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# Stochastic Rank-1 Bandits

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**Sumeet Katariya**  
Department of ECE  
University of Wisconsin-Madison  
katariya@wisc.edu

**Branislav Kveton**  
Adobe Research  
San Jose, CA  
kveton@adobe.com

**Csaba Szepesvári**  
Department of Computing Science  
University of Alberta  
szepesva@cs.ualberta.ca

**Claire Vernade**  
Telecom ParisTech  
Paris, France  
claire.vernade@telecom-paristech.fr

**Zheng Wen**  
Adobe Research  
San Jose, CA  
zwen@adobe.com

## Abstract

We propose stochastic rank-1 bandits, a class of online learning problems where at each step a learning agent chooses a pair of row and column arms, and receives the product of their values as a reward. The main challenge of the problem is that the individual values of the row and column are unobserved. We assume that these values are stochastic and drawn independently. We propose a computationally-efficient algorithm for solving our problem, which we call Rank1Elim. We derive a  $O((K + L)(1/\Delta) \log n)$  upper bound on its  $n$ -step regret, where  $K$  is the number of rows,  $L$  is the number of columns, and  $\Delta$  is the minimum of the row and column gaps; under the assumption that the mean row and column rewards are bounded away from zero. To the best of our knowledge, we present the first bandit algorithm that finds the maximum entry of a rank-1 matrix whose regret is linear in  $K + L$ ,  $1/\Delta$ , and  $\log n$ . We also derive a nearly matching lower bound. Finally, we evaluate Rank1Elim empirically on multiple problems. We observe that it leverages the structure of our problems and can learn near-optimal solutions even if our modeling assumptions are mildly violated.

## 1 Introduction

We study the problem of finding the maximum entry of a stochastic rank-1 matrix from noisy and adaptively-chosen observations. This problem is motivated by two problems, ranking in the position-based model [26] and online advertising.

The *position-based model (PBM)* [26] is one of the most fundamental click models [5], a model of how people click on a list of  $K$  items out of  $L$ . Each item is associated with its *attraction* and each position in the list is associated with its *examination*. The attraction of any item and the examination of any position are i.i.d. Bernoulli random variables. The item in the list is *clicked* only if it is attractive and its position is examined. Then the item and position with the highest probability of clicking is the maximum entry of a rank-1 matrix, which is the outer product of the attraction probabilities of items and the examination probabilities of positions.

As another example, consider a marketer of a product who has two sets of actions,  $K$  population *segments* and  $L$  marketing *channels*. Given a product, some segments are *easier to market to* and some channels are *more appropriate*. Let each action affect conversions *independently*. Then the pair of the segment and channel that lead to most conversions is the maximum entry of a rank-1 matrix.

We propose an online learning model for solving our motivating problems, which we call a *stochastic rank-1 bandit*. The learning agent interacts with our problem as follows. At time  $t$ , the agent selects a pair of row and column arms, and receives the product of their individual values as a reward. The values are stochastic, drawn independently, and not observed. The goal of the agent is to maximize its expected cumulative reward, or equivalently to minimize its expected cumulative regret with respect to the optimal so-

lution, the most rewarding pair of row and column arms.

We make five contributions. First, we precisely formulate the online learning problem of *stochastic rank-1 bandits*. Second, we design an elimination algorithm for solving it, which we call Rank1Elim. The key idea in Rank1Elim is to explore all remaining rows and columns randomly over all remaining columns and rows, respectively, to estimate their expected rewards; and then eliminate those rows and columns that seem suboptimal. This algorithm is computationally efficient and easy to implement. Third, we derive a  $O((K+L)(1/\Delta)\log n)$  gap-dependent upper bound on its  $n$ -step regret, where  $K$  is the number of rows,  $L$  is the number of columns, and  $\Delta$  is the minimum of the row and column gaps; under the assumption that the mean row and column rewards are bounded away from zero. Fourth, we derive a nearly matching gap-dependent lower bound. Finally, we evaluate our algorithm empirically. In particular, we validate the scaling of its regret, compare it to multiple baselines, and show that it can learn near-optimal solutions even if our modeling assumptions are mildly violated.

We denote random variables by boldface letters and define  $[n] = \{1, \dots, n\}$ . For any sets  $A$  and  $B$ , we denote by  $A^B$  the set of all vectors whose entries are indexed by  $B$  and take values from  $A$ .

## 2 Setting

We formulate our online learning problem as a *stochastic rank-1 bandit*. An instance of this problem is defined by a tuple  $(K, L, P_U, P_V)$ , where  $K$  is the number of rows,  $L$  is the number of columns,  $P_U$  is a probability distribution over a unit hypercube  $[0, 1]^K$ , and  $P_V$  is a probability distribution over a unit hypercube  $[0, 1]^L$ .

Let  $(\mathbf{u}_t)_{t=1}^n$  be an i.i.d. sequence of  $n$  vectors drawn from distribution  $P_U$  and  $(\mathbf{v}_t)_{t=1}^n$  be an i.i.d. sequence of  $n$  vectors drawn from distribution  $P_V$ , such that  $\mathbf{u}_t$  and  $\mathbf{v}_t$  are drawn independently at any time  $t$ . The learning agent interacts with our problem as follows. At time  $t$ , it chooses arm  $(\mathbf{i}_t, \mathbf{j}_t) \in [K] \times [L]$  based on its history up to time  $t$ ; and then observes  $\mathbf{u}_t(\mathbf{i}_t)\mathbf{v}_t(\mathbf{j}_t)$ , which is also its reward.

The goal of the agent is to maximize its expected cumulative reward in  $n$  steps. This is equivalent to minimizing the *expected cumulative regret* in  $n$  steps

$$R(n) = \mathbb{E} \left[ \sum_{t=1}^n R(\mathbf{i}_t, \mathbf{j}_t, \mathbf{u}_t, \mathbf{v}_t) \right],$$

where  $R(\mathbf{i}_t, \mathbf{j}_t, \mathbf{u}_t, \mathbf{v}_t) = \mathbf{u}_t(i^*)\mathbf{v}_t(j^*) - \mathbf{u}_t(\mathbf{i}_t)\mathbf{v}_t(\mathbf{j}_t)$  is the *instantaneous stochastic regret* of the agent at time  $t$  and

$$(i^*, j^*) = \arg \max_{(i,j) \in [K] \times [L]} \mathbb{E} [\mathbf{u}_1(i)\mathbf{v}_1(j)]$$

is the *optimal solution* in hindsight of knowing  $P_U$  and  $P_V$ . Since  $\mathbf{u}_1$  and  $\mathbf{v}_1$  are drawn independently, and  $\mathbf{u}_1(i) \geq 0$

for all  $i \in [K]$  and  $\mathbf{v}_1(j) \geq 0$  for all  $j \in [L]$ , we get that

$$i^* = \arg \max_{i \in [K]} \mu \bar{u}(i), \quad j^* = \arg \max_{j \in [L]} \mu \bar{v}(j),$$

for any  $\mu > 0$ , where  $\bar{u} = \mathbb{E} [\mathbf{u}_1]$  and  $\bar{v} = \mathbb{E} [\mathbf{v}_1]$ . This is the key idea in our solution.

Note that the problem of learning  $\bar{u}$  and  $\bar{v}$  from stochastic observations  $\{\mathbf{u}_t(\mathbf{i}_t)\mathbf{v}_t(\mathbf{j}_t)\}_{t=1}^n$  is a special case of *matrix completion from noisy observations* [15]. This problem is harder than that of learning  $(i^*, j^*)$ . The most popular approach to matrix completion [17] is alternating minimization of a non-convex function, where the noise is assumed to be Gaussian. In comparison, our algorithm in Section 4 learns the optimal solution  $(i^*, j^*)$  with a high probability, and does not make any strong assumptions on  $P_U$  and  $P_V$ .

## 3 Naive Solutions

Our learning problem is a  $KL$ -arm bandit with  $K+L$  parameters,  $\bar{u} \in [0, 1]^K$  and  $\bar{v} \in [0, 1]^L$ . The main challenge is to leverage this structure to learn efficiently. In this section, we discuss the challenges of solving our problem by existing algorithms. We conclude that a new algorithm is necessary and present it in Section 4.

Rank-1 bandits can be formulated as a multi-armed bandit with  $KL$  arms, where each pair  $(i, j)$  is treated as a separate arm, and then solved by UCB1 [2]. The  $n$ -step regret of UCB1 in our problem is  $O(KL(1/\Delta)\log n)$ . Therefore, UCB1 is impractical when both  $K$  and  $L$  are large.

Note that  $\log(\bar{u}(i)\bar{v}(j)) = \log(\bar{u}(i)) + \log(\bar{v}(j))$  for any  $\bar{u}(i), \bar{v}(j) > 0$ . Therefore, it seems that we can formulate our problem as a stochastic linear bandit [8, 1] and solve it by LinUCB, where the reward of arm  $(i, j)$  is  $\log(\mathbf{u}_t(i)) + \log(\mathbf{v}_t(j))$  and its features  $x_{i,j} \in \{0, 1\}^{K+L}$  are

$$x_{i,j}(e) = \begin{cases} \mathbb{1}\{e = i\}, & e \leq K; \\ \mathbb{1}\{e - K = j\}, & e > K, \end{cases} \quad (1)$$

for any  $e \in [K+L]$ . This approach is problematic for at least two reasons. First, the reward is not properly defined when either  $\mathbf{u}_t(i) = 0$  or  $\mathbf{v}_t(j) = 0$ . Second,

$$\mathbb{E} [\log(\mathbf{u}_t(i)) + \log(\mathbf{v}_t(j))] \neq \log(\bar{u}(i)) + \log(\bar{v}(j)).$$

On the other hand, both sides of the above inequality have maxima at  $(i^*, j^*)$ , and therefore LinUCB should perform well. Therefore, we compare to it in Section 6.2.

Also note that  $\bar{u}(i)\bar{v}(j) = \exp[\log(\bar{u}(i)) + \log(\bar{v}(j))]$  for any  $\bar{u}(i), \bar{v}(j) > 0$ . Therefore, we can view rank-1 bandits as generalized linear bandits [9], where the mean function is  $\exp[\cdot]$  and the feature vector of arm  $(i, j)$  is in (1); and then solve them by GLM-UCB. This approach is impractical for three reasons. First, the parameter space is unbounded,

since  $\log(\bar{u}(i)) \rightarrow -\infty$  as  $\bar{u}(i) \rightarrow 0$  and  $\log(\bar{v}(j)) \rightarrow -\infty$  as  $\bar{v}(j) \rightarrow 0$ . Second, the confidence radii in GLM-UCB are scaled by the reciprocal of the minimum derivative of the mean function  $c_\mu^{-1}$ , which can be very large in our setting. In particular,  $c_\mu = \min_{(i,j) \in [K] \times [L]} \bar{u}(i)\bar{v}(j)$ . In addition, the gap-dependent upper bound on the regret of GLM-UCB is  $O((K+L)^2 c_\mu^{-2})$ , which further indicates that GLM-UCB is not practical. Our upper bound in Theorem 1 scales much better with all quantities of interest. Third, GLM-UCB needs to compute the maximum-likelihood estimates of  $\bar{u}$  and  $\bar{v}$  at each step, which is a non-convex problem (Section 2).

Some variants of our problem can be solved trivially. For instance, let  $\mathbf{u}_t(i) \in \{0.1, 0.5\}$  for all  $i \in [K]$  and  $\mathbf{v}_t(j) \in \{0.5, 0.9\}$  for all  $j \in [L]$ . Then  $(\mathbf{u}_t(i), \mathbf{v}_t(j))$  can be identified from  $\mathbf{u}_t(i)\mathbf{v}_t(j)$ . Therefore, this learning problem is as hard as a multi-armed bandit with  $K+L$  arms. We do not focus on such degenerate cases in this paper.

#### 4 Rank1Elim Algorithm

Our algorithm, Rank1Elim, is presented in Algorithm 1. It is an elimination algorithm [3] with UCB1 confidence intervals [2] on the expected rewards of rows and columns. The algorithm operates in stages. In stage  $\ell$ , all remaining rows and columns are explored over random remaining columns and rows, respectively. At the end of the stage, Rank1Elim eliminates some rows and columns that cannot be optimal. Then a new stage starts.

The eliminated rows and columns are tracked as follows. We denote by  $\mathbf{h}_\ell^u(i)$  the index of the most rewarding row whose expected reward is believed by Rank1Elim to be at least as high as that of row  $i$  in stage  $\ell$ . Initially,  $\mathbf{h}_0^u(i) = i$ . When row  $i$  is eliminated by row  $\mathbf{i}_\ell$  in stage  $\ell$ ,  $\mathbf{h}_{\ell+1}^u(i)$  is set to  $\mathbf{i}_\ell$ ; then when row  $\mathbf{i}_\ell$  is eliminated by row  $\mathbf{i}_{\ell'}$  in stage  $\ell' > \ell$ ,  $\mathbf{h}_{\ell'+1}^u(i)$  is set to  $\mathbf{i}_{\ell'}$ ; and so on. The corresponding column quantity,  $\mathbf{h}_\ell^v(j)$ , is defined and updated analogously. The *remaining rows and columns in stage  $\ell$* ,  $\mathbf{I}_\ell$  and  $\mathbf{J}_\ell$ , are then the unique values in  $\mathbf{h}_\ell^u$  and  $\mathbf{h}_\ell^v$ , respectively; and we set these in line 7 of Algorithm 1.

Each stage of Algorithm 1 has two main steps: exploration (lines 9–20) and elimination (lines 22–41). In the row exploration step, each row  $i \in \mathbf{I}_\ell$  is explored randomly over all remaining columns  $\mathbf{J}_\ell$  such that its expected reward up to stage  $\ell$  is at least  $\mu\bar{u}(i)$ , where  $\mu$  is in (4). To guarantee this, we sample column  $j \in [L]$  randomly and then substitute it with column  $\mathbf{h}_\ell^v(j)$ , which is at least as rewarding as column  $j$ . This is critical to avoid  $1/\min_{j \in [L]} \bar{v}(j)$  in our regret bound, which can be large and is not necessary. Our observations are stored in reward matrix  $\mathbf{C}_\ell^u \in \mathbb{R}^{K \times L}$ . As all rows are explored similarly, their expected rewards are scaled similarly, and this permits elimination. The column exploration step is analogous.

In the elimination step, the confidence intervals of all re-

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#### Algorithm 1 Rank1Elim for stochastic rank-1 bandits.

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1: // Initialization
2:  $t \leftarrow 1$ ,  $\tilde{\Delta}_0 \leftarrow 1$ ,  $\mathbf{C}_0^u \leftarrow \{0\}^{K \times L}$ ,  $\mathbf{C}_0^v \leftarrow \{0\}^{K \times L}$ ,
3:  $\mathbf{h}_0^u \leftarrow (1, \dots, K)$ ,  $\mathbf{h}_0^v \leftarrow (1, \dots, L)$ ,  $n_{-1} \leftarrow 0$ 
4:
5: for all  $\ell = 0, 1, \dots$  do
6:    $n_\ell \leftarrow \lceil 4\tilde{\Delta}_\ell^{-2} \log n \rceil$ 
7:    $\mathbf{I}_\ell \leftarrow \bigcup_{i \in [K]} \{\mathbf{h}_\ell^u(i)\}$ ,  $\mathbf{J}_\ell \leftarrow \bigcup_{j \in [L]} \{\mathbf{h}_\ell^v(j)\}$ 
8:
9:   // Row and column exploration
10:  for  $n_\ell - n_{\ell-1}$  times do
11:    Choose uniformly at random column  $j \in [L]$ 
12:     $j \leftarrow \mathbf{h}_\ell^v(j)$ 
13:    for all  $i \in \mathbf{I}_\ell$  do
14:       $\mathbf{C}_\ell^u(i, j) \leftarrow \mathbf{C}_\ell^u(i, j) + \mathbf{u}_t(i)\mathbf{v}_t(j)$ 
15:       $t \leftarrow t + 1$ 
16:    Choose uniformly at random row  $i \in [K]$ 
17:     $i \leftarrow \mathbf{h}_\ell^u(i)$ 
18:    for all  $j \in \mathbf{J}_\ell$  do
19:       $\mathbf{C}_\ell^v(i, j) \leftarrow \mathbf{C}_\ell^v(i, j) + \mathbf{u}_t(i)\mathbf{v}_t(j)$ 
20:       $t \leftarrow t + 1$ 
21:
22:  // UCBs and LCBs on the expected rewards of all
  remaining rows and columns
23:  for all  $i \in \mathbf{I}_\ell$  do
24:     $\mathbf{U}_\ell^u(i) \leftarrow \frac{1}{n_\ell} \sum_{j=1}^L \mathbf{C}_\ell^u(i, j) + \sqrt{\frac{\log n}{n_\ell}}$ 
25:     $\mathbf{L}_\ell^u(i) \leftarrow \frac{1}{n_\ell} \sum_{j=1}^L \mathbf{C}_\ell^u(i, j) - \sqrt{\frac{\log n}{n_\ell}}$ 
26:  for all  $j \in \mathbf{J}_\ell$  do
27:     $\mathbf{U}_\ell^v(j) \leftarrow \frac{1}{n_\ell} \sum_{i=1}^K \mathbf{C}_\ell^v(i, j) + \sqrt{\frac{\log n}{n_\ell}}$ 
28:     $\mathbf{L}_\ell^v(j) \leftarrow \frac{1}{n_\ell} \sum_{i=1}^K \mathbf{C}_\ell^v(i, j) - \sqrt{\frac{\log n}{n_\ell}}$ 
29:
30:  // Row and column elimination
31:   $\mathbf{i}_\ell \leftarrow \arg \max_{i \in \mathbf{I}_\ell} \mathbf{L}_\ell^u(i)$ 
32:   $\mathbf{h}_{\ell+1}^u \leftarrow \mathbf{h}_\ell^u$ 
33:  for all  $i = 1, \dots, K$  do
34:    if  $\mathbf{U}_\ell^u(\mathbf{h}_\ell^u(i)) \leq \mathbf{L}_\ell^u(\mathbf{i}_\ell)$  then
35:       $\mathbf{h}_{\ell+1}^u(i) \leftarrow \mathbf{i}_\ell$ 
36:
37:   $\mathbf{j}_\ell \leftarrow \arg \max_{j \in \mathbf{J}_\ell} \mathbf{L}_\ell^v(j)$ 
38:   $\mathbf{h}_{\ell+1}^v \leftarrow \mathbf{h}_\ell^v$ 
39:  for all  $j = 1, \dots, L$  do
40:    if  $\mathbf{U}_\ell^v(\mathbf{h}_\ell^v(j)) \leq \mathbf{L}_\ell^v(\mathbf{j}_\ell)$  then
41:       $\mathbf{h}_{\ell+1}^v(j) \leftarrow \mathbf{j}_\ell$ 
42:
43:   $\tilde{\Delta}_{\ell+1} \leftarrow \tilde{\Delta}_\ell/2$ ,  $\mathbf{C}_{\ell+1}^u \leftarrow \mathbf{C}_\ell^u$ ,  $\mathbf{C}_{\ell+1}^v \leftarrow \mathbf{C}_\ell^v$ 

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maining rows,  $[\mathbf{L}_\ell^u(i), \mathbf{U}_\ell^u(i)]$  for any  $i \in \mathbf{I}_\ell$ , are estimated

from matrix  $\mathbf{C}_\ell^u \in \mathbb{R}^{K \times L}$ ; and the confidence intervals of all remaining columns,  $[\mathbf{L}_\ell^v(j), \mathbf{U}_\ell^v(j)]$  for any  $j \in \mathbf{J}_\ell$ , are estimated from  $\mathbf{C}_\ell^v \in \mathbb{R}^{K \times L}$ . This separation is needed to guarantee that the expected rewards of all remaining rows and columns are scaled similarly. The confidence intervals are designed such that

$$\mathbf{U}_\ell^u(i) \leq \mathbf{L}_\ell^u(\mathbf{i}_\ell) = \max_{i \in \mathbf{I}_\ell} \mathbf{L}_\ell^u(i)$$

implies that row  $i$  is suboptimal with a high probability for any column elimination policy up to the end of stage  $\ell$ , and

$$\mathbf{U}_\ell^v(j) \leq \mathbf{L}_\ell^v(\mathbf{j}_\ell) = \max_{j \in \mathbf{J}_\ell} \mathbf{L}_\ell^v(j)$$

implies that column  $j$  is suboptimal with a high probability for any row elimination policy up to the end of stage  $\ell$ . As a result, all suboptimal rows and columns are eliminated correctly with a high probability.

## 5 Analysis

This section has three subsections. In Section 5.1, we derive a gap-dependent upper bound on the  $n$ -step regret of Rank1Elim. In Section 5.2, we derive a gap-dependent lower bound that nearly matches our upper bound. In Section 5.3, we discuss the results of our analysis.

### 5.1 Upper Bound

The hardness of our learning problem is measured by two sets of metrics. The first metrics are gaps. The *gaps* of row  $i \in [K]$  and column  $j \in [L]$  are defined as

$$\Delta_i^u = \bar{u}(i^*) - \bar{u}(i), \quad \Delta_j^v = \bar{v}(j^*) - \bar{v}(j), \quad (2)$$

respectively; and the *minimum row and column gaps* are defined as

$$\Delta_{\min}^u = \min_{i \in [K]: \Delta_i^u > 0} \Delta_i^u, \quad \Delta_{\min}^v = \min_{j \in [L]: \Delta_j^v > 0} \Delta_j^v, \quad (3)$$

respectively. Roughly speaking, the smaller the gaps, the harder the problem. The second metric is the minimum of the average of entries in  $\bar{u}$  and  $\bar{v}$ , which is defined as

$$\mu = \min \left\{ \frac{1}{K} \sum_{i=1}^K \bar{u}(i), \frac{1}{L} \sum_{j=1}^L \bar{v}(j) \right\}. \quad (4)$$

The smaller the value of  $\mu$ , the harder the problem. This quantity appears in our regret bound due to the averaging character of Rank1Elim (Section 4). Our upper bound on the regret of Rank1Elim is stated and proved below.

**Theorem 1.** *The expected  $n$ -step regret of Rank1Elim is bounded as*

$$R(n) \leq \frac{1}{\mu^2} \left( \sum_{i=1}^K \frac{384}{\bar{\Delta}_i^u} + \sum_{j=1}^L \frac{384}{\bar{\Delta}_j^v} \right) \log n + 3(K + L),$$

where

$$\begin{aligned} \bar{\Delta}_i^u &= \Delta_i^u + \mathbb{1}\{\Delta_i^u = 0\} \Delta_{\min}^u, \\ \bar{\Delta}_j^v &= \Delta_j^v + \mathbb{1}\{\Delta_j^v = 0\} \Delta_{\min}^v. \end{aligned}$$

The proof of Theorem 1 is organized as follows. First, we bound the probability that at least one confidence interval is violated. The corresponding regret is small,  $O(K + L)$ . Second, by the design of Rank1Elim and because all confidence intervals hold, the expected reward of any explored row  $i \in [K]$  is at least  $\mu \bar{u}(i)$ . Since all rows are explored in the same way, any suboptimal row  $i \in [K]$  is eliminated after  $O([1/(\mu \Delta_i^u)]^2 \log n)$  observations. Third, we bound the expected regret from above by the sum of row and column gaps (2), and account for both of them before row  $i$  is eliminated. This step is possible because Rank1Elim eliminates rows and columns simultaneously. Finally, we sum up the regret of all rows and columns, and get our result.

Note that the gaps in Theorem 1,  $\bar{\Delta}_i^u$  and  $\bar{\Delta}_j^v$ , are slightly different from those in (2). More specifically, all zero row and column gaps in (2) are substituted with the minimum column and row gap, respectively. The reason for the new gaps is that even optimal rows and columns incur regret, until all suboptimal columns and rows are eliminated, respectively. The proof of Theorem 1 is below.

*Proof.* Let  $\mathbf{R}_\ell^u(i)$  and  $\mathbf{R}_\ell^v(j)$  be the stochastic regret associated with exploring row  $i$  and column  $j$ , respectively, in stage  $\ell$ . Then the expected  $n$ -step regret of Rank1Elim is bounded as

$$R(n) \leq \mathbb{E} \left[ \sum_{\ell=0}^{n-1} \left( \sum_{i=1}^K \mathbf{R}_\ell^u(i) + \sum_{j=1}^L \mathbf{R}_\ell^v(j) \right) \right],$$

where the outer sum is over possibly  $n$  stages. Let

$$\begin{aligned} \bar{\mathbf{u}}_\ell(i) &= \sum_{t=0}^{\ell} \mathbb{E} \left[ \sum_{j=1}^L \frac{\mathbf{C}_t^u(i, j) - \mathbf{C}_{t-1}^u(i, j)}{n_\ell} \middle| \mathbf{h}_t^u \right] \\ &= \bar{u}(i) \sum_{t=0}^{\ell} \frac{n_t - n_{t-1}}{n_\ell} \sum_{j=1}^L \frac{\bar{v}(\mathbf{h}_t^u(j))}{L} \end{aligned}$$

be the expected reward of row  $i \in \mathbf{I}_\ell$  in the first  $\ell$  stages, where  $n_{-1} = 0$  and  $\mathbf{C}_{-1}^u(i, j) = 0$ ; and let

$$\mathcal{E}_\ell^u = \{\forall i \in \mathbf{I}_\ell : \bar{\mathbf{u}}_\ell(i) \in [\mathbf{L}_\ell^u(i), \mathbf{U}_\ell^u(i)], \bar{\mathbf{u}}_\ell(i) \geq \mu \bar{u}(i)\}$$

be the event that for all remaining rows  $i \in \mathbf{I}_\ell$  at the end of stage  $\ell$ , the confidence interval on the expected reward holds and that this reward is at least  $\mu \bar{u}(i)$ . Let  $\bar{\mathcal{E}}_\ell^u$  be the complement of event  $\mathcal{E}_\ell^u$ . Let

$$\begin{aligned} \bar{\mathbf{v}}_\ell(j) &= \sum_{t=0}^{\ell} \mathbb{E} \left[ \sum_{i=1}^K \frac{\mathbf{C}_t^v(i, j) - \mathbf{C}_{t-1}^v(i, j)}{n_\ell} \middle| \mathbf{h}_t^v \right] \\ &= \bar{v}(j) \sum_{t=0}^{\ell} \frac{n_t - n_{t-1}}{n_\ell} \sum_{i=1}^K \frac{\bar{u}(\mathbf{h}_t^v(i))}{K} \end{aligned}$$

denote the expected reward of column  $j \in \mathbf{J}_\ell$  in the first  $\ell$  stages, where  $n_{-1} = 0$  and  $\mathbf{C}_{-1}^v(i, j) = 0$ ; and let

$$\mathcal{E}_\ell^v = \{\forall j \in \mathbf{J}_\ell : \bar{\mathbf{v}}_\ell(j) \in [\mathbf{L}_\ell^v(j), \mathbf{U}_\ell^v(j)], \bar{\mathbf{v}}_\ell(j) \geq \mu \bar{v}(j)\}$$

be the event that for all remaining columns  $j \in \mathbf{J}_\ell$  at the end of stage  $\ell$ , the confidence interval on the expected reward holds and that this reward is at least  $\mu \bar{v}(j)$ . Let  $\bar{\mathcal{E}}_\ell^v$  be the complement of event  $\mathcal{E}_\ell^v$ . Let  $\mathcal{E}$  be the event that all events  $\mathcal{E}_\ell^u$  and  $\mathcal{E}_\ell^v$  happen; and  $\bar{\mathcal{E}}$  be the complement of  $\mathcal{E}$ , the event that at least one of  $\mathcal{E}_\ell^u$  and  $\mathcal{E}_\ell^v$  does not happen. Then the expected  $n$ -step regret of Rank1Elim is bounded from above as

$$\begin{aligned} R(n) &\leq \mathbb{E} \left[ \left( \sum_{\ell=0}^{n-1} \left( \sum_{i=1}^K \mathbf{R}_\ell^u(i) + \sum_{j=1}^L \mathbf{R}_\ell^v(j) \right) \right) \mathbb{1}\{\mathcal{E}\} \right] + \\ &\quad nP(\bar{\mathcal{E}}) \\ &\leq \sum_{i=1}^K \mathbb{E} \left[ \sum_{\ell=0}^{n-1} \mathbf{R}_\ell^u(i) \mathbb{1}\{\mathcal{E}\} \right] + \\ &\quad \sum_{j=1}^L \mathbb{E} \left[ \sum_{\ell=0}^{n-1} \mathbf{R}_\ell^v(j) \mathbb{1}\{\mathcal{E}\} \right] + 2(K+L), \end{aligned}$$

where the last inequality is from Lemma 1 in Appendix A.

Let  $\mathcal{H}_\ell = (\mathbf{I}_\ell, \mathbf{J}_\ell)$  be the rows and columns in stage  $\ell$ , and

$$\mathcal{F}_\ell = \left\{ \forall i \in \mathbf{I}_\ell, j \in \mathbf{J}_\ell : \Delta_i^u \leq \frac{2\tilde{\Delta}_{\ell-1}}{\mu}, \Delta_j^v \leq \frac{2\tilde{\Delta}_{\ell-1}}{\mu} \right\}$$

be the event that all rows and columns with ‘‘large gaps’’ are eliminated by the beginning of stage  $\ell$ . By Lemma 2 in Appendix A, event  $\mathcal{F}_\ell$  happens when  $\mathcal{E}$  happens. Now note that the expected regret in stage  $\ell$  does not depend on  $\mathcal{F}_\ell$  given  $\mathcal{H}_\ell$ . Therefore, the regret can be further bounded as

$$\begin{aligned} R(n) &\leq \sum_{i=1}^K \mathbb{E} \left[ \sum_{\ell=0}^{n-1} \mathbb{E} [\mathbf{R}_\ell^u(i) \mid \mathcal{H}_\ell] \mathbb{1}\{\mathcal{F}_\ell\} \right] + \\ &\quad \sum_{j=1}^L \mathbb{E} \left[ \sum_{\ell=0}^{n-1} \mathbb{E} [\mathbf{R}_\ell^v(j) \mid \mathcal{H}_\ell] \mathbb{1}\{\mathcal{F}_\ell\} \right] + \\ &\quad 2(K+L). \end{aligned} \quad (5)$$

By Lemma 3 in Appendix A,

$$\begin{aligned} \mathbb{E} \left[ \sum_{\ell=0}^{n-1} \mathbb{E} [\mathbf{R}_\ell^u(i) \mid \mathcal{H}_\ell] \mathbb{1}\{\mathcal{F}_\ell\} \right] &\leq \frac{384}{\mu^2 \bar{\Delta}_i^u} \log n + 1, \\ \mathbb{E} \left[ \sum_{\ell=0}^{n-1} \mathbb{E} [\mathbf{R}_\ell^v(j) \mid \mathcal{H}_\ell] \mathbb{1}\{\mathcal{F}_\ell\} \right] &\leq \frac{384}{\mu^2 \bar{\Delta}_j^v} \log n + 1, \end{aligned}$$

for any row  $i \in [K]$  and column  $j \in [L]$ . Finally, we apply the above upper bounds to (5) and get our main claim. ■

## 5.2 Lower Bound

We derive a gap-dependent lower bound on the family of rank-1 bandits where  $P_u$  and  $P_v$  are products of independent Bernoulli variables, which are parameterized by their means  $\bar{u}$  and  $\bar{v}$ , respectively. The lower bound is derived for any *uniformly efficient algorithm*  $\mathcal{A}$ , which is any algorithm such that for any  $(\bar{u}, \bar{v}) \in [0, 1]^K \times [0, 1]^L$  and any  $\alpha \in (0, 1)$ ,  $R(n) = o(n^\alpha)$ :

**Theorem 2.** *For any problem  $(\bar{u}, \bar{v}) \in [0, 1]^K \times [0, 1]^L$  with a unique best arm and any uniformly efficient algorithm  $\mathcal{A}$  whose regret is  $R(n)$ ,*

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{R(n)}{\log n} &\geq \sum_{i \in [K] \setminus \{i^*\}} \frac{\bar{u}(i^*)\bar{v}(j^*) - \bar{u}(i)\bar{v}(j^*)}{d(\bar{u}(i)\bar{v}(j^*), \bar{u}(i^*)\bar{v}(j^*))} + \\ &\quad \sum_{j \in [L] \setminus \{j^*\}} \frac{\bar{u}(i^*)\bar{v}(j^*) - \bar{u}(i^*)\bar{v}(j)}{d(\bar{u}(i^*)\bar{v}(j), \bar{u}(i^*)\bar{v}(j^*))}, \end{aligned}$$

where  $d(p, q)$  is the Kullback-Leibler (KL) divergence between Bernoulli random variables with means  $p$  and  $q$ .

The lower bound involves two terms. The first term is the regret due to learning the optimal row  $i^*$ , while playing the optimal column  $j^*$ . The second term is the regret due to learning the optimal column  $j^*$ , while playing the optimal row  $i^*$ . We do not know whether this lower bound is tight. We discuss its tightness in Section 5.3.

*Proof.* The proof is based on the change-of-measure techniques from Kaufmann *et al.* [13] and Lagree *et al.* [20], who ultimately build on Graves and Lai [11]. Let

$$w^*(\bar{u}, \bar{v}) = \max_{(i,j) \in [K] \times [L]} \bar{u}(i)\bar{v}(j)$$

be the maximum reward in model  $(\bar{u}, \bar{v})$ . We consider the set of models where  $\bar{u}(i^*)$  and  $\bar{v}(j^*)$  remain the same, but the optimal arm changes,

$$\begin{aligned} B(\bar{u}, \bar{v}) &= \{(\bar{u}', \bar{v}') \in [0, 1]^K \times [0, 1]^L : \bar{u}(i^*) = \bar{u}'(i^*), \\ &\quad \bar{v}(j^*) = \bar{v}'(j^*), w^*(\bar{u}, \bar{v}) < w^*(\bar{u}', \bar{v}')\}. \end{aligned}$$

By Theorem 17 of Kaufmann *et al.* [13],

$$\liminf_{n \rightarrow \infty} \frac{\sum_{i=1}^K \sum_{j=1}^L \mathbb{E} [\mathbf{T}_n(i, j)] d(\bar{u}(i)\bar{v}(j), \bar{u}'(i)\bar{v}'(j))}{\log n} \geq 1$$

for any  $(\bar{u}', \bar{v}') \in B(\bar{u}, \bar{v})$ , where  $\mathbb{E} [\mathbf{T}_n(i, j)]$  is the expected number of times that arm  $(i, j)$  is chosen in  $n$  steps in problem  $(\bar{u}, \bar{v})$ . From this and the regret decomposition

$$R(n) = \sum_{i=1}^K \sum_{j=1}^L \mathbb{E} [\mathbf{T}_n(i, j)] (\bar{u}(i^*)\bar{v}(j^*) - \bar{u}(i)\bar{v}(j)),$$

we get that

$$\liminf_{n \rightarrow \infty} \frac{R(n)}{\log n} \geq f(\bar{u}, \bar{v}),$$

where

$$f(\bar{u}, \bar{v}) = \inf_{c \in [0, \infty)^{K \times L}} \sum_{i=1}^K \sum_{j=1}^L (\bar{u}(i^*) \bar{v}(j^*) - \bar{u}(i) \bar{v}(j)) c_{i,j}$$

s.t.  $\forall (\bar{u}', \bar{v}') \in B(\bar{u}, \bar{v}) :$

$$\sum_{i=1}^K \sum_{j=1}^L d(\bar{u}(i) \bar{v}(j), \bar{u}'(i) \bar{v}'(j)) c_{i,j} \geq 1$$

To obtain our lower bound, we relax the constraints of the above problem carefully, so that we do not lose much in the bound. The details are presented in Appendix B. In the relaxed problem, only  $K + L - 1$  entries in the optimal solution  $c^*$  are non-zero, as in Combes *et al.* [6]. In particular,  $c_{i^*, j}^* > 0$  for  $j \in [L]$  and  $c_{i, j^*}^* > 0$  for  $i \in [K]$ , and these entries are

$$c_{i,j} = \begin{cases} 1/d(\bar{u}(i) \bar{v}(j^*), \bar{u}(i^*) \bar{v}(j^*)), & j = j^*, i \neq i^*; \\ 1/d(\bar{u}(i^*) \bar{v}(j), \bar{u}(i^*) \bar{v}(j^*)), & i = i^*, j \neq j^*; \\ 0, & \text{otherwise.} \end{cases}$$

Our lower bound is obtained by substituting  $c^*$  into the objective of the above optimization problem. ■

### 5.3 Discussion

We derive a gap-dependent upper bound on the  $n$ -step regret of Rank1Elim in Theorem 1, which is

$$O((K + L)(1/\mu^2)(1/\Delta) \log n),$$

where  $K$  denotes the number of rows,  $L$  denotes the number of columns,  $\Delta = \min\{\Delta_{\min}^u, \Delta_{\min}^v\}$  is the minimum of the row and column gaps in (3), and  $\mu$  is the minimum of the average of entries in  $\bar{u}$  and  $\bar{v}$ , as defined in (4).

We argue that our upper bound is nearly tight on the following class of problems. The  $i$ -th entry of  $\mathbf{u}_t$ ,  $\mathbf{u}_t(i)$ , is an independent Bernoulli variable with mean

$$\bar{u}(i) = p_u + \Delta_u \mathbf{1}\{i = 1\}$$

for some  $p_u \in [0, 1]$  and row gap  $\Delta_u \in (0, 1 - p_u]$ . The  $j$ -th entry of  $\mathbf{v}_t$ ,  $\mathbf{v}_t(j)$ , is an independent Bernoulli variable with mean

$$\bar{v}(j) = p_v + \Delta_v \mathbf{1}\{j = 1\}$$

for  $p_v \in [0, 1]$  and column gap  $\Delta_v \in (0, 1 - p_v]$ . Note that the optimal arm is  $(1, 1)$  and that the expected reward for choosing it is  $(p_u + \Delta_u)(p_v + \Delta_v)$ . We refer to the instance of this problem by  $B_{\text{SPIKE}}(K, L, p_u, p_v, \Delta_u, \Delta_v)$ ; and parameterize it by  $K, L, p_u, p_v, \Delta_u$ , and  $\Delta_v$ .

Let  $p_u = 0.5 - \Delta_u$  for  $\Delta_u \in [0, 0.25]$ , and  $p_v = 0.5 - \Delta_v$  for  $\Delta_v \in [0, 0.25]$ . Then the upper bound in Theorem 1 is

$$O([K(1/\Delta_u) + L(1/\Delta_v)] \log n)$$

since  $1/\mu^2 \leq 1/0.25^2 = 16$ . On the other hand, the lower bound in Theorem 2 is

$$\Omega([K(1/\Delta_u) + L(1/\Delta_v)] \log n)$$

since  $d(p, q) \leq [q(1 - q)]^{-1}(p - q)^2$  and  $q = 1 - q = 0.5$ . Note that the bounds match in  $K, L$ , the gaps, and  $\log n$ .

We conclude with the observation that Rank1Elim is sub-optimal in problems where  $\mu$  in (4) is small. In particular, consider the above problem, and choose  $\Delta_u = \Delta_v = 0.5$  and  $K = L$ . In this problem, the regret of Rank1Elim is  $O(K^3 \log n)$ ; because Rank1Elim eliminates  $O(K)$  rows and columns with  $O(1/K)$  gaps, and the regret for choosing any suboptimal arm is  $O(1)$ . This is much higher than the regret of UCB1 in Section 3,  $O(K^2 \log n)$ . We note that the upper bound in Theorem 1 is also  $O(K^3 \log n)$ . Therefore, the upper bound is not loose, and a new algorithm is needed to improve over UCB1 in this particular problem.

## 6 Experiments

We conduct three experiments. In Section 6.1, we validate that the regret of Rank1Elim grows as suggested by Theorem 1. In Section 6.2, we compare Rank1Elim to three baselines. Finally, in Section 6.3, we evaluate Rank1Elim on a real-world problem where our modeling assumptions are violated.

### 6.1 Regret Bound

The first experiment shows that the regret of Rank1Elim scales as suggested by our upper bound in Theorem 1. We experiment with the class of synthetic problems from Section 5.3,  $B_{\text{SPIKE}}(K, L, p_u, p_v, \Delta_u, \Delta_v)$ . We vary its parameters and report the  $n$ -step regret in 2 million (M) steps.

Table 1 shows the  $n$ -step regret of Rank1Elim for various choices of  $K, L, p_u, p_v, \Delta_u$ , and  $\Delta_v$ . In each table, we vary two parameters and keep the rest fixed. We observe that the regret increases as  $K$  and  $L$  increase, and  $\Delta_u$  and  $\Delta_v$  decrease; as suggested by Theorem 1. Specifically, the regret doubles when  $K$  and  $L$  are doubled, and when  $\Delta_u$  and  $\Delta_v$  are halved. We also observe that the regret is not quadratic in  $1/\mu$ , where  $\mu \approx \min\{p_u, p_v\}$ . This indicates that the upper bound in Theorem 1 is loose in  $\mu$  when  $\mu$  is bounded away from zero. We argue in Section 5.3 that this is not the case as  $\mu \rightarrow 0$ .

### 6.2 Comparison to Alternative Solutions

In the second experiment, we compare Rank1Elim to the three alternative methods in Section 3: UCB1, LinUCB, and GLM-UCB. The confidence radii of LinUCB and GLM-UCB are set as suggested by Abbasi-Yadkori *et al.* [1] and Filippi *et al.* [9], respectively. The maximum-likelihood estimates of  $\bar{u}$  and  $\bar{v}$  in GLM-UCB are computed using the online EM [4],

$K$	$L$	Regret	$p_U$	$p_V$	Regret	$\Delta_U$	$\Delta_V$	Regret
8	8	17491 ± 384	0.700	0.700	17744 ± 466	0.20	0.20	17653 ± 307
8	16	29628 ± 1499	0.700	0.350	23983 ± 594	0.20	0.10	22891 ± 912
8	32	50030 ± 1931	0.700	0.175	24776 ± 2333	0.20	0.05	30954 ± 787
16	8	28862 ± 585	0.350	0.700	22963 ± 205	0.10	0.20	20958 ± 614
16	16	41823 ± 1689	0.350	0.350	38373 ± 71	0.10	0.10	33642 ± 1089
16	32	62451 ± 2268	0.350	0.175	57401 ± 68	0.10	0.05	45511 ± 3257
32	8	46156 ± 806	0.175	0.700	27440 ± 2011	0.05	0.20	30688 ± 482
32	16	61992 ± 2339	0.175	0.350	57492 ± 67	0.05	0.10	44390 ± 2542
32	32	85208 ± 3546	0.175	0.175	95586 ± 99	0.05	0.05	68412 ± 2312

$p_U = p_V = 0.7, \Delta_U = \Delta_V = 0.2$      
  $K = L = 8, \Delta_U = \Delta_V = 0.2$      
  $K = L = 8, p_U = p_V = 0.7$

Table 1: The  $n$ -step regret of Rank1Elim in  $n = 2M$  steps as  $K$  and  $L$  increase (left),  $p_U$  and  $p_V$  decrease (middle), and  $\Delta_U$  and  $\Delta_V$  decrease (right). The results are averaged over 20 runs.

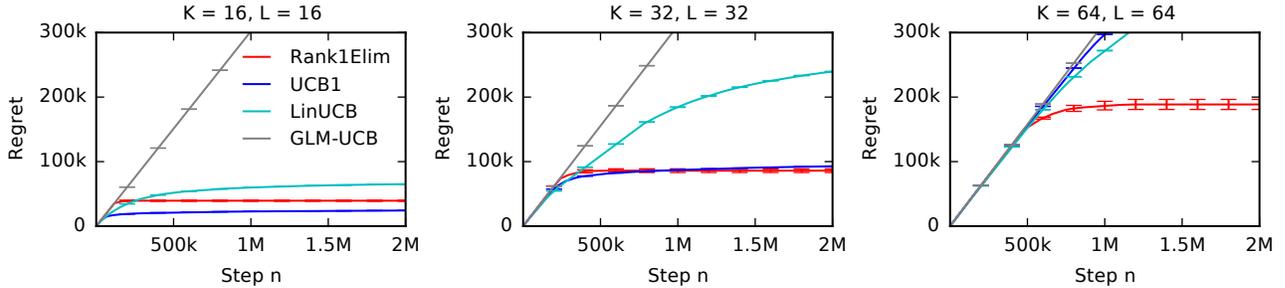


Figure 1: The  $n$ -step regret of Rank1Elim, UCB1, LinUCB, and GLM-UCB on three synthetic problems in up to  $n = 2M$  steps. The results are averaged over 20 runs.

which is observed to converge to  $\bar{u}$  and  $\bar{v}$  in our problems. We experiment with the problem from Section 6.1, where  $p_U = p_V = 0.7, \Delta_U = \Delta_V = 0.2$ , and  $K = L$ .

Our results are reported in Figure 1. We observe that the regret of Rank1Elim flattens in all three problems, which indicates that Rank1Elim learns the optimal arm. When  $K = 16$ , UCB1 has a lower regret than Rank1Elim. However, because the regret of UCB1 is  $O(KL)$  and the regret of Rank1Elim is  $O(K + L)$ , Rank1Elim can outperform UCB1 on larger problems. When  $K = 32$ , both algorithms already perform similarly; and when  $K = 64$ , Rank1Elim clearly outperforms UCB1. This shows that Rank1Elim can leverage the structure of our problem. Neither LinUCB nor GLM-UCB are competitive on any of our problems.

We investigated the poor performance of both LinUCB and GLM-UCB. When the confidence radii of LinUCB are multiplied by  $1/3$ , LinUCB becomes competitive on all problems. When the confidence radii of GLM-UCB are multiplied by  $1/100$ , GLM-UCB is still not competitive on any of our problems. We conclude that LinUCB and GLM-UCB perform poorly because their theory-suggested confidence intervals are too wide. In contrast, Rank1Elim is implemented with its theory-suggested intervals in all experiments.

### 6.3 MovieLens Experiment

In our last experiment, we evaluate Rank1Elim on a recommender system problem. The goal is to find the pair of users and items with the highest expected rating. We experiment with the *MovieLens* dataset from February 2003 [21], where 6k people give 1M ratings to 6k movies.

Our learning problem is formulated as follows. We define a user group for each unique combination of gender, age group, and occupation in the *MovieLens* dataset. The total number of groups is 241. For each user group and movie, we average the ratings of all users in that group that rated that movie, and learn a low-rank approximation to the underlying rating matrix by a state-of-the-art algorithm [15]. The algorithm automatically detects the rank of the matrix to be 5. We randomly choose  $K = 128$  user groups and  $L = 128$  movies. We report the average ratings of these user groups and movies in Figure 2a, and the corresponding completed rating matrix in Figure 2b. The reward for choosing row  $i \in [K]$  and column  $j \in [L]$  is a categorical variable whose domain is [5]. We estimate its parameters by discretizing the Gaussian variables in our factorization. We use Rank1Elim to find the maximum entry of our matrix. Since Rank1Elim has no guarantees beyond rank 1, our model is misspecified in this experiment.

Our results are reported in Figure 2c. We observe that the regret of Rank1Elim is concave in the number of steps  $n$ ,

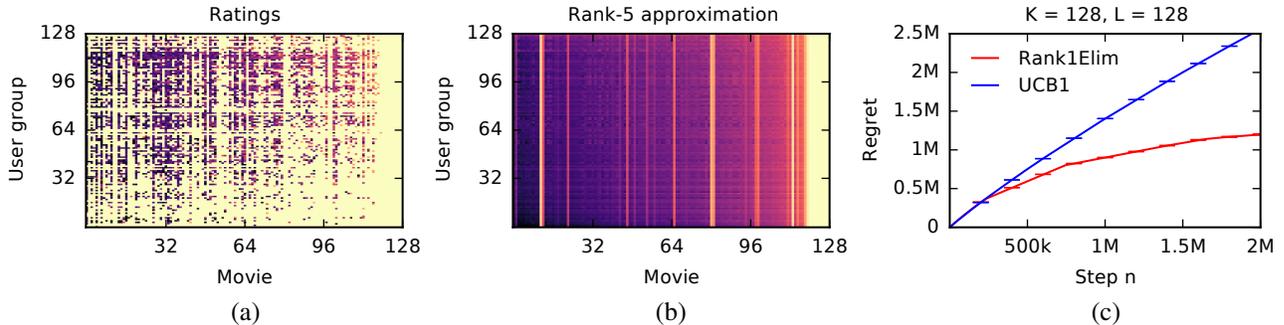


Figure 2: **a.** Ratings from the MovieLens dataset. The darker the color, the higher the rating. The rows and columns are ordered by their average ratings. The missing ratings are shown in yellow. **b.** Rank-5 approximation to the ratings. **c.** The  $n$ -step regret of Rank1Elim and UCB1 in up to  $n = 2M$  steps.

and flattens. This indicates that Rank1Elim learns a near-optimal solution. This is possible because of the structure of our rating matrix. Although it is rank 5, its first eigenvalue is an order of magnitude larger than the remaining four non-zero eigenvalues. This structure is not surprising because the ratings of items are often subject to significant *user and item biases* [17]. Therefore, our rating matrix is nearly rank 1, and Rank1Elim learns a good solution. Our theory cannot explain this result and we leave it for future work. Finally, we note that UCB1 explores throughout because our problem has more than 10k arms.

## 7 Related Work

Zhao *et al.* [28] proposed a bandit algorithm for low-rank matrix completion, where the posterior of latent item factors is approximated by its point estimate. This algorithm is not analyzed. Kawale *et al.* [14] proposed a Thompson sampling (TS) algorithm for low-rank matrix completion, where the posterior of low-rank matrices is approximated by particle filtering. A computationally-inefficient variant of the algorithm has  $O((1/\Delta^2) \log n)$  regret in rank-1 matrices. In contrast, note that Rank1Elim is computationally efficient and its  $n$ -step regret is  $O((1/\Delta) \log n)$ .

The problem of learning to recommend in the bandit setting was studied in several recent papers. Valko *et al.* [27] and Kocak *et al.* [16] proposed content-based recommendation algorithms, where the features of items are derived from a known similarity graph over the items. Gentile *et al.* [10] proposed an algorithm that clusters users based on their preferences, under the assumption that the features of items are known. Li *et al.* [22] extended this algorithm to the clustering of items. Maillard *et al.* [24] studied a multi-armed bandit problem where the arms are partitioned into latent groups. The problems in the last three papers are a special form of low-rank matrix completion, where some rows are identical. In this work, we do not make any such assumptions, but our results are limited to rank 1.

Rank1Elim is motivated by the structure of the position-

based model [7]. Lagree *et al.* [20] proposed a bandit algorithm for this model under the assumption that the examination probabilities of all positions are known. Online learning to rank in click models was studied in several recent papers [18, 6, 19, 12, 23, 29]. In practice, the probability of clicking on an item depends on both the item and its position, and this work is a major step towards learning to rank from such heterogeneous effects.

## 8 Conclusions

In this work, we propose stochastic rank-1 bandits, a class of online learning problems where the goal is to learn the maximum entry of a rank-1 matrix. This problem is challenging because the reward is a product of latent random variables, which are not observed. We propose a practical algorithm for solving this problem, Rank1Elim, and prove a gap-dependent upper bound on its regret. We also prove a nearly matching gap-dependent lower bound. Finally, we evaluate Rank1Elim empirically. In particular, we validate the scaling of its regret, compare it to baselines, and show that it learns high-quality solutions even when our modeling assumptions are mildly violated.

We conclude that Rank1Elim is a practical algorithm for finding the maximum entry of a stochastic rank-1 matrix. It is surprisingly competitive with various baselines (Section 6.2) and can be applied to higher-rank matrices (Section 6.3). On the other hand, we show that Rank1Elim can be suboptimal on relatively simple problems (Section 5.3). We plan to address this issue in our future work. We note that our results can be generalized to other reward models, such as  $\mathbf{u}_t(i)\mathbf{v}_t(j) \sim \mathcal{N}(\bar{u}(i)\bar{v}(j), \sigma)$  for  $\sigma > 0$ .

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## A Upper Bound

**Lemma 1.** Let  $\bar{\mathcal{E}}$  be defined as in the proof of Theorem 1. Then

$$P(\bar{\mathcal{E}}) \leq \frac{2(K+L)}{n}.$$

*Proof.* Let  $\mathcal{E}_\ell = \mathcal{E}_\ell^u \cap \mathcal{E}_\ell^v$ . Then from the definition of  $\bar{\mathcal{E}}$ ,

$$\bar{\mathcal{E}} = \bar{\mathcal{E}}_0 \cup (\bar{\mathcal{E}}_1 \cap \mathcal{E}_0) \cup \dots \cup (\bar{\mathcal{E}}_{n-1} \cap \mathcal{E}_{n-2} \cap \dots \cap \mathcal{E}_0),$$

and from the definition of  $\mathcal{E}_\ell$ ,

$$\bar{\mathcal{E}}_\ell \cap \mathcal{E}_{\ell-1} \cap \dots \cap \mathcal{E}_0 = (\bar{\mathcal{E}}_\ell^u \cap \mathcal{E}_{\ell-1} \cap \dots \cap \mathcal{E}_0) \cup (\bar{\mathcal{E}}_\ell^v \cap \mathcal{E}_\ell^u \cap \mathcal{E}_{\ell-1} \cap \dots \cap \mathcal{E}_0).$$

It follows that the probability of event  $\bar{\mathcal{E}}$  is bounded as

$$\begin{aligned} P(\bar{\mathcal{E}}) &\leq \sum_{\ell=0}^{n-1} P(\bar{\mathcal{E}}_\ell^u, \mathcal{E}_0^u, \dots, \mathcal{E}_{\ell-1}^u, \mathcal{E}_0^v, \dots, \mathcal{E}_{\ell-1}^v) + \sum_{\ell=0}^{n-1} P(\bar{\mathcal{E}}_\ell^v, \mathcal{E}_0^u, \dots, \mathcal{E}_{\ell-1}^u, \mathcal{E}_0^v, \dots, \mathcal{E}_{\ell-1}^v) \\ &\leq \sum_{\ell=0}^{n-1} P(\bar{\mathcal{E}}_\ell^u, \mathcal{E}_0^v, \dots, \mathcal{E}_{\ell-1}^v) + \sum_{\ell=0}^{n-1} P(\bar{\mathcal{E}}_\ell^v, \mathcal{E}_0^u, \dots, \mathcal{E}_{\ell-1}^u). \end{aligned}$$

From the definition of  $\bar{\mathcal{E}}_\ell^u$ , it follows that

$$\begin{aligned} P(\bar{\mathcal{E}}_\ell^u, \mathcal{E}_0^v, \dots, \mathcal{E}_{\ell-1}^v) &\leq P(\exists i \in \mathbf{I}_\ell \text{ s.t. } \bar{\mathbf{u}}_\ell(i) \notin [\mathbf{L}_\ell^u(i), \mathbf{U}_\ell^u(i)]) + \\ &\quad P(\exists i \in \mathbf{I}_\ell \text{ s.t. } \bar{\mathbf{u}}_\ell(i) < \mu \bar{u}(i), \mathcal{E}_0^v, \dots, \mathcal{E}_{\ell-1}^v). \end{aligned}$$

Now we bound the probability of the above two events. The probability  $P(\bar{\mathcal{E}}_\ell^v, \mathcal{E}_0^u, \dots, \mathcal{E}_{\ell-1}^u)$  can be bounded similarly and we omit this proof.

**Event 1:**  $\exists i \in \mathbf{I}_\ell$  s.t.  $\bar{\mathbf{u}}_\ell(i) \notin [\mathbf{L}_\ell^u(i), \mathbf{U}_\ell^u(i)]$

Fix any  $i \in \mathbf{I}_\ell$ . Let  $\mathbf{c}_k$  be the  $k$ -th observation of row  $i$  in the row exploration stage of Rank1Elim and  $\ell(k)$  be the index of that stage. Then

$$\left( \mathbf{c}_k - \bar{u}(i) \sum_{j=1}^L \frac{\bar{v}(\mathbf{h}_{\ell(k)}^v(j))}{L} \right)_{k=1}^n$$

is a martingale difference sequence with respect to history  $\mathbf{h}_0^v, \dots, \mathbf{h}_{\ell(k)}^v$  in step  $k$ . This follows from the observation that

$$\mathbb{E} \left[ \mathbf{c}_k \mid \mathbf{h}_0^v, \dots, \mathbf{h}_{\ell(k)}^v \right] = \bar{u}(i) \sum_{j=1}^L \frac{\bar{v}(\mathbf{h}_{\ell(k)}^v(j))}{L},$$

because column  $j \in [L]$  in stage  $\ell(k)$  is chosen randomly and then mapped to at least as rewarding column  $\mathbf{h}_{\ell(k)}^v(j)$ . By the definition of our sequence and from Remark 2.2.1 of Raginsky and Sason [25],

$$\begin{aligned} P(\bar{\mathbf{u}}_\ell(i) \notin [\mathbf{L}_\ell^u(i), \mathbf{U}_\ell^u(i)]) &= P \left( \left| \frac{1}{n_\ell} \sum_{j=1}^L \mathbf{C}_\ell^u(i, j) - \bar{\mathbf{u}}_\ell(i) \right| > \sqrt{\frac{\log n}{n_\ell}} \right) \\ &= P \left( \left| \sum_{k=1}^{n_\ell} \left[ \mathbf{c}_k - \bar{u}(i) \sum_{j=1}^L \frac{\bar{v}(\mathbf{h}_{\ell(k)}^v(j))}{L} \right] \right| > \sqrt{n_\ell \log n} \right) \\ &\leq 2 \exp[-2 \log n] \\ &= 2n^{-2} \end{aligned}$$

for any stage  $\ell$ . By the union bound,

$$P(\exists i \in \mathbf{I}_\ell \text{ s.t. } \bar{\mathbf{u}}_\ell(i) \notin [\mathbf{L}_\ell^{\mathbf{U}}(i), \mathbf{U}_\ell^{\mathbf{U}}(i)]) \leq 2Kn^{-2}$$

for any stage  $\ell$ .

**Event 2:**  $\exists i \in \mathbf{I}_\ell$  s.t.  $\bar{\mathbf{u}}_\ell(i) < \mu \bar{u}(i)$ ,  $\mathcal{E}_0^{\mathbf{V}}$ ,  $\dots$ ,  $\mathcal{E}_{\ell-1}^{\mathbf{V}}$

We claim that this event cannot happen. Fix any  $i \in \mathbf{I}_\ell$ . When  $\ell = 0$ , we get that  $\bar{\mathbf{u}}_0(i) = \bar{u}(i)(1/L) \sum_{j=1}^L \bar{v}(j) = \mu \bar{u}(i)$  from the definitions of  $\bar{\mathbf{u}}_0(i)$  and  $\mu$ , and event 2 obviously does not happen. When  $\ell > 0$  and events  $\mathcal{E}_0^{\mathbf{V}}, \dots, \mathcal{E}_{\ell-1}^{\mathbf{V}}$  happen, any eliminated column  $j$  up to stage  $\ell$  is substituted with column  $j'$  such that  $\bar{v}(j') \geq \bar{v}(j)$ , by the design of Rank1Elim. From this fact and the definition of  $\bar{\mathbf{u}}_\ell(i)$ ,  $\bar{\mathbf{u}}_\ell(i) \geq \mu \bar{u}(i)$ . Therefore, event 2 does not happen when  $\ell > 0$ .

### Total probability

Finally, we sum all probabilities up and get that

$$P(\bar{\mathcal{E}}) \leq n \left( \frac{2K}{n^2} \right) + n \left( \frac{2L}{n^2} \right) \leq \frac{2(K+L)}{n}.$$

This concludes our proof. ■

**Lemma 2.** *Let event  $\mathcal{E}$  happen and  $m$  be the first stage where  $\tilde{\Delta}_m < \mu \Delta_i^{\mathbf{U}}/2$ . Then row  $i$  is guaranteed to be eliminated by the end of stage  $m$ . Moreover, let  $m$  be the first stage where  $\tilde{\Delta}_m < \mu \Delta_j^{\mathbf{V}}/2$ . Then column  $j$  is guaranteed to be eliminated by the end of stage  $m$ .*

*Proof.* We only prove the first claim. The other claim is proved analogously.

Before we start, note that by the design of Rank1Elim and from the definition of  $m$ ,

$$\tilde{\Delta}_m = 2^{-m} < \frac{\mu \Delta_i^{\mathbf{U}}}{2} \leq 2^{-(m-1)} = \tilde{\Delta}_{m-1}. \quad (6)$$

By the design of our confidence intervals,

$$\begin{aligned} \frac{1}{n_m} \sum_{j=1}^K \mathbf{C}_m^{\mathbf{U}}(i, j) + \sqrt{\frac{\log n}{n_m}} &\stackrel{(a)}{\leq} \bar{\mathbf{u}}_m(i) + 2\sqrt{\frac{\log n}{n_m}} \\ &= \bar{\mathbf{u}}_m(i) + 4\sqrt{\frac{\log n}{n_m}} - 2\sqrt{\frac{\log n}{n_m}} \\ &\stackrel{(b)}{\leq} \bar{\mathbf{u}}_m(i) + 2\tilde{\Delta}_m - 2\sqrt{\frac{\log n}{n_m}} \\ &\stackrel{(c)}{\leq} \bar{\mathbf{u}}_m(i) + \mu \Delta_i^{\mathbf{U}} - 2\sqrt{\frac{\log n}{n_m}} \\ &= \bar{\mathbf{u}}_m(i^*) + \mu \Delta_i^{\mathbf{U}} - [\bar{\mathbf{u}}_m(i^*) - \bar{\mathbf{u}}_m(i)] - 2\sqrt{\frac{\log n}{n_m}}, \end{aligned}$$

where inequality (a) is from  $\mathbf{L}_m^{\mathbf{U}}(i) \leq \bar{\mathbf{u}}_m(i)$ , inequality (b) is from  $n_m \geq 4\tilde{\Delta}_m^{-2} \log n$ , and inequality (c) is by (6). Now note that

$$\bar{\mathbf{u}}_m(i^*) - \bar{\mathbf{u}}_m(i) = q(\bar{u}(i^*) - \bar{u}(i)) \geq \mu \Delta_i^{\mathbf{U}}$$

for some  $q \in [0, 1]$ . The equality holds because  $\bar{\mathbf{u}}_m(i^*)$  and  $\bar{\mathbf{u}}_m(i)$  are estimated from the same sets of random columns. The inequality follows from the fact that events  $\mathcal{E}_0^{\mathbf{V}}, \dots, \mathcal{E}_{\ell-1}^{\mathbf{V}}$  happen. The events imply that any eliminated column  $j$  up to stage  $m$  is substituted with column  $j'$  such that  $\bar{v}(j') \geq \bar{v}(j)$ , and thus  $q \geq \mu$ . From the above inequality, we get that

$$\bar{\mathbf{u}}_m(i^*) + \mu \Delta_i^{\mathbf{U}} - [\bar{\mathbf{u}}_m(i^*) - \bar{\mathbf{u}}_m(i)] - 2\sqrt{\frac{\log n}{n_m}} \leq \bar{\mathbf{u}}_m(i^*) - 2\sqrt{\frac{\log n}{n_m}}.$$

Finally,

$$\begin{aligned} \bar{\mathbf{u}}_m(i^*) - 2\sqrt{\frac{\log n}{n_m}} &\stackrel{(a)}{\leq} \frac{1}{n_m} \sum_{j=1}^K \mathbf{C}_m^{\mathbf{u}}(i^*, j) - \sqrt{\frac{\log n}{n_m}} \\ &\stackrel{(b)}{\leq} \frac{1}{n_m} \sum_{j=1}^K \mathbf{C}_m^{\mathbf{u}}(\mathbf{i}_m, j) - \sqrt{\frac{\log n}{n_m}}, \end{aligned}$$

where inequality (a) follows from  $\bar{\mathbf{u}}_m(i^*) \leq \mathbf{U}_m^{\mathbf{u}}(i^*)$  and inequality (b) follows from  $\mathbf{L}_m^{\mathbf{u}}(i^*) \leq \mathbf{L}_m^{\mathbf{u}}(\mathbf{i}_m)$ , since  $i^* \in \mathbf{I}_\ell$  and  $\mathbf{i}_\ell = \arg \max_{i \in \mathbf{I}_\ell} \mathbf{L}_\ell^{\mathbf{u}}(i)$ . Now we chain all inequalities and get our final claim. ■

**Lemma 3.** *The expected regret associated with any row  $i \in [K]$  is bounded as*

$$\mathbb{E} \left[ \sum_{\ell=0}^{n-1} \mathbb{E} [\mathbf{R}_\ell^{\mathbf{u}}(i) \mid \mathcal{H}_\ell] \mathbb{1}\{\mathcal{F}_\ell\} \right] \leq \frac{384}{\mu^2 \tilde{\Delta}_i^{\mathbf{u}}} \log n + 1.$$

Moreover, the expected regret associated with any column  $j \in [L]$  is bounded as

$$\mathbb{E} \left[ \sum_{\ell=0}^{n-1} \mathbb{E} [\mathbf{R}_\ell^{\mathbf{v}}(j) \mid \mathcal{H}_\ell] \mathbb{1}\{\mathcal{F}_\ell\} \right] \leq \frac{384}{\mu^2 \tilde{\Delta}_j^{\mathbf{v}}} \log n + 1.$$

*Proof.* We only prove the first claim. The other claim is proved analogously.

This proof has two parts. In the first part, we assume that row  $i$  is suboptimal,  $\tilde{\Delta}_i^{\mathbf{u}} > 0$ . In the second part, we assume that row  $i$  is optimal,  $\tilde{\Delta}_i^{\mathbf{u}} = 0$ .

### Row $i$ is suboptimal

Let row  $i$  be suboptimal and  $m$  be the first stage where  $\tilde{\Delta}_m < \mu \tilde{\Delta}_i^{\mathbf{u}}/2$ . Then row  $i$  is guaranteed to be eliminated by the end of stage  $m$  (Lemma 2), and thus

$$\mathbb{E} \left[ \sum_{\ell=0}^{n-1} \mathbb{E} [\mathbf{R}_\ell^{\mathbf{u}}(i) \mid \mathcal{H}_\ell] \mathbb{1}\{\mathcal{F}_\ell\} \right] \leq \mathbb{E} \left[ \sum_{\ell=0}^m \mathbb{E} [\mathbf{R}_\ell^{\mathbf{u}}(i) \mid \mathcal{H}_\ell] \mathbb{1}\{\mathcal{F}_\ell\} \right].$$

By Lemma 4, the expected regret of choosing row  $i$  in stage  $\ell$  can be bounded from above as

$$\mathbb{E} [\mathbf{R}_\ell^{\mathbf{u}}(i) \mid \mathcal{H}_\ell] \mathbb{1}\{\mathcal{F}_\ell\} \leq (\tilde{\Delta}_i^{\mathbf{u}} + \max_{j \in \mathbf{J}_\ell} \tilde{\Delta}_j^{\mathbf{v}})(n_\ell - n_{\ell-1}),$$

where  $\max_{j \in \mathbf{J}_\ell} \tilde{\Delta}_j^{\mathbf{v}}$  is the maximum column gap in stage  $\ell$ ,  $n_\ell$  is the number of steps by the end of stage  $\ell$ , and  $n_{-1} = 0$ . From the definition of  $\mathcal{F}_\ell$  and  $\tilde{\Delta}_\ell$ , if column  $j$  is not eliminated before stage  $\ell$ , we have that

$$\tilde{\Delta}_j^{\mathbf{v}} \leq \frac{2\tilde{\Delta}_{\ell-1}}{\mu} = \frac{2 \cdot 2^{m-\ell+1} \tilde{\Delta}_m}{\mu} < 2^{m-\ell+1} \tilde{\Delta}_i^{\mathbf{u}}.$$

From the above inequalities and the definition of  $n_\ell$ , it follows that

$$\begin{aligned}
 \mathbb{E} \left[ \sum_{\ell=0}^m \mathbb{E} [\mathbf{R}_\ell^{\mathbf{u}}(i) \mid \mathcal{H}_\ell] \mathbf{1}\{\mathcal{F}_\ell\} \right] &\leq \sum_{\ell=0}^m (\Delta_i^{\mathbf{u}} + \max_{j \in \mathcal{J}_\ell} \Delta_j^{\mathbf{v}}) (n_\ell - n_{\ell-1}) \\
 &\leq \sum_{\ell=0}^m (\Delta_i^{\mathbf{u}} + 2^{m-\ell+1} \Delta_i^{\mathbf{u}}) (n_\ell - n_{\ell-1}) \\
 &\leq \Delta_i^{\mathbf{u}} \left( n_m + \sum_{\ell=0}^m 2^{m-\ell+1} n_\ell \right) \\
 &\leq \Delta_i^{\mathbf{u}} \left( 2^{2m+2} \log n + 1 + \sum_{\ell=0}^m 2^{m-\ell+1} (2^{2\ell+2} \log n + 1) \right) \\
 &= \Delta_i^{\mathbf{u}} \left( 2^{2m+2} \log n + 1 + \sum_{\ell=0}^m 2^{m+\ell+3} \log n + \sum_{\ell=0}^m 2^{m-\ell+1} \right) \\
 &\leq \Delta_i^{\mathbf{u}} (5 \cdot 2^{2m+2} \log n + 2^{m+2}) + 1 \\
 &\leq 6 \cdot 2^4 \cdot 2^{2m-2} \Delta_i^{\mathbf{u}} \log n + 1,
 \end{aligned}$$

where the last inequality follows from  $\log n \geq 1$  for  $n \geq 3$ . From the definition of  $\tilde{\Delta}_{m-1}$  in (6), we have that

$$2^{m-1} = \frac{1}{\tilde{\Delta}_{m-1}} \leq \frac{2}{\mu \Delta_i^{\mathbf{u}}}.$$

Now we chain all above inequalities and get that

$$\mathbb{E} \left[ \sum_{\ell=0}^{n-1} \mathbb{E} [\mathbf{R}_\ell^{\mathbf{u}}(i) \mid \mathcal{H}_\ell] \mathbf{1}\{\mathcal{F}_\ell\} \right] \leq 6 \cdot 2^4 \cdot 2^{2m-2} \Delta_i^{\mathbf{u}} \log n + 1 \leq \frac{384}{\mu^2 \Delta_i^{\mathbf{u}}} \log n + 1.$$

This concludes the first part of our proof.

#### Row $i$ is optimal

Let row  $i$  be optimal and  $m$  be the first stage where  $\tilde{\Delta}_m < \mu \Delta_{\min}^{\mathbf{v}}/2$ . Then similarly to the first part of the analysis,

$$\mathbb{E} \left[ \sum_{\ell=0}^{n-1} \mathbb{E} [\mathbf{R}_\ell^{\mathbf{u}}(i) \mid \mathcal{H}_\ell] \mathbf{1}\{\mathcal{F}_\ell\} \right] \leq \sum_{\ell=0}^m (\max_{j \in \mathcal{J}_\ell} \Delta_j^{\mathbf{v}}) (n_\ell - n_{\ell-1}) \leq \frac{384}{\mu^2 \Delta_{\min}^{\mathbf{v}}} \log n + 1.$$

This concludes our proof. ■

**Lemma 4.** Let  $\mathbf{u} \sim P_{\mathbf{U}}$  and  $\mathbf{v} \sim P_{\mathbf{V}}$  be drawn independently. Then the expected regret of choosing any row  $i \in [K]$  and column  $j \in [L]$  is bounded from above as

$$\mathbb{E} [\mathbf{u}(i^*) \mathbf{v}(j^*) - \mathbf{u}(i) \mathbf{v}(j)] \leq \Delta_i^{\mathbf{u}} + \Delta_j^{\mathbf{v}}.$$

*Proof.* Note that for any  $x, y, x^*, y^* \in [0, 1]$ ,

$$x^* y^* - xy = x^* y^* - xy^* + xy^* - xy = y^* (x^* - x) + x (y^* - y) \leq (x^* - x) + (y^* - y).$$

By the independence of the entries of  $\mathbf{u}$  and  $\mathbf{v}$ , and from the above inequality,

$$\mathbb{E} [\mathbf{u}(i^*) \mathbf{v}(j^*) - \mathbf{u}(i) \mathbf{v}(j)] = \bar{u}(i^*) \bar{v}(j^*) - \bar{u}(i) \bar{v}(j) \leq (\bar{u}(i^*) - \bar{u}(i)) + (\bar{v}(j^*) - \bar{v}(j)).$$

This concludes our proof. ■

## B Lower Bound

In this section we present the missing details of the proof of Theorem 2. Recall that we need to bound from below the value of  $f(\bar{u}, \bar{v})$  where

$$\begin{aligned} f(\bar{u}, \bar{v}) &= \inf_{c \in [0, \infty)^{K \times L}} \sum_{i=1}^K \sum_{j=1}^L (\bar{u}(i^*) \bar{v}(j^*) - \bar{u}(i) \bar{v}(j)) c_{i,j} \\ \text{s.t. for all } (\bar{u}', \bar{v}') &\in B(\bar{u}, \bar{v}) : \\ &\sum_{i=1}^K \sum_{j=1}^L d(\bar{u}(i) \bar{v}(j), \bar{u}'(i) \bar{v}'(j)) c_{i,j} \geq 1 \end{aligned}$$

and where

$$B(\bar{u}, \bar{v}) = \{(\bar{u}', \bar{v}') \in [0, 1]^K \times [0, 1]^L : \bar{u}(i^*) = \bar{u}'(i^*), \bar{v}(j^*) = \bar{v}'(j^*), w^*(\bar{u}, \bar{v}) < w^*(\bar{u}', \bar{v}')\}.$$

Without loss of generality we assume that the optimal action in the original model  $(\bar{u}, \bar{v})$  is  $(i^*, j^*) = (1, 1)$ . Also recall that by our assumption, the optimal action is unique. To get a lower bound, we will consider the same optimization than above but with  $B$  replaced with a subset of it. Clearly, this way we can only decrease the optimal value.

Concretely, we propose to consider only those models in  $B(\bar{u}, \bar{v})$  where only one parameter is changed at a time. Letting

$$\begin{aligned} B_U(\bar{u}, \bar{v}) &= \{(\bar{u}', \bar{v}') : \bar{u}' \in [0, 1]^K, \exists i_0 \in \{2, \dots, K\}, \epsilon \in [0, 1] \text{ s.t. } \forall i \neq i_0 : \bar{u}(i) = \bar{u}'(i) \text{ and } \bar{u}'(i_0) = \bar{u}(1) + \epsilon\}, \\ B_V(\bar{u}, \bar{v}) &= \{(\bar{u}, \bar{v}') : \bar{v}' \in [0, 1]^L, \exists j_0 \in \{2, \dots, L\}, \epsilon \in [0, 1] \text{ s.t. } \forall j \neq j_0 : \bar{v}(j) = \bar{v}'(j) \text{ and } \bar{v}'(j_0) = \bar{v}(1) + \epsilon\}, \end{aligned}$$

we define  $f'(\bar{u}, \bar{v})$  to be the optimal value of the above optimization problem when  $B(\bar{u}, \bar{v})$  is replaced by  $B_U(\bar{u}, \bar{v}) \cup B_V(\bar{u}, \bar{v}) \subset B(\bar{u}, \bar{v})$ . Now, say  $(\bar{u}', \bar{v}') \in B_U(\bar{u}, \bar{v})$  where, say, the index  $i_0$  where  $\bar{u}'(i_0) \neq \bar{u}(i_0)$  is 2. Then, for any  $j \in [L]$  and  $i \neq 2$ ,  $d(\bar{u}(i) \bar{v}(j), \bar{u}'(i) \bar{v}'(j)) = 0$  and for  $j \in [L]$ ,  $i = 2$ ,  $d(\bar{u}(i) \bar{v}(j), \bar{u}'(i) \bar{v}'(j)) = d(\bar{u}(2) \bar{v}(j), (\bar{u}(1) + \epsilon) \bar{v}(j))$ . Hence,

$$\sum_{i=1}^K \sum_{j=1}^L d(\bar{u}(i) \bar{v}(j), \bar{u}'(i) \bar{v}'(j)) = \sum_{j=1}^L d(\bar{u}(2) \bar{v}(j), (\bar{u}(1) + \epsilon) \bar{v}(j)).$$

Reasoning similarly for  $B_V(\bar{u}, \bar{v})$ , we see that  $f'(\bar{u}, \bar{v})$  satisfies

$$\begin{aligned} f'(\bar{u}, \bar{v}) &= \inf_{c \in [0, \infty)^{K \times L}} \sum_{i=1}^K \sum_{j=1}^L (\bar{u}(i^*) \bar{v}(j^*) - \bar{u}(i) \bar{v}(j)) c_{i,j} \\ \text{s.t. } \forall \epsilon_V &\in (0, 1 - \bar{v}(1)], \epsilon_U \in (0, 1 - \bar{u}(1)] \\ \forall j \neq 1, &\sum_{i=1}^K d(\bar{u}(i) \bar{v}(j), \bar{u}(i) (\bar{v}(1) + \epsilon_V)) c_{i,j} \geq 1 \\ \forall i \neq 1, &\sum_{j=1}^L d(\bar{u}(i) \bar{v}(j), (\bar{u}(1) + \epsilon_U) \bar{v}(j)) c_{i,j} \geq 1. \end{aligned}$$

Clearly, the smallest the coefficients of  $(c_{i,j})$  are in the constraints, the more constraining are the constraints. We obtain the smallest coefficients when  $\epsilon_V, \epsilon_U \rightarrow 0$ . By continuity, we get

$$\begin{aligned} f'(\bar{u}, \bar{v}) &= \inf_{c \in [0, \infty)^{K \times L}} \sum_{i=1}^K \sum_{j=1}^L (\bar{u}(i^*) \bar{v}(j^*) - \bar{u}(i) \bar{v}(j)) c_{i,j} \\ \text{s.t. } \forall j \neq 1, &\sum_{i=1}^K d(\bar{u}(i) \bar{v}(j), \bar{u}(i) \bar{v}(1)) c_{i,j} \geq 1 \\ \forall i \neq 1, &\sum_{j=1}^L d(\bar{u}(i) \bar{v}(j), \bar{u}(1) \bar{v}(j)) c_{i,j} \geq 1. \end{aligned}$$

Let

$$c_{i,j} = \begin{cases} 1/d(\bar{u}(i)\bar{v}(1), \bar{u}(1)\bar{v}(1)), & j = 1 \text{ and } i > 1; \\ 1/d(\bar{u}(1)\bar{v}(j), \bar{u}(1)\bar{v}(1)), & i = 1 \text{ and } j > 1; \\ 0, & \text{otherwise.} \end{cases}$$

We claim that  $(c_{i,j})$  is an optimal solution for the problem defining  $f'$ .

First, we show that  $(c_{i,j})$  is feasible. Let  $i \neq 1$ . Then  $\sum_{j=1}^L d(\bar{u}(i)\bar{v}(j), \bar{u}(1)\bar{v}(j))c_{i,j} = d(\bar{u}(i)\bar{v}(1), \bar{u}(1)\bar{v}(1))c_{i,1} = 1$ . Similarly, we can verify the other constraint, too, showing that  $(c_{i,j})$  is indeed feasible.

Now, it remains to show that the proposed solution is indeed optimal. We prove this by contradiction, following the ideas of [6]. We suppose that there exists a solution  $c$  of the optimization problem such that  $c_{i_0, j_0} > 0$  for  $i_0 \neq 1$  and  $j_0 \neq 1$ . Then, we prove that it is possible to find another feasible solution  $c'$  but with an objective lower than that obtained with  $c$ , contradicting the assumption of optimality of  $c$ .

We define  $c'$  as follows, redistributing the mass of  $c_{i_0, j_0}$  on the first row and the first column:

$$c'_{i,j} = \begin{cases} 0, & i = i_0 \text{ and } j = j_0; \\ c_{i_0,1} + c_{i_0, j_0} \frac{d(\bar{u}(i_0)\bar{v}(j_0), \bar{u}(1)\bar{v}(j_0))}{d(\bar{u}(i_0)\bar{v}(1), \bar{u}(1)\bar{v}(1))}, & i = i_0 \text{ and } j = 1; \\ c_{1, j_0} + c_{i_0, j_0} \frac{d(\bar{u}(i_0)\bar{v}(j_0), \bar{u}(i_0)\bar{v}(1))}{d(\bar{u}(1)\bar{v}(j_0), \bar{u}(1)\bar{v}(1))}, & i = 1 \text{ and } j = j_0; \\ c_{i,j}, & \text{otherwise.} \end{cases}$$

It is easily verified that if  $c$  satisfies the constraints, then so does  $c'$  because the missing mass of  $c_{i_0, j_0}$  is simply redistributed on  $c'_{i_0,1}$  and  $c'_{1, j_0}$ . For example, for  $i = i_0$  we have

$$\begin{aligned} & \sum_{j=1}^L d(\bar{u}(i_0)\bar{v}(j), \bar{u}(1)\bar{v}(j))c'_{i_0, j} - \sum_{j=1}^L d(\bar{u}(i_0)\bar{v}(j), \bar{u}(1)\bar{v}(j))c_{i_0, j} \\ &= d(\bar{u}(i_0)\bar{v}(1), \bar{u}(1)\bar{v}(1))c_{i_0, j_0} \frac{d(\bar{u}(i_0)\bar{v}(j_0), \bar{u}(1)\bar{v}(j_0))}{d(\bar{u}(i_0)\bar{v}(1), \bar{u}(1)\bar{v}(1))} - c_{i_0, j_0} d(\bar{u}(i_0)\bar{v}(j_0), \bar{u}(1)\bar{v}(j_0)) \\ &= 0 \end{aligned}$$

while for  $i \notin \{1, i_0\}$ ,  $c'_{i,j} = c_{i,j}$ , so  $\sum_{j=1}^L d(\bar{u}(i)\bar{v}(j), \bar{u}(1)\bar{v}(j))c'_{i,j} = \sum_{j=1}^L d(\bar{u}(i)\bar{v}(j), \bar{u}(1)\bar{v}(j))c_{i,j}$ .

Now, we prove that the objective function is lower for  $c'$  than for  $c$  by showing that the difference between them is negative:

$$\begin{aligned} \Delta &\doteq \sum_{i=1}^K \sum_{j=1}^L (\bar{u}(1)\bar{v}(1) - \bar{u}(i)\bar{v}(j))c'_{i,j} - \sum_{i=1}^K \sum_{j=1}^L (\bar{u}(1)\bar{v}(1) - \bar{u}(i)\bar{v}(j))c_{i,j} \\ &= c_{i_0, j_0} (\bar{u}(1)\bar{v}(1) - \bar{u}(i_0)\bar{v}(1)) \frac{d(\bar{u}(i_0)\bar{v}(j_0), \bar{u}(1)\bar{v}(j_0))}{d(\bar{u}(i_0)\bar{v}(1), \bar{u}(1)\bar{v}(1))} \\ &\quad + c_{i_0, j_0} (\bar{u}(1)\bar{v}(1) - \bar{u}(1)\bar{v}(j_0)) \frac{d(\bar{u}(i_0)\bar{v}(j_0), \bar{u}(i_0)\bar{v}(1))}{d(\bar{u}(1)\bar{v}(j_0), \bar{u}(1)\bar{v}(1))} \\ &\quad - c_{i_0, j_0} (\bar{u}(1)\bar{v}(1) - \bar{u}(i_0)\bar{v}(j_0)) \\ &= c_{i_0, j_0} \left\{ (\bar{u}(1) - \bar{u}(i_0))\bar{v}(1) \frac{d(\bar{u}(i_0)\bar{v}(j_0), \bar{u}(1)\bar{v}(j_0))}{d(\bar{u}(i_0)\bar{v}(1), \bar{u}(1)\bar{v}(1))} \right. \\ &\quad \left. + (\bar{v}(1) - \bar{v}(j_0))\bar{u}(1) \frac{d(\bar{u}(i_0)\bar{v}(j_0), \bar{u}(i_0)\bar{v}(1))}{d(\bar{u}(1)\bar{v}(j_0), \bar{u}(1)\bar{v}(1))} \right. \\ &\quad \left. - (\bar{u}(1)\bar{v}(1) - \bar{u}(i_0)\bar{v}(j_0)) \right\} \end{aligned}$$

Writing

$$\bar{u}(1)\bar{v}(1) - \bar{u}(i_0)\bar{v}(j_0) = (\bar{u}(1) - \bar{u}(i_0))\bar{v}(j_0) + (\bar{v}(1) - \bar{v}(j_0))\bar{u}(1)$$

we get

$$\begin{aligned} \Delta &= c_{i_0, j_0}(\bar{u}(1) - \bar{u}(i_0)) \left( \bar{v}(1) \frac{d(\bar{u}(i_0)\bar{v}(j_0), \bar{u}(1)\bar{v}(j_0))}{d(\bar{u}(i_0)\bar{v}(1), \bar{u}(1)\bar{v}(1))} - \bar{v}(j_0) \right) \\ &\quad + c_{i_0, j_0}(\bar{v}(1) - \bar{v}(j_0)) \left( \bar{u}(1) \frac{d(\bar{u}(i_0)\bar{v}(j_0), \bar{u}(i_0)\bar{v}(1))}{d(\bar{u}(1)\bar{v}(j_0), \bar{u}(1)\bar{v}(1))} - \bar{u}(1) \right). \end{aligned}$$

To finish the proof, it suffices to prove that both terms of the above sum are negative. First,  $\bar{u}(1) - \bar{u}(i_0), \bar{v}(1) - \bar{v}(j_0), c_{i_0, j_0} > 0$ , hence it remain to consider the terms involving the ratios of KL divergences. For the first such term, note that the difference is increased if  $-\bar{v}(j_0)$  is replaced by  $\bar{v}(1)$ . Thus it remains to see whether the ratios of the KL divergences are below one. Note that both ratios take the form  $\frac{d(\alpha p, \alpha q)}{d(p, q)}$  with  $\alpha < 1$ . Lemma 5 proven below shows that the function  $\alpha \mapsto d(\alpha p, \alpha q)$  is increasing on  $(0, 1)$ , showing that

$$\frac{d(\bar{u}(i_0)\bar{v}(j_0), \bar{u}(1)\bar{v}(j_0))}{d(\bar{u}(i_0)\bar{v}(1), \bar{u}(1)\bar{v}(1))}, \quad \frac{d(\bar{u}(i_0)\bar{v}(j_0), \bar{u}(i_0)\bar{v}(1))}{d(\bar{u}(1)\bar{v}(j_0), \bar{u}(1)\bar{v}(1))} < 1.$$

Thus, the proof is finished once we prove Lemma 5.

**Lemma 5.** *Let  $p, q$  be any fixed real numbers in  $(0, 1)$ . The function  $f : \alpha \mapsto d(\alpha p, \alpha q)$  is convex and increasing on  $(0, 1)$ . As a consequence, for any  $\alpha < 1$ ,  $d(\alpha p, \alpha q) < d(p, q)$ .*

*Proof.* We first re-parametrize our problem into polar coordinates  $(r, \theta)$  :

$$\begin{cases} p &= r \cos \theta \\ q &= r \sin \theta \end{cases}$$

In order to prove the statement of the lemma, it now suffices to prove that  $f_\theta : r \mapsto d(r \sin \theta, r \cos \theta)$  is increasing. We have

$$f_\theta(r) = r \cos \theta \log \left( \frac{\cos \theta}{\sin \theta} \right) + (1 - r \cos \theta) \log \left( \frac{1 - r \cos \theta}{1 - r \sin \theta} \right)$$

which can be differentiated along  $r$  for a fixed  $\theta$  :

$$f'_\theta(r) = \cos \theta \log \left( \frac{1 - r \sin \theta}{1 - r \cos \theta} \right) + \frac{\sin \theta - \cos \theta}{1 - r \sin \theta} + \cos \theta \log \left( \frac{\cos \theta}{\sin \theta} \right).$$

Now, we can differentiate again along  $r$  and after some calculations we obtain

$$f''_\theta(r) = \frac{(\sin \theta - \cos \theta)^2}{(1 - r \sin \theta)^2(1 - r \cos \theta)} > 0$$

which proves that the function  $f_\theta$  is convex. It remains to prove that  $f'_\theta(0) \geq 0$  for any  $\theta \in (0, \pi/2)$ . We rewrite  $f'_\theta(0)$  as a function of  $\theta$  :

$$\begin{aligned} f'_\theta(0) &= \cos \theta \log \left( \frac{\cos \theta}{\sin \theta} \right) + \sin \theta - \cos \theta \\ &:= \phi(\theta) \end{aligned}$$

Let us assume that there exists  $\theta_0 \in (0, \pi/2)$  such that  $\phi(\theta_0) < 0$ . Then, in this direction  $f'_\theta(0) < 0$  and as  $f_\theta(0) = 0$  for any  $\theta \in (0, \pi/2)$ , it means that there exists  $r_0 > 0$  such that  $f_{\theta_0}(r_0) < 0$ . Yet,  $f_{\theta_0}(r_0) = d(r_0 \cos \theta_0, r_0 \sin \theta_0) > 0$  because of the positivity of the KL divergence.

So by contradiction, we proved that for all  $\theta \in (0, \pi/2)$ ,  $f'_\theta(0) = \phi(\theta) \geq 0$  and by convexity  $f_\theta$  is non-negative and non-decreasing on  $[0, +\infty)$ .

■

### B.1 Gaussian payoffs

The lower bound naturally extends to other classes of distributions, such as Gaussians. For illustration here we show the lower bound for this case. We still assume that the means are in  $[0, 1]$ , as before. We also assume that all payoffs have a common variance  $\sigma^2 > 0$ . Recall that the Kullback-Leibler divergence between two distributions with fixed variance  $\sigma^2$  is  $d(p, q) = (p - q)^2 / (2\sigma^2)$ . Then, the proof of Theorem 2 can be repeated with minor differences (in particular, the proof of the analogue of Lemma 5 becomes trivial) and we get the following result:

**Theorem 3.** *For any  $(\bar{u}, \bar{v}) \in [0, 1]^K \times [0, 1]^L$  with a unique optimal action and any uniformly efficient algorithm  $\mathcal{A}$  whose regret is  $R(n)$ , assuming Gaussian row and column rewards with common variance  $\sigma^2$ ,*

$$\liminf_{n \rightarrow \infty} \frac{R(n)}{\log(n)} \geq \frac{2\sigma^2}{\bar{v}(j^*)} \sum_{i \in [K] \setminus \{i^*\}} \frac{1}{\Delta_i^{\bar{u}}} + \frac{2\sigma^2}{\bar{u}(i^*)} \sum_{j \in [L] \setminus \{j^*\}} \frac{1}{\Delta_j^{\bar{v}}}.$$