and **H** over the oceans is also discussed, and it is shown that some results obtained recently by Fonarev need extending and amending.

1. Introduction

The basic ideas underlying magnetotelluric methods of prospecting were presented in a classic paper by *Cagniard* [1953], in which he investigated the amplitude and phase relations that hold between the horizontal components of **E** and **H** when a uniform oscillating electromagnetic field exists at the surface of a stratified earth. He indicated how these relations depended on the electrical conductivities of the strata and the period of the oscillating field, and he suggested that new methods of prospecting could be based on the measurements of the horizontal components of **E** and **H** over a range of frequencies of natural geomagnetic oscillations.

These methods have been developed in recent years and applied extensively to geophysical explorations of limited areas [*Wait*, 1954; *Tikhonov and Shakhsuvarov*, 1956; *Garland*, 1960; *Cantwell and Madden*, 1960; *Niblett and Sayn-Wittgenstein*, 1960; *Smith, Provazek, and Bostick*, 1961]. They have also been used by Tikhonov and others in attempts to infer the distribution of conductivity down to depths as great as 950 km; some of the results obtained for these great depths are quoted by *Migaux, Astier, and Revol* [1960]. These results do not, however, agree very well with those obtained by spherical harmonic analyses of magnetic variations and associated studies of currents induced in nonuniform spherical conductors [*Lahiri and Price*, 1939]. It is the purpose of the present paper to examine the basic theory of magnetotelluric methods more closely to see whether an explanation of the discordant results is to be found therein.

One notable feature of magnetotelluric methods is that they apparently indicate the conductivity distribution with depth from measurements (over a suitable range of frequencies of the oscillations) made at one station only, whereas the magnetic variations method requires the analysis of the magnetic field over the entire earth, or over a suitable portion of it if the inducing field is a local one. This suggests the desirability of considering the influence of the distribution of the source field when using magnetotelluric methods.

Apart from a short but important discussion by *Wait* [1954], little attention appears to have been paid to a fundamental assumption on which Cagniard's calculations of the amplitude and phase relations of **E** and **H** are based, namely, that the electromagnetic field is uniform over any horizontal plane. Wait showed that, if this condition is not satisfied, the simple relations found by Cagniard are not exact, and he calculated corrections to those relations in terms of second-order space derivatives of the field. He concluded that the corrections would be necessary

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if the horizontal magnetic field changed appreciably in a distance of 35 km, when the period (T) of the oscillations was greater than about 10 seconds and the ground conductivity (σ) of the order of 10^{-3} mho/m (10^{-14} emu). (The ground conductivity is given in Wait's paper as 10^{-1} mho/m, but this is apparently a misprint.) The relevant distance varies as $\sqrt{(T/\sigma)}$. Wait pointed out that, if the source field arises from ionospheric currents flowing at heights of about 100 km, the corrections would be important for many of the natural geomagnetic oscillations that might be used in magnetotelluric methods. We may note in particular that this would be true for the fluctuations often observed in the fields of auroral or equatorial jets.

Cagniard [1953], in a reply to Wait's discussion, argued that 'most magnetotelluric perturbations are generated by vast systems of ionospheric electric currents whose dimensions are on a global scale,' and that consequently his original formulas can be applied in the great generality of cases. This conclusion would be fairly reasonable if the actual effect of the source field distribution were limited to that disclosed by Wait's discussion, but it seems that Wait himself made a simplifying assumption that causes his calculated effect to be smaller than the true one in certain cases. He assumed that for the purpose of estimating the corrections to Cagniard's formulas the earth could be treated as a semi-infinite conductor of uniform conductivity. We find that, if the conductivity is assumed to vary with depth from the surface, the dimensions and distribution of the inducing field cannot be ignored, even when the field is on a global scale and the depths being probed are quite moderate.

2. The General Theory of Magnetotelluric Methods

We first develop the general theory of magnetotelluric methods for any source field and any distribution of conductivity with depth. We then illustrate the importance of the spatial dimensions of the source field by considering a simple example.

It will be sufficient for the present purpose to treat the earth as a semi-infinite conductor occupying the half-space $z > 0$ of Cartesian coordinates, z being vertically *downward* from the surface, though for the application of magnetotelluric methods to deep probing (to depths that are a significant fraction of the earth's radius) it will be necessary to develop the corresponding theory for a sphere. The conductivity σ is a function of z only, and for simplicity we take the permeability μ to be unity. We assume that a varying magnetic field of arbitrary distribution in the region $z < 0$ induces electric currents in the conductor. We must find the tangential components of the electric and magnetic fields at the surface $z = 0$.

The theory of magnetotelluric probing is frequently expressed in terms of concepts drawn from wave propagation theory and transmiss[illegible] line analogies. For example, Cagniard's basic formulas can be derived by considering plane-polarized electromagnetic waves incident normally on the surface of the earth; the required complex ratio E/H is then simply the field impedance at the surface. This can be found readily by using the transmission and reflection coefficients at the boundaries between the different strata. Some caution is needed, however, in drawing deductions from the physical picture provided by real wave propagation. An arbitrary oscillating magnetic field at the earth's surface cannot, in general, be built up physically from plane-polarized waves incident at all (real) angles α on the surface, though such a field can be built up *mathematically* if the angles α can take complex values. But then we no longer have real waves in the physical sense but only the so-called evanescent waves. Physically we are, in fact, concerned not with electromagnetic wave propagation but with the diffusion of the electromagnetic field through the conductor. The problem may be treated as one in pure diffusion if the displacement current in the conductor is negligible compared with the conduction current. This condition is certainly satisfied when the geomagnetic oscillations considered have periods greater than 1 second, and the oscillations used in magnetotelluric methods generally have periods much longer than this. Hence, in developing the general theory of magnetotelluric probing it is convenient to ignore the displacement current from the start.

The field equations (in electromagnetic units) for a periodic field, when the displacement current is ignored and the permeability is unity, are

$$\text{curl}\ \mathbf{H} = 4\pi \mathbf{i} = 4\pi\sigma(z)\mathbf{E} \tag{1}$$

$$\text{curl}\ \mathbf{E} = -i\omega\mathbf{H} \tag{2}$$

where all the field vectors contain the same time factor $e^{i\omega t}$.

Taking the curl of (2), we obtain .

$$\operatorname{grad} \operatorname{div} \mathbf{E} - \nabla^2\mathbf{E} = -4\pi i\omega\sigma(z)\mathbf{E} \tag{3}$$

It follows from (1) that div **i** is zero, and therefore

$$\sigma(z)\{\partial E_x/\partial x + \partial E_y/\partial y + \partial E_z/\partial z\} + E_z\, \partial\sigma/\partial z = 0 \tag{4}$$

Now the currents induced in the conductor by a varying external magnetic field necessarily flow parallel to the surface of the conductor. This is because the layers of equal conductivity are parallel to the surface [*Lahiri and Price*, 1939]. Current flows having a component normal to the surface are, of course, possible, but they cannot be produced or affected by electromagnetic induction from outside the conductor (*Price*, 1950].

It follows that in the conductor

$$i_z = 0 \qquad E_z = 0 \tag{5}$$

and therefore, from (4),

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0 \tag{6}$$

and from (3)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\mathbf{E} = 4\pi i\omega\sigma(z)\mathbf{E} \tag{7}$$

We shall now show that the solution of the general problem, corresponding to an inducing magnetic field of arbitrary distribution, can be built up from elementary solutions of (7) of the form

$$\mathbf{E} = e^{i\omega t}Z(z)\mathbf{F}(x, y) \tag{8}$$

in which

$$\mathbf{F}(x, y) = \left(\frac{\partial P}{\partial y}, \frac{\partial P}{\partial x}, 0\right) \tag{9}$$

in virtue of (5) and (6). Equation 7 then shows that

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \nu^2 P = 0 \tag{10}$$

and

$$\frac{\partial^2 Z}{\partial z^2} = \{\nu^2 + 4\pi i\omega\sigma(z)\}Z \tag{11}$$

where ν is a constant.

$= 2\pi/\lambda$ [illegible]

Hence the electric field inside the conductor is given by

$$\mathbf{E} = e^{i\omega t}Z(z)\left(\frac{\partial P}{\partial y}, \frac{-\partial P}{\partial x}, 0\right) \tag{12}$$

where P and Z satisfy (10) and (11).

From (2) we find that the corresponding magnetic field is given by

$$i\omega\mathbf{H} = -e^{i\omega t}\left(\frac{dZ}{dz}\frac{\partial P}{\partial x}, \frac{dZ}{dz}\frac{\partial P}{\partial y}, \nu^2 ZP\right) \tag{13}$$

Outside the conductor, where $\sigma = 0$, the solution of (11) is of the form

$$Z = \mathcal{A}e^{-\nu z} + \mathcal{B}e^{\nu z} \tag{14}$$

and, since the tangential components of **E** are continuous at the boundary $z = 0$, the function $Z(z)$ of (12) must have the surface value

$$Z(0) = \mathcal{A} + \mathcal{B}$$

The above implies that E_z is zero outside the conductor as well as inside, for if we assume that in the dielectric

$$\mathbf{E} = e^{i\omega t}Z(z)(F_x, F_y, F_z)$$

and there is no space-charge distribution, so that

$$\operatorname{div} \mathbf{E} = 0$$

then $Z\, \partial F_x/\partial x + Z\, \partial F_y/\partial y + F_z\, \partial Z/\partial z = 0$. But the tangential components of **E** are continuous at $z = 0$, so that

$$\partial F_x/\partial x = -\partial F_y/\partial y = \partial^2 P/\partial x\, \partial y \tag{15}$$

and therefore $F_z\, \partial Z/\partial z = 0$, whence $F_z = 0$.

This raises the question whether the assumed elementary solutions represented by (8) are sufficiently general to permit the solution for any inducing field to be found from them (see section 3), since such a field may not have E_z zero. The currents induced in the conductor depend, however, only on the varying *magnetic* field, and any magnetic inducing field can always be represented by a suitable current system flowing in a plane parallel to $z = 0$. Such a current system would make $E_z = 0$ everywhere; we may conclude that a nonzero value of E_z in the actual source field would not affect the result.

If the actual source field has an oscillating normal component of **E** at the surface, it will induce a surface-charge distribution whose field will practically extinguish E_z inside the conductor and double it outside. This is not exactly true, because oscillating currents having a nonzero normal component are required to vary the surface charge. These currents, however, are of the order of $1/(3.10^{10})$ times the magnetically induced currents flowing parallel to the surface and are therefore negligible. It follows from the above that the tangential components of **E** and **H** at the surface will be practically unaffected by E_z, which can therefore be taken as zero without loss of generality. It may be noted in passing that the tangential component of **E** is mainly determined by the conductivity and will in general be much smaller than the tangential component of **E** arising from the source alone.

The magnetic field outside the conductor is given by substituting the value of Z from (14) into (13), which can then be reduced to the form

$$i\omega\mathbf{H} = -\nu \operatorname{grad} \{(-\mathcal{A}e^{-\nu z} + \mathcal{B}e^{\nu z})P(x, y, \nu)\} \quad (16)$$

Hence the scalar potential of the magnetic field in the nonconducting region $z > 0$ is

$$\Omega = (Ae^{-\nu z} + Be^{\nu z})P(x, y, \nu) \quad (17)$$

where

$$A = -\mathcal{A}\nu/i\omega \qquad B = \mathcal{B}\nu/i\omega \quad (18)$$

If ν is now taken as real and positive, the term involving $Ae^{-\nu z}$ in (17) corresponds to a source in the region $z < h < 0$ and therefore represents an inducing field, whereas the term involving $Be^{\nu z}$ represents the field of the induced currents.

Since the tangential components of **H** are continuous at $z = 0$, we have from (13), (16), and (18)

$$\left(\frac{\partial Z}{\partial z}\right)_{z=0} = \nu(-\mathcal{A} + \mathcal{B}) = i\omega(A + B) \quad (19)$$

The other boundary condition for **H** at $z = 0$, namely, the continuity of the normal components of **H**, is already satisfied in virtue of (15).

Since the field originates from a varying magnetic source in the region $z < h < 0$, all the field vectors must tend to zero as z tends to infinity. Hence the appropriate solution of equation 11 for $Z(z)$ is the one that satisfies the condition

$$Z(z) \to 0 \quad \text{as} \quad z \to \infty \quad (20)$$

Solutions of (11) satisfying (20) can be found in analytic form for several different functions $\sigma(z)$; for example, there are solutions in terms of Bessel functions when $\sigma(z) = k$ (a positive constant), $\sigma(z) = kz^{-2}$, and $\sigma(z) = ke^{\pm\alpha z}$. In the general case, when $\sigma(z)$ is an arbitrary positive real function of z (not necessarily continuous), the solution can be obtained to any degree of approximation by numerical methods. The values of **E** and **H** at all points can then be found in terms of the inducing field represented by the first term of (17).

It will be noted that the solution for this particular inducing field (corresponding to any one value of ν) makes $\mathbf{E}\cdot\mathbf{H} = 0$; that is, the field vectors are orthogonal. This is one of the necessary conditions for the application of Cagniard's method in its original form. It has, however, already been pointed out by *Wait* [1954] that, for an arbitrary inducing field, this condition is not in general satisfied. This point is considered further in a later section.

The amplitude and phase relations between the tangential components of **E** and **H** at the surface, which form the basis of the magnetotelluric methods, are found from the complex ratios E_x/H_y or E_y/H_x, and from (12) and (13) we find that

$$E_x/i\omega H_y = -E_y/i\omega H_x = -Z(0)/(\partial Z/\partial z)_0$$
$$= (\mathcal{A} + \mathcal{B})/\nu(\mathcal{A} - \mathcal{B}) \quad (21)$$

Since $Z(z)$ depends on the parameter ν through equations 11, it is evident that the ratio E_x/H_y also depends on ν. The reciprocal of this parameter is a measure of the horizontal scale of the source field, represented by the term $Ae^{-\nu z} P(x, y, \nu)$ in expression 17 for the potential Ω of the magnetic field outside the conductor. A simple example is $Ae^{-\nu z} \cos \nu x$, corresponding to a field having a wavelength $2\pi/\nu$ in the x direction. The case treated by Cagniard is obtained by taking $\nu = 0$, $P = 1$. This corresponds to a uniform oscillating field parallel to the surface of the conductor. Actually, the problem of finding the induced field (and the corresponding induced currents) becomes indeterminate if the inducing field is of this simple form, but if the *total* field (inducing plus induced) is assumed to have this form a determinate problem results and, for a uniform conductor, gives the well-known formula for the 'skin effect' as well as the result quoted by Cagniard for

E_x/H_y. The result can be regarded as giving the approximate value of E_x/H_y over a limited region of the surface, of linear dimensions L, where $L\nu/2\pi$ is small, and the vertical component of the total field is nearly zero over this region. It is important to notice that, without further information about the field outside this region, it is not possible to separate the induced and inducing fields. In any actual physical situation the induced currents will flow in closed horizontal loops and will be determined by the electromotive forces generated by the changing *vertical* component of the inducing field. If this were zero everywhere, no currents would be induced.

It is sometimes argued that, if the surface vectors E_x and H_y are practically uniform over the limited area of the earth's surface being investigated, their amplitude ratio and phase difference depend only on the conductivity of the earth strata, and it is therefore unnecessary to know the nature of the inducing field over a wider area in order to determine the conductivity distribution. The argument is based on the fact that E_x depends on the surface values of the current density and conductivity, while H_y is a measure of the integrated current density throughout the entire depth of the conductor, being given by

$$H_y = 4\pi \int_0^\infty J_x \, dz$$

[*Cagniard,* 1953, p. 608], from which it has been assumed that H_y is due solely to the currents in the conductor. But this assumption is unjustified, because Cagniard's derivation of the above expression for H_y depends on taking the field to be strictly uniform over the entire horizontal surface, and this in turn implies that the inducing field is similarly uniform. Such a field will not decrease as z increases to infinity but will remain constant. Hence, although the *total* H_y tends to zero as z tends to infinity, it does so because the induced field becomes equal and opposite to the inducing field. Hence the H_y in Cagniard's formula contains an (unknown) contribution from the inducing field. Of course, in any real situation the inducing field will tend to zero as z tends to infinity, but then the field will not be strictly uniform and will contain a vertical component. No matter how small this component is, it will render Cagniard's expression for H_y invalid.

3. The General Inducing Field

The equations of the last section give the solution for the particular case when the potential of the source field has the form $Ae^{-\nu z} P(x, y, \nu)$. The general solution corresponding to an arbitrary source field can be derived by summation from this, because the potential of any such field can always be expressed as a sum (or integral) of terms of this type, summation of the solutions being permissible because all the equations are linear. For example, the field of a line current of intensity $Je^{i\omega t}$, flowing parallel to the surface of the conductor along $z = -h$, $x = 0$ can be written

$$\Omega_e = -2Je^{i\omega t} \int_0^\infty e^{-\nu(z+h)} \sin \nu x \frac{d\nu}{\nu} \qquad z > -h \qquad (22)$$

corresponding to a summation of terms of the above form in which $P(x, y, \nu) = \sin \nu x$ and $A = -2Je^{\nu h}\, d\nu/\nu$.

In the general case it is convenient to express equation 10 in polar coordinates in the form

$$\frac{\partial^2 P}{\partial r^2} + \frac{1}{r}\frac{\partial P}{\partial r} + \frac{1}{r^2}\frac{\partial^2 P}{\partial \theta} + \nu^2 P = 0 \qquad (23)$$

of which elementary solutions are $J_s(\nu r) \cos (s\theta + \xi)$, where s is any integer. Any arbitrary function Ω_0, given over the plane of (r, θ), can then be expressed in terms of these functions by means of Fourier's theorem and the Fourier-Bessel integral.

Where the inducing field involves only a single value of ν, it has been seen that the horizontal components of **E** and **H** are orthogonal. This is not, however, true in general when a range of values of ν is involved, though it can be true in special cases such as that of the straight line current represented by (22). If the conductivity of the earth is isotropic, the angle between **E** and **H** is determined by the inducing field. Deviations from perpendicularity of the **E** and **H** vectors that are found in practical applications of the method have usually been attributed to anisotropic conductivity of the ground, but they could also arise from the nature of the distributions of the inducing field. Consider, for example, the field whose potential is

$$\Omega_e = (A_1 e^{-\nu_1 z} \cos \nu_1 x + A_2 e^{-\nu_2 z} \cos \nu_2 y)e^{i\omega t} \qquad (24)$$

The induced field will then have a potential

$$\Omega_i = (B_1 e^{\nu_1 z} \cos \nu_1 x + B_2 e^{\nu_2 z} \cos \nu_2 y) e^{i\omega t} \quad (25)$$

where B_1 and B_2 are obtained from the equations of the previous section. The horizontal components of **H** at $z = 0$ are therefore

$$\begin{aligned} H_x &= \nu_1(A_1 + B_1) \sin \nu_1 x \\ H_y &= \nu_2(A_2 + B_2) \sin \nu_2 y \end{aligned} \quad (26)$$

and the corresponding horizontal components of **E** are

$$\begin{aligned} E_x &= i\omega(A_2 - B_2) \sin \nu_2 y \\ E_y &= -i\omega(A_1 - B_1) \sin \nu_1 x \end{aligned} \quad (27)$$

Hence

$$E_x H_x + E_y H_y = i\omega\{\nu_1(A_1 + B_1)(A_2 - B_2) - \nu_2(A_1 - B_1)(A_2 + B_2)\} \sin \nu_1 x \sin \nu_2 y \quad (28)$$

and therefore the horizontal components of **E** and **H** are perpendicular if (1) ν_1 or ν_2 is zero, (2) $\nu_1 x$ or $\nu_2 y$ is a multiple of π, or (3) $\nu_1 = \nu_2$ (because then $B_2/A_2 = B_1/A_1$).

This suggests that apart from (1) when the source field is unidirectional, or (2) when the point (x, y) has a special position in relation to the source field, the field vectors will approach perpendicularity if the band width of ν for the inducing field is small. This is probably true for many world-wide geomagnetic fluctuations having a definite time period $2\pi/\omega$. Even when it is not, as for a jet current, represented approximately by (22), the vectors may be perpendicular for other reasons already noted.

Values of ν for geomagnetic fields. It has already been pointed out that the general equations obtained above reduce to Cagniard's equations when $\nu = 0$, and that $2\pi/\nu$ may be taken as a measure of the linear dimensions of the source field. Cagniard has argued that, since the geomagnetic fields are generally world-wide in character, their spatial variations can be ignored for local geophysical prospecting; this is equivalent to taking $\nu = 0$. On the face of it this would seem reasonable, but calculations show that the dimensions of the source field may be quite important in certain circumstances. To obtain an estimate of the least value of ν that can occur, we may equate $2\pi/\nu$ to the circumference of the earth, which would correspond to the wavelength of a field represented in polar coordinates by a spherical harmonic of the first order; this gives $\nu = 1.57 \times 10^{-9}$ cm^{-1}. The fields of most geomagnetic fluctuations would contain spherical harmonics of an order somewhat higher than the first, and $\nu = 10^{-8}$ cm^{-1} would probably be a good representative value for many such fields.

For more local fields, like the field of an ionospheric jet current, the relevant values of ν are larger. The greatest value of ν likely to be of importance may be found by equating the wavelength $2\pi/\nu$ to, say, 4 times the height (about 100 km) of the ionospheric currents; this gives a maximum for ν of about 1.57×10^{-7} cm^{-1}. For example, in the expression 22 for the potential of the field of a straight jet current at $z = -h$, although all values of ν from 0 to infinity occur in the integration, the magnitude of the integrand drops off sharply when ν increases beyond $\pi/(2h)$ because of the factor $e^{-\nu h}/\nu$. Hence the important values of ν are those less than $\pi/(2h)$. We may conclude that the values of ν of interest in magnetotelluric investigations will generally lie somewhere in the range 1.57×10^{-9} cm^{-1} to 1.57×10^{-7} cm^{-1}.

4. The Variation of E_x/H_y with ν in a Simple Case

We shall now show that for certain distributions of ground conductivity the quantity E_x/H_y used in magnetotelluric methods will be considerably affected by the value of ν assumed for the inducing field and may be very different from that obtained when ν is assumed zero.

A simple illustrative case is that in which the conductivity has a constant value σ down to a certain depth D and is zero below this depth. This will represent very roughly a common actual situation, since it is probable that the conductivity of the earth at a depth of a few kilometers is much smaller than at the surface (though at considerably greater depths it may rise again, owing to increase of temperature).

For this simple case the function $Z(z)$ of equation 11 in the region $0 > z > D$ is easily found to be

$$Z = ae^{-\theta z} + be^{\theta z} \quad (29)$$

where a and b are constants and

$$\theta^2 = \nu^2 + 4\pi i\omega\sigma \quad (30)$$

and therefore

$$\theta = 2^{-1/2}[\{(\alpha^4 + \nu^4)^{1/2} + \nu^2\}^{1/2} + i\{(\alpha^4 + \nu^4)^{1/2} - \nu^2\}^{1/2}] \qquad (31)$$

where $\alpha^2 = 4\pi\sigma\omega$. In the region $z < 0$, Z is given by (14); and in the region $z > D$, Z is of the form

$$Z = \mathcal{C}e^{-\nu z} \qquad (32)$$

The field vectors $\mathbf{E}$, $\mathbf{H}$ in each of the three regions can then be written from (12) and (13).

The tangential components of $\mathbf{E}$ and all the components of $\mathbf{H}$ are continuous at each of the boundaries $z = 0$ and $z = D$. These boundary conditions lead to the relations

$$a + b = \mathcal{A} + \mathcal{B} \qquad (33)$$

$$ae^{-\theta D} + be^{\theta D} = \mathcal{C} \qquad (34)$$

$$\nu(\mathcal{A} - \mathcal{B}) = \theta(a - b) \qquad (35)$$

$$\theta(ae^{-\theta D} - be^{\theta D}) = \nu\mathcal{C} \qquad (36)$$

If the inducing field were known, $\mathcal{A}$ would be known, and the above four equations would determine the other coefficients $\mathcal{B}$, $\mathcal{C}$ a, and b in terms of $\mathcal{A}$; the total field could then be evaluated everywhere.

For our purpose we need only the ratio E_x/H_y. From equations 21, 33, and 35 we have

$$\frac{E_x}{i\omega H_y} = -\frac{E_y}{i\omega H_x} = \frac{\mathcal{A} + \mathcal{B}}{\nu(\mathcal{A} - \mathcal{B})} = \frac{a + b}{\theta(a - b)} \qquad (37)$$

Eliminating $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ from equations 33 to 36 we find

$$b = ae^{-2\theta D}(\theta - \nu)/(\theta + \nu) \qquad (38)$$

Hence

$$\frac{E_x}{i\omega H_y} = \frac{\theta + \nu + (\theta - \nu)e^{-2\theta D}}{\theta\{\theta + \nu - (\theta - \nu)e^{-2\theta D}\}} \qquad (39)$$

The modulus and argument of the complex quantity E_x/H_y given by this expression are equal to the amplitude ratio and phase difference of E_x and H_y. In the expression, ν and D are real but θ is the complex function given by (31). In considering the dependence of the ratio E_x/H_y on ν, it should be noted that, apart from ν appearing explicitly on the right-hand side of (39), the quantity θ also depends on ν.

When $\nu = 0$, (39) reduces to

$$\frac{E_x}{i\omega H_y} = \frac{1 + e^{-2D\theta_0}}{\theta_0(1 - e^{-2D\theta_0})} \qquad (40)$$

where

$$\theta_0 = (1 + i)\sqrt{(2\pi\omega\sigma)} \qquad (41)$$

which is equivalent to result 44 given by *Cagniard* [1953]. For $D\theta_0$ small, the right-hand side of (40) reduces to $1/\theta_0^2 D$ approximately, so that

$$E_x/H_y \sim 1/4\pi\sigma D \qquad (42)$$

showing that E_x/H_y is then practically independent of the period $T = 2\pi/\omega$.

Cagniard obtains a corresponding result, his equations 45, in the form

$$\frac{E_x}{H_y} = \frac{1}{2\sqrt{(\sigma T)}}\frac{P_1}{D} \qquad \theta = 0$$

Here θ is the phase difference, and P_1 the depth of penetration given by

$$P = (1/2\pi)\sqrt{T/\sigma}$$

He does not point out, however, that T disappears from the formula.

When D is infinite, that is, for a semi-infinite conductor, equation 39 reduces to

$$\frac{E_x}{i\omega H_y} = \frac{1}{\theta} = \frac{1}{\sqrt{(\nu^2 + 4\pi i\omega\sigma)}} \qquad (43)$$

Thus the dependence on ν is now only through the quantity θ. For small values of ν we have, approximately,

$$\frac{E_x}{H_y} = \sqrt{\frac{i\omega}{4\pi\sigma}}\left(1 - \frac{\nu^2}{8\pi i\omega\sigma}\right) \qquad (44)$$

Wait [1954] has obtained the correction to Cagniard's formula in this case by a quite different method. He finds

$$E_x = \eta H_y + \frac{\eta}{2\gamma^2} \cdot\left(\frac{\partial^2 H_y}{\partial y^2} - \frac{\partial^2 Hy}{\partial x^2} + 2\frac{\partial^2 Hx}{\partial x \partial y}\right) + 0(\gamma^{-4}) \qquad (45)$$

and a similar equation for E_y, where, on taking $\mu = 1$, ignoring displacement currents, and using emu,

$$\gamma^2 = 4\pi i\sigma\omega \qquad \eta^2 = i\omega/4\pi\sigma \qquad (46)$$

Wait actually writes $\gamma = i\sigma\mu\omega - \epsilon\mu\omega^2$ and $\eta = i\omega\mu/\gamma$, but in the first equation γ should read γ^2, and the term $\epsilon\mu\omega^2$ is negligible; the factor 4π is accounted for by the different units used. It may be noted that, since there are no vertical currents, we have

$$\frac{\partial H_x}{\partial y} = \frac{\partial H_y}{\partial x} \tag{47}$$

so that (45) may be reduced to the more symmetrical form

$$E_x = \eta H_y + \frac{\eta}{2\gamma^2}\left(\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2}\right) + 0(\gamma^{-4}) \tag{48}$$

To compare this with (44), we note that using (13) and (29) we can express H_y at the surface in the form

$$H_y = -(i\theta/\omega)Z(0)\,\partial P/\partial y \tag{49}$$

Substituting this value in (48) and using (10), it is easily seen that Wait's formula 48 reduces to (44). It should be emphasized, however, that this formula is based on the assumption that the conductivity has the same constant value at all depths extending to infinity. In the expression 39 for a different distribution of conductivity, Wait's correction effectively takes account only of the factor θ in the denominator. It will now be shown that the correction introduced by the other factors in this expression will sometimes be of greater importance.

For numerical illustration, calculations have been made of the modulus and argument of the expression on the right of (39) for a range of values of ν from 0 to 10^{-6} cm^{-1}, and for a range of depths D from 0 to 10^8 cm. The conductivity σ has been taken as 10^{-14} emu, and the period T ($=2\pi/\omega$) as 100 seconds. Since σ and T enter into the expression only as the ratio σ/T, it follows that the values calculated for $E_x/i\omega H_y$ will be unaltered if σ and T are altered by the same factor; for example, the calculations will apply equally well to $\sigma = 10^{-13}$ emu, $T = 10^3$ seconds. We also note from (39) that the quantity $\theta E_x/i\omega H_y$ is nondimensional, and its value will be unchanged if $\sqrt{(\sigma/T)}$ and ν are changed by a factor k while D is changed by the factor $1/k$. The calculations are therefore easily extended to other values of the parameters.

The results are shown in Figures 1, 2, and 3.

T = 100 s

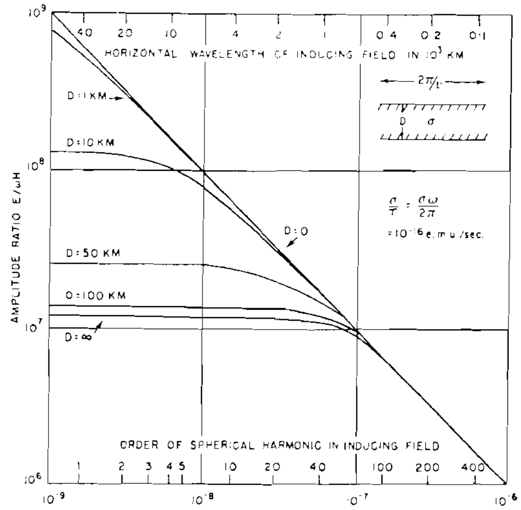

Fig. 1. Variation of amplitude ratio $E/\omega H$ with change of dimensions of inducing field, for a conducting stratum of thickness D overlying a stratum of high resistivity. $\sigma/T = \sigma\omega/2\pi = 10^{-16}$ emu/sec $= 10^{-5}$ in mks units.

In Figure 1 the value of mod $(E_x/i\omega H_y)$ is plotted against ν for various thicknesses D of the conductor. The wavelength $(2\pi/\nu)$ of the inducing field is shown at the top of the figure, and the corresponding order of the spherical harmonic for a field varying in the same degree over the earth's surface, but expressed in spherical harmonics, is shown at the bottom. It will be seen that for moderate values of D there is a considerable change in the value of the modulus as ν ranges over the values from 1.57×10^{-9} cm^{-1} to 1.57×10^{-7} cm^{-1}, which are those we expect to find in the fields of natural geomagnetic fluctuations. The values of mod $(E_x/i\omega H_y)$ for different values of D when ν is assumed zero, as in Cagniard's calculations, would be obtained by continuing the curves to infinity on the left of the diagram. (Note that the scales for both ν and mod $(E_x/i\omega H_y)$ are logarithmic. Each curve will level off at some value of mod $(E_x/i\omega H_y)$ as ν is decreased and will then remain practically constant for smaller values of ν.

Wait's corrected value corresponds to the curve labeled $D = \infty$, and his correction becomes important only when ν increases beyond 10^{-7}; that is, the wavelength $2\pi/\nu$ is less than 628 km. This agrees with Wait's own estimate if we take into account the different values we have assumed for σ and T. It is clear from the figure that, for

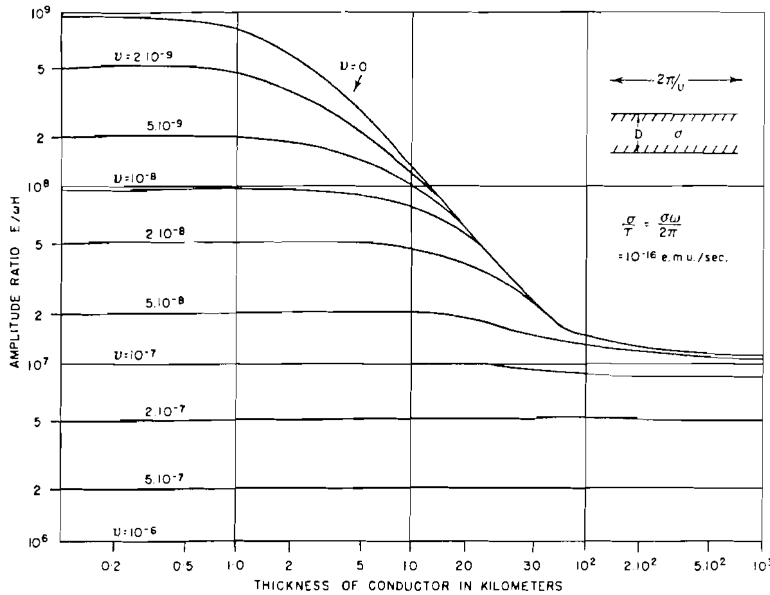

Fig. 2. Variation of amplitude ratio $E/\omega H$ with thickness D of conducting stratum for different values of ν, which determines the scale of the inducing field.

Grad T
v 2.4 'D'

geomagnetic fluctuations having a period of 100 seconds and originating from fields of large dimensions corresponding to wavelengths $2\pi/\nu$ ranging from 40,000 km to, say, 1000 km, Wait's correction would not be important.

If, however, the actual distribution of conductivity is represented more closely by the curve $D = 10$ km, that is, if the conductivity decreases sharply with depth at about 10 km, to such an extent that we can neglect it below this depth, then the graph shows that mod $(E_x/i\omega H_y)$ depends very markedly on the value of ν, and a correction to Cagniard's formula would be necessary when considering *all* geomagnetic fluctuations, including those having a global distribution.

Another way of looking at the results is to plot the values of mod $(E_x/\omega H_y)$ against D for different values of ν. This is done in Figure 2, both scales being again logarithmic. The results obtained by Cagniard now correspond to the curve $\nu = 0$, and this curve would be used to estimate the thickness of the conducting layer from a knowledge of mod $(E_x/\omega H_y)$ found from

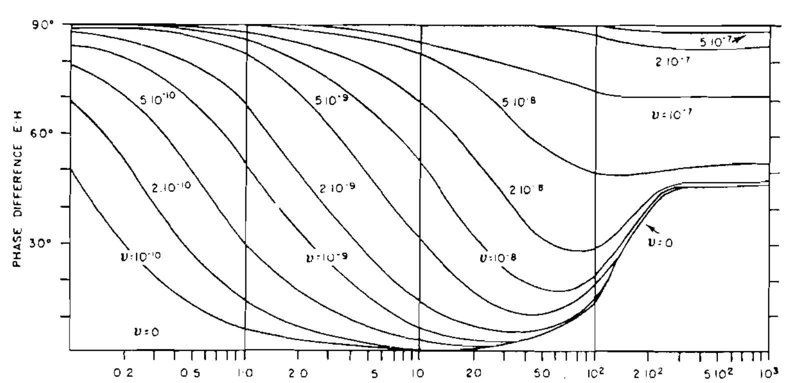

Fig. 3. Variation of phase difference $E - H$ with thickness D of conducting stratum for different values of ν.

measurements of E_x and H_y. For example, if mod $(E_x/\omega H_y)$ were found to be 5.10^7, using emu, the thickness would be estimated as about 25 km. But this estimate would be in serious error if the source field were such that $\nu = 2.10^{-8}$ cm^{-1}. In this case the thickness could not be greater than 4 km, but might have any value less than this, because the value of mod $(E_x/\omega H_y)$ would now be due almost entirely to the inducing field.

The phase difference, given by arg (E_x/H_y), is plotted against D for various values of ν in Figure 3. The importance of the linear dimensions of the field source, represented by the reciprocal of the parameter ν, is again very obvious. Only when D is greater than 300 km is the influence of ν much reduced. In this case the phase difference is always greater than 45°, but may still vary from 45° to 75° for values of ν well within the range appropriate to natural geomagnetic fluctuations (see section 3). For smaller values of D, for example, $D = 20$ km, the phase difference varies from near 0° to about 85°.

Although the above calculations have been made only for an especially simple distribution of conductivity, they are sufficient to show that the dimensions of the source field will sometimes have an important influence on both the amplitude ratio and the phase difference found between the horizontal components of **E** and **H** in magnetotelluric prospecting. This is so even when the source field has a global distribution and when only moderate depths within the earth are involved. The corrections to Cagniard's formulas to take account of this may in certain cases be considerably greater than those calculated from Wait's formula.

It is perhaps worth noting that, if both the amplitude ratio and the phase difference of **E** and **H** are known for a given ω, the value of D for the model considered above would be uniquely determined; for example, if mod $(E/\omega H)$ were 5.10^7 and arg (E/H) were 60°, D would be about 13 km. This suggests the possibility that, for a general model with σ an unknown function of z, a knowledge of both the amplitude ratios and the phase differences for a whole range of values of ω might be sufficient to determine both ν and $\sigma(z)$. In this way an extension of Cagniard's theory could be made, with a corresponding useful development of magnetotelluric methods. It would, however, be necessary to consider the practical limitations imposed by the assumption that the ground conductivity is uniform and isotropic over any horizontal plane.

5. Application to Measurements at Sea

The above figures will also represent the values of mod $(\omega E_x/H_y)$ and arg (E_x/H_y) found at the surface of a sea of depth D, if suitable changes are made in the values of the parameters and if the bottom conductivity can be ignored (or allowed for approximately by taking an effective D somewhat greater than the true value). To obtain a value of σ corresponding to the conductivity of sea water (about 3.10^{-11} emu) the original σ must be multiplied by a factor 3000. If the period T is kept the same at 100 seconds, α is increased by a factor $\sqrt{(3000)} = 55$ approximately. Hence if ν is multiplied by 55 and D divided by 55, the curves of the diagrams will give the value of 55 mod $(E_x/\omega H_y)$. The curve labeled $D = 10$ km in Figure 1 will correspond to a sea of depth rather less than 200 meters, and the figure therefore shows that for seas deeper than about 200 meters the value of mod (E_x/H_y) is scarcely affected by the dimensions of the source field unless ν is greater than about $3 \times 10^{-9} \times 55 = 1.65 \times 10^{-7}$, that is, unless the field is of a decidedly local character.

For slower geomagnetic fluctuations, corresponding to larger values of T, the dimensions of the source field become more important. Thus for $T = 86{,}400$ seconds (1 day) and the same value 3×10^{-11} emu for σ, the values of ν and of mod $(E_x/\omega H_y)$ in the diagrams must each be multiplied by $\sqrt{(30/864)} = 0.19$ approximately, and D multiplied by 5.26. Since the average depth of the oceans is near 5 km, the curve labeled $D = 1$ km in Figure 1, which now corresponds to an ocean of depth 5.26 km, would be fairly representative of the results obtained at sea for daily oscillations of the source field. This shows that the amplitude ratio of E_x to H_y and their phase difference would now be largely dependent on ν throughout the whole range of possible values of ν. In fact, it appears that the ratio will be almost entirely determined by the source field, and practically independent of the induced field. It therefore appears that the calculations made by *Fonarev* [1961] of mod (E/H) and arg (E/H) over an ocean of depth 5 km for geomagnetic oscillations of periods 3, 6, 12, and 24 hours need modifying and extending to take account of the source field.

6. The Determination of the Conductivity at Great Depths

In the previous sections we have been concerned mainly with distributions of conductivity at moderate depths within the earth. Some information about the distribution at much greater depths can be obtained from analyses of relatively long-period fluctuations of the geomagnetic field components [*Chapman and Price*, 1930; *Lahiri and Price*, 1939]. It is worth while considering whether and in what way magnetotelluric methods can be used to supplement this information [*Garland*, 1960]. To answer this question completely it would be necessary to develop the theory of magnetotelluric methods for a spherical conductor, but we can get some idea of what to expect from the equations of the present theory.

In the first place, it is clear that in order to gain information about the conditions at a great depth it is necessary that the measured surface field components be significantly affected by the induced currents flowing at that depth. Hence the induced currents must penetrate deeply, and this in turn requires that the field fluctuations have a correspondingly long period. We have already seen that, in the particular case treated in section 4, the influence of the dimensions of the source field on the amplitude ratio E_x/H_y increases as the period is made longer. In the general case the amplitude ratio and phase difference of E_x and H_y are given by (21), in which the ratio $Z(0)/(\partial Z/\partial y)_0$ is determined by (11) and (20). The problem is to decide to what extent this ratio is affected by the value of ν in (11). From equation 11 it is evident that ν will be important if ν^2 is comparable in magnitude to $4\pi\omega\sigma(z)$ over all or some appreciable part of the range of z involved. Since $\sigma(z)$ is the quantity we are seeking, we have to proceed, to some extent, by trial and error. Moreover, it is quite likely that $\sigma(z)$ varies by several orders of magnitude within the range of z we wish to consider. The values near the surface will have the greatest effect, but the solution of the problem treated in section 4 shows that it is not permissible merely to take the surface value $\sigma(0)$.

For the purpose of numerical calculation we shall adopt the representative value $\sigma = 10^{-14}$ emu with the proviso that it may be altered by one or two orders of magnitude either way. Then for a period of 1 day the value of $4\pi\omega\sigma$ is approximately 4.6×10^{-18}, so that ν will be important if it is near the value 2.15×10^{-9}. This is near the lower limit (1.57×10^{-9}) of the values of ν estimated in section 3 to be present in natural geomagnetic fields. Hence, for this value of σ and all smaller values the distribution of the inducing field would have to be taken into account in using magnetotelluric methods. Even if σ is as high as 10^{-12} emu (and the results of the magnetic field analyses already referred to suggest that this is too high until a depth of several hundred kilometers is reached), ν would be important when near 2.15×10^{-8}, which is still within the range of values expected in the natural fields.

We may conclude that the application of magnetotelluric methods to the determination of the conductivity at great depths would first require the analysis of the horizontal surface magnetic field components (H_x, H_y) over the earth to determine the distribution of the field. This shows that the methods are comparable with those based on the analysis of the magnetic components (H_x, H_y, H_z) without using E_x and E_y. The only question that remains is whether a knowledge of E_x or E_y over the earth is equivalent to a knowledge of H_z. That this is so may be inferred from equations 12 and 13, which show that

$$\frac{E_x}{H_z} = \frac{i\omega\,\partial P/\partial y}{\nu^2 P}$$

$$\frac{E_y}{H_z} = \frac{i\omega\,\partial P/\partial x}{\nu^2 P} \qquad (50)$$

so that, if the horizontal distribution of the field represented by $P(x, y, \nu)$ is known, E_x and E_y can be found from H_z. It will be noted that these relations depend only on the distribution of the field and the period $2\pi/\omega$, and that they are independent of the conductivity.

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