

Uncertainty and Performance of Adaptive Controllers for Functionally Uncertain Output Feedback Systems

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Abstract

We consider nonlinear systems in an output feedback form which are functionally known up to a L^2 measure of uncertainty. The control task is to drive the output of the system to some neighbourhood of the origin. A modified L^2 measure of transient performance (penalising both state and control effort) is given, and the performance of a class of model based adaptive controllers is studied. An upper performance bound is derived.

1 Introduction

Until recently a major deficit in adaptive control theory has been the complete lack of performance analysis, where the costs are chosen to penalise both the state or output transient *and the control effort*. For Lyapunov designs some recent results have begun to rectify this situation. In [1] an optimal adaptive controller was given, but the construction of this controller involves the solution to a generally intractable HJI PDE. In [7] an inverse optimal design was constructed. Inverse optimal results controllers have desirable robustness properties, but the cost functionals are not-specified a-priori. In the thesis [3], upper bounds are given on a-priori given cost functionals for Lyapunov based designs.

The results given in this paper are in the framework of approximate adaptive control, where the uncertainty is assumed to be static and functional, and an approximate parametric model is utilised to give an adaptive design. We consider output feedback models [8], [10], [6]:

$$\begin{aligned}\Sigma_{\Delta} : \dot{x}_i &= x_{i+1} + f_i(y) \quad 1 \leq i \leq n-1 \\ \dot{x}_n &= u + f_n(y) \\ y &= x_1.\end{aligned}\tag{1}$$

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The state vector $x = (x_1, \dots, x_n)^T \in \mathcal{X} = \mathbb{R}^n$ is assumed to be unavailable for measurement, only the output $y \in \mathcal{O} = \mathbb{R}$ can be measured. f denotes the uncertain function $f = (f_1, \dots, f_n)^T$, and $f^0 = (f_1^0, \dots, f_n^0)^T$ represents the (known) nominal system. We assume that $f, f^0 \in C^{n+3}(\mathcal{O}, \mathcal{X})$. It should also be observed that, for simplicity, as in [10], we are considering a slightly simpler normal form than [8], [6]. The uncertainty set we consider is:

$$\begin{aligned}\Delta &= \Delta(L^2(\Omega; w), f^0, \delta) \\ &= \{f \in C^{n+3}(\mathcal{O}, \mathcal{X}) \mid f - f^0 \in K, \\ &\quad \|f_i - f_i^0\|_{L^2(\Omega; w_i)} \leq \delta_i \quad 1 \leq i \leq n\},\end{aligned}\tag{2}$$

where $\Omega \subset \mathcal{O}$ is not necessarily compact. K denotes an approximation theoretic smoothness class, such as a bound in a Sobolev space or a known moduli of continuity. Knowledge of such a class K , allows us to determine the required resolution of the adaptive model using a result such as Jackson's theorem [9].

For simplicity we consider the control task of driving y to a small neighbourhood of 0, $[-\sqrt{2}\eta, +\sqrt{2}\eta]$, (ie. to guarantee the existence of a $\tau < \infty$ s.t. $y(\tau) = \pm\sqrt{2}\eta$), whilst keeping x bounded. Using appropriate robust modifications to the controller, these results can be extended to stabilization results [4]. Performance will be measured as a modified LQ cost functional penalising the output and the control:

$$\mathcal{P} = \mathcal{P}(Q, k, \eta) = \sup_{f \in \Delta} \int_0^\tau z(t)^T Q z(t) + k u^2(t) dt, \tag{4}$$

where $z(t)$ is the solution of the error system to be defined below; $Q = \text{diag}\{c_1, \dots, c_n\}$ and is positive definite, and $\tau = \inf\{t \geq 0 \mid y(t)^2 = 2\eta^2\}$. Noting that the transformed state z_1 will be equal to $x_1 = y$, the performance $\mathcal{P}$ bounds the important cost:

$$\mathcal{P}' = \sup_{f \in \Delta} \int_0^\tau c_1 y^2(t) + k u^2(t) dt. \tag{5}$$

We give both semi-global (for finite dimensional models) and global (for so called SFD models) results. Importantly, we obtain upper bounds on the performance. **Notation.** $\mathcal{W}$, $\mathcal{X}$, $\mathcal{O}$ denote the weight, state and output spaces respectively, all are taken to be Euclidean spaces. L_p denotes the standard Lebesgue space. If the eigenvalues of a matrix R

are $\lambda_1, \dots, \lambda_n$, then $\bar{\lambda}(R), \underline{\lambda}(R)$ are defined to be $\max_{1 \leq i \leq n} |\lambda_i|, \min_{1 \leq i \leq n} |\lambda_i|$ respectively. $m(\Omega)$ denotes the Lebesgue measure of Ω , and the weighted L^2 space $L^2(\Omega; w)$ has the inner product $\langle f, g \rangle = \int_{\Omega} f g w$.

2 Adaptive Control Design

The adaptive control methodology is based on that of [8, 6] and robust backstepping [2, 5]. First we introduce the model $\Phi = (\phi_1, \phi_2, \dots, \phi_n)^T$, $\phi_i: P_i \mathcal{X} \rightarrow \mathcal{W}_i$, $\phi_i = (\varphi_{i1}, \varphi_{i2}, \dots, \varphi_{im_i})^T$ where $\mathcal{W}_i = \mathbf{R}^{m_i}$ and where for convenience we assume that $\varphi_{ij} \in C^{n+3}(\mathcal{O}, \mathbf{R})$. Then we write the system in the form:

$$\begin{aligned} \dot{x}_i &= x_{i+1} + f_i^0(y) + \theta_i^T \phi_1(y) + d_i^f(y) \quad 1 \leq i \leq n-1 \\ \dot{x}_n &= u + f_n^0(y) + \theta_n^T \phi_n(y) + d_n^f(y) \end{aligned} \quad (6)$$

where $d_i^f = f_i - f_i^0 - \theta_i^T \phi_i$. Where f is clear from the context, we often write d_i for d_i^f . $d = d^f$ defines the vector $d^f = (d_1^f, \dots, d_n^f)^T$.

However, to define filters for the system, it is convenient to write the system in a similar form to [6]. We reparameterize the system as follows. Let

$$\theta' = (\theta'_1, \theta'_2, \dots, \theta'_m)^T = (\theta_1^T | \theta_2^T | \dots | \theta_n^T)^T \in \mathcal{W} = \mathbf{R}^m, \quad (7)$$

where $m = \sum_{i=1}^n m_i$ and define

$$\begin{aligned} \phi'_1 &= (\phi_1^T | 0 | \dots | 0)^T, \\ \phi'_2 &= (0 | \phi_2^T | \dots | 0)^T, \\ &\vdots \\ \phi'_n &= (0 | \dots | 0 | \phi_n^T)^T. \end{aligned} \quad (8)$$

Then define

$$\Phi'_j = (\phi'_{1j}, \phi'_{2j}, \dots, \phi'_{nj})^T. \quad (9)$$

Note that by definition, $\Phi'_j: \mathcal{O} \rightarrow \mathcal{X}$ only has one non-zero entry. Then we can rewrite the system in the alternative form:

$$\dot{x} = Ax + e_n u + f^0(y) + \sum_{j=1}^m \theta'_j \Phi'_j(y) + d^f(y) \quad (10)$$

where A is the matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}. \quad (11)$$

The system is then in the required form but with additional disturbance terms. We now follow the definition

of the filters for state estimation in [6]: Suppose the gain vector $\gamma = (\gamma_1, \dots, \gamma_n)^T \in \mathbf{R}^n$ is such that

$$A_0 = A - \gamma e_1^T \quad (12)$$

is Hurwitz. Such a gain vector γ is said to be admissible. The nominal (ω), model (ζ) and control (ν) filters are defined as follows:

$$\dot{\omega} = A_0 \omega + \gamma y + f^0(y), \quad \omega(0) = 0, \quad \omega \in \mathcal{X} \quad (13)$$

$$\begin{aligned} \dot{\zeta}_j &= A_0 \zeta_j + \Phi'_j(y), \quad \zeta_j(0) = 0, \quad \zeta_j \in \mathcal{X} \\ 1 &\leq j \leq m \end{aligned} \quad (14)$$

$$\dot{\nu} = A_0 \nu + e_n u, \quad \nu(0) = 0, \quad \nu \in \mathcal{X}. \quad (15)$$

For convenience ζ denotes the vector $\zeta = (\zeta_1, \dots, \zeta_m)^T$. The state estimation error $\varepsilon \in \mathcal{X}$ is defined to be:

$$\varepsilon = x - \left(\omega + \sum_{j=1}^m \theta'_j \zeta_j + \nu \right). \quad (16)$$

The error system recursively defines a vector $z = (z_1, \dots, z_n)^T \in \mathcal{Z} = \mathbf{R}^n$ as follows:

$$\begin{aligned} z_1 &= y \\ z_i &= \nu_i - \alpha_{i-1} \quad 2 \leq i \leq n \end{aligned} \quad (17)$$

where

$$\alpha_i = \alpha_i(y, \nu_1, \dots, \nu_i, \omega, \zeta; \tilde{\theta}_k, \hat{\theta}'_k \quad 1 \leq k \leq i) \quad (18)$$

with $\tilde{\theta}_k \in \mathcal{W}_1$, $\hat{\theta}'_k = (\hat{\theta}'_{1k}, \dots, \hat{\theta}'_{mk})^T \in \mathcal{W}$ denoting the parameter estimates of $\theta_1 \in \mathcal{W}_1$ and $\theta' = (\theta'_1, \dots, \theta'_m)^T \in \mathcal{W}$ at step k $1 \leq k \leq n$ respectively. The functions α_i $1 \leq i \leq n$ are defined:

$$\begin{aligned} \alpha_0 &= 0, \\ \alpha_1 &= -\omega_2 - \sum_{j=1}^m \hat{\theta}'_{j1} \zeta_{j,2} - \tilde{\theta}_1^T \phi_1(y) - \\ &\quad - f_1^0(y) - c_1 z_1 - (nl^2/3 + \kappa^2) z_1, \end{aligned} \quad (19)$$

and for $2 \leq i \leq n$,

$\alpha_i =$

$$\begin{aligned} &\frac{\partial \alpha_{i-1}}{\partial y} \left(\omega_2 + \sum_{j=1}^m \hat{\theta}'_{ji} \zeta_{j,2} + \nu_2 + \tilde{\theta}_i^T \phi_1(y) + f_1^0(y) \right) \\ &+ \frac{\partial \alpha_{i-1}}{\partial \omega} (A_0 \omega + \gamma y + f^0(y)) \\ &+ \sum_{j=1}^m \frac{\partial \alpha_{i-1}}{\partial \zeta_j} (A_0 \zeta_j + \Phi'_j(y)) \\ &+ \sum_{k=1}^{i-1} \left(\frac{\partial \alpha_{i-1}}{\partial \hat{\theta}'_k} \hat{\beta}_k + \frac{\partial \alpha_{i-1}}{\partial \tilde{\theta}_k} \tilde{\beta}_k \right) \\ &+ \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \nu_j} (\nu_{j+1} - \gamma_j \nu_1) \\ &- \left(\frac{nl^2}{3} + \kappa^2 \right) \left(\frac{\partial \alpha_{i-1}}{\partial y} \right)^2 z_i - c_i z_i - z_{i-1} + \gamma_i \nu_1 \end{aligned} \quad (20)$$

where

$$\begin{aligned}\tilde{\beta}_1 &= \alpha z_1 \tilde{G} \phi_1(y) \quad \tilde{\beta}_1 \in \mathcal{W}_1, \\ \tilde{\beta}_k &= -\alpha z_k \frac{\partial \alpha_{k-1}}{\partial y} \tilde{G} \phi_1(y) \quad \tilde{\beta}_k \in \mathcal{W}_1 \quad 2 \leq k \leq n \\ \hat{\beta}_1 &= \alpha z_1 \hat{G}(\zeta_{1,2}, \dots, \zeta_{m,2})^T \quad \hat{\beta}_1 \in \mathcal{W} \\ \hat{\beta}_k &= -\alpha z_k \frac{\partial \alpha_{k-1}}{\partial y} \hat{G}(\zeta_{1,2}, \dots, \zeta_{m,2})^T \\ &\quad \hat{\beta}_k \in \mathcal{W}, \quad 2 \leq k \leq n\end{aligned}\quad (21)$$

and where $\tilde{G}, \hat{G}$ are defined from adaptive structure $G = (G_1, \dots, G_n)$ as

$$\begin{aligned}\tilde{G} &= G_1 \\ \hat{G} &= \text{diag}(G_1, \dots, G_n),\end{aligned}\quad (22)$$

where G_i $1 \leq i \leq n$ are positive definite matrices. $\alpha > 0$ is the adaptive gain, $\kappa > 0$ is the robust gain and $l > 0$ is the state estimation robust gain. Note that by the differentiability assumptions on f^0, Φ , it follows that for $1 \leq i \leq n$, α_i is defined and at least C^1 , hence locally Lipschitz.

The controller, $\Xi = \Xi(G, Q, \alpha, \Phi, \kappa, l, \gamma)$, (where recall that $Q = \text{diag}(c_1, \dots, c_n)$), is taken to be:

$$u = \alpha_n(y, \nu, \omega, \zeta_j \quad 1 \leq j \leq n, \tilde{\theta}_k, \hat{\theta}'_k \quad 1 \leq k \leq n) \quad (23)$$

with parameter update laws $1 \leq i \leq n$:

$$\begin{aligned}\dot{\tilde{\theta}}_i &= \tilde{\beta}_i, \quad \tilde{\theta}_i(0) = 0, \tilde{\theta}_i \in \mathcal{W}_1 \\ \dot{\hat{\theta}}'_i &= \hat{\beta}_i, \quad \hat{\theta}'_i(0) = 0, \hat{\theta}'_i = (\hat{\theta}'_{1i}, \dots, \hat{\theta}'_{mi})^T \in \mathcal{W}\end{aligned}\quad (24)$$

and filters 13, 14, 15.

It will be convenient to partition $\hat{\theta}'_i \in \mathcal{W}$ as follows:

$$\begin{aligned}\hat{\theta}'_i &= (\hat{\theta}_{1i}, \hat{\theta}_{2i}, \dots, \hat{\theta}_{mi})^T \\ &= (\hat{\theta}_{1i}^T \mid \hat{\theta}_{2i}^T \mid \dots \mid \hat{\theta}_{ni}^T)^T\end{aligned}\quad (25)$$

where $\hat{\theta}_{ki} \in \mathcal{W}_k$ for $1 \leq i, k \leq n$. $\hat{\theta}_{ki}$ is then the adaptive estimate at step k of the parameter θ_i . Note that θ_1 plays a special role, it has two parameter estimates constructed for it at each step k , namely $\tilde{\theta}_k, \hat{\theta}_{1k}$. The parameter estimate vectors, $\tilde{\theta}_i, \hat{\theta}'_i, 1 \leq i \leq n$ are concatenated into the vectors $\tilde{\Theta} = (\tilde{\theta}_1, \dots, \tilde{\theta}_n)^T \in \mathcal{W}_1^n$, $\hat{\Theta}' = (\hat{\theta}'_1, \dots, \hat{\theta}'_n)^T \in \mathcal{W}^n$. Let T denote the map:

$$((y, \nu_2, \dots, \nu_n), \nu_1, \omega, \zeta, \tilde{\Theta}, \hat{\Theta}') \mapsto (z, \nu_1, \omega, \zeta, \tilde{\Theta}, \hat{\Theta}') \quad (26)$$

defined by equations 17, 19.

The model error constants are $q_i, 1 \leq i \leq n, g$ and s and are such that:

$$\begin{aligned}\sup_{f \in \Delta} \|d_i^f\|_{L^2(\Omega, w)} &\leq q_i, \\ \sup_{f \in \Delta} \sup_{y \in \Omega} \|d^f T(y) P_0 + P_0 d^f(y)\|_2 &\leq g \\ \sup_{f \in \Delta} \|d_1^f\|_{C(\Omega)} &\leq s,\end{aligned}\quad (27)$$

where P_0 is the solution to the Lyapunov equation $A_0^T P_0 + P_0 A_0 = -I$.

3 Stability and Performance

Before we proceed to the performance bound, we first give an elementary lemma:

Lemma 1 Consider the following system on the time interval $[0, \tau], \tau < \infty$:

$$\dot{x} = Ax + d(t) \quad x \in \mathbb{R}^n, \quad x(0) = 0, \quad (28)$$

where A is a Hurwitz matrix. If P is the solution to the Lyapunov equation $A^T P + PA = -I$, then:

$$x^T(t)Px(t) \leq 4\bar{\lambda}(P)^3 \sup_{s \in [0, \tau]} \|d(s)\|^2. \quad (29)$$

Proof: The proof is elementary and omitted for brevity. ■

We now give the key proposition.

Proposition 2 Consider the system Σ , given by equation 1, with initial conditions $x_0 \in \mathcal{X}$ and suppose further that $y^2(0) > 2\eta^2$. Let Q be a positive diagonal matrix and suppose $\eta > 0, k > 0$ and consider the performance $\mathcal{P}_f(Q, k, \eta)$. Let the adaption gain be $\alpha > 0$ and let the control and state estimation robust gains be $\kappa > 0, l > 0$ respectively. Let $G_1, \dots, G_n$ be positive definite adaptive structure matrices, and let γ be an admissible filter gain vector. Define P_0 as the solution to the Lyapunov equation:

$$A_0^T P_0 + P_0 A_0 = -I, \quad (30)$$

where A_0 is defined by equation 12. and let V be defined:

$$\begin{aligned}V(z, \varepsilon, \hat{\Theta}', \tilde{\Theta}) &= \frac{1}{2} z^T z + \frac{1}{l^2} \varepsilon^T P_0 \varepsilon \\ &\quad + \frac{1}{2\alpha} \sum_{i=1}^n (\theta_1 - \tilde{\theta}_i)^T G_1^{-1} (\theta_1 - \tilde{\theta}_i) \\ &\quad + \frac{1}{2\alpha} \sum_{i=1}^n \sum_{j=1}^n (\theta_j - \hat{\theta}_{ji})^T G_j^{-1} (\theta_j - \hat{\theta}_{ji}).\end{aligned}\quad (31)$$

Let $V_0 = V(z(0), \varepsilon(0), \hat{\Theta}'(0), \tilde{\Theta}(0))$, and suppose the approximation region Ω is such that i) $[-\sqrt{2V_0}, \sqrt{2V_0}] \subset \Omega^\circ$, and ii) $p = \frac{n s^2}{4 \kappa^2} + \frac{g^2}{l^2} < 2\lambda(Q)\eta^2$. Implement the controller Ξ 23, 24, 21 with adaptive gain $\alpha > 0$ and robustness gain $\kappa > 0$. Then there exists $\tau < \infty$ such that: 1. (Σ, Ξ) is well posed on $[0, \tau]$, and all signals

are uniquely defined, 2. $y(\tau) = \pm\sqrt{2}\eta$, and 3.

$$\begin{aligned} \mathcal{P}(Q, k, \eta) &\leq \frac{2\lambda(Q)\eta^2}{2\lambda(Q)\eta^2 - p}(V_0 - \eta^2) + \\ &+ \frac{1}{2\lambda(Q)\eta^2 - p} \int_{\eta^2}^{V_0} \tilde{u}^2(v) dv \quad (32) \end{aligned}$$

where $\tilde{u}: \mathbb{R} \rightarrow \mathbb{R}$ is defined:

$$\begin{aligned} \tilde{u}(r) &= \sup \left\{ |u(y, \nu, \zeta, \omega, \hat{\theta}', \tilde{\theta})| \in \mathbb{R} \mid \right. \\ &\quad \zeta_j^T P_0 \zeta_j \leq 4\bar{\lambda}(P_0)^3 \sup_{y^2 \leq 2V_0} \|\Phi_j'(y)\|; \\ &\quad \omega^T P_0 \omega \leq 4\bar{\lambda}(P_0)^3 \sup_{y^2 \leq 2V_0} \|\gamma y + f^0(y)\|; \\ &\quad (y, \nu_2, \dots, \nu_n) = z; \\ &\quad \nu_1 = y - \omega_1 - \sum_{j=1}^m \theta_j' \zeta_{j,1} - \varepsilon_1; \\ &\quad \left. \eta^2 \leq V(z, \varepsilon, \hat{\theta}', \tilde{\theta}) \leq r \right\}. \end{aligned}$$

Proof: (Some details omitted). Some algebraic manipulations (which we omit) yield the system in the z coordinates. $\forall z \in \mathcal{Z}$ we have:

$$\begin{aligned} \dot{z}_1 &= -c_1 z_1 + z_2 + \sum_{j=1}^m (\theta_j' - \hat{\theta}_{j1}) \zeta_{j,2} \\ &+ (\theta_1 - \tilde{\theta}_1)^T \phi_1(y) + (d_1(y) - \kappa^2 z_1) + (\varepsilon_2 - \frac{nl^2}{3} z_1). \end{aligned} \quad (33)$$

For $2 \leq i \leq n-1$ we have:

$$\begin{aligned} \dot{z}_i &= z_{i+1} - c_i z_i - z_{i-1} - \frac{\partial \alpha_{i-1}}{\partial y} \sum_{j=1}^m (\theta_j' - \hat{\theta}_{ji}) \zeta_{j,2} - \\ &- \frac{\partial \alpha_{i-1}}{\partial y} (\theta_1 - \tilde{\theta}_i)^T \phi_1(y) \\ &+ \frac{\partial \alpha_{i-1}}{\partial y} (-\varepsilon_2 - \frac{nl^2}{3} \frac{\partial \alpha_{i-1}}{\partial y} z_i) \\ &+ \frac{\partial \alpha_{i-1}}{\partial y} (-d_1(y) - \kappa^2 \frac{\partial \alpha_{i-1}}{\partial y} z_i) \end{aligned} \quad (34)$$

Similarly for $i = n$ we have:

$$\begin{aligned} \dot{z}_n &= -c_n z_n - z_{n-1} - \frac{\partial \alpha_{n-1}}{\partial y} \sum_{j=1}^m (\theta_j' - \hat{\theta}_{jn}) \zeta_{j,2} - \\ &- \frac{\partial \alpha_{n-1}}{\partial y} (\theta_1 - \tilde{\theta}_n)^T \phi_1(y) \\ &+ \frac{\partial \alpha_{n-1}}{\partial y} (-\varepsilon_2 - \frac{nl^2}{3} \frac{\partial \alpha_{n-1}}{\partial y} z_n) + \\ &+ \frac{\partial \alpha_{n-1}}{\partial y} (-d_1(y) - \kappa^2 \frac{\partial \alpha_{n-1}}{\partial y} z_n). \end{aligned} \quad (35)$$

The dynamics of the state estimation error ε are easily shown to be given by:

$$\dot{\varepsilon} = A_0 \varepsilon + d(y). \quad (36)$$

We can now obtain the fundamental inequality. Let $L = [-2V_0, 2V_0]$ and $\Omega_0 = [-2\eta^2, 2\eta^2]$. By some estimates which again we must omit for brevity we obtain the following inequality $\forall y(t) \in L \setminus \Omega_0$:

$$\dot{V} \leq -(2\lambda(Q)\eta^2 - p). \quad (37)$$

By ii) we have $\dot{V} < 0 \forall y(t) \in L \setminus \Omega_0$, so since $y(0) \in L$ and $z_1 = y$, it follows from the definition of V ($\frac{1}{2}y^2(t) \leq V(t)$), that $y(t)$ is bounded by L , and the above inequality 37 holds $\forall t \in [0, \tau]$, where τ is defined to be the first time at which $y(\tau) = \pm\sqrt{2}\eta$. Since V is decreasing on $[0, \tau]$, it also follows that $z, \varepsilon, \hat{\theta}, \tilde{\theta}$ are bounded. We now show the boundedness of the other signals (ω, ζ, ν and x). By the definition of the filter 13, and Lemma 1 we have:

$$\begin{aligned} \omega^T P_0 \omega &\leq 4\bar{\lambda}(P_0)^3 \sup_{t \in [0, \tau]} \|\gamma y(t) + f^0(y(t))\| \\ &\leq \sup_{y^2 \leq 2V_0} \|\gamma y + f^0(y)\| \end{aligned} \quad (38)$$

since $y^2(t) \in L$. Similarly from the definition of the filter 14 we have $\zeta_j = A_0 \zeta_j + \Phi_j'(y)$, so by Lemma 1 we have for each $1 \leq j \leq m$:

$$\zeta_j^T P_0 \zeta_j \leq 4\bar{\lambda}(P_0)^3 \sup_{y^2 \leq 2V_0} \|\Phi_j'(y)\|. \quad (39)$$

We now show ν is bounded. Since $x_1 = y$, equation 16 implies:

$$\varepsilon_1 = y - \left(\omega_1 + \sum_{j=1}^m \theta_j' \zeta_{j,1} + \nu_1 \right), \quad (40)$$

from which the boundedness of ν_1 follows from the boundedness of $\theta', \zeta_1, \omega_1, y, \varepsilon_1$. If $\nu_1, \dots, \nu_i$ are bounded, then α_i is bounded, and hence by the boundedness of z and equation 17 it follows that ν_{i+1} is bounded. Hence ν is bounded. Boundedness of u follows from the boundedness of α_n . Boundedness of x follows from equation 16. Well posedness now follows from the standard existence and uniqueness results, which also guarantees the uniqueness of x, z, ω, ν and ζ . From the uniqueness of x it follows that y is uniquely defined.

Since $\dot{V} \leq -2\lambda(Q)\eta^2 + p \forall t \in [0, \tau]$, we have

$$\tau \leq \frac{\sup_{t \in [0, \tau]} V(t) - \inf_{t \in [0, \tau]} V(t)}{\inf_{t \in T_1} |\dot{V}(t)|} \leq \frac{V(0) - \eta^2}{2\lambda(Q)\eta^2 - p} < \infty. \quad (41)$$

Now we bound the state performance. We can show that

$$\dot{V} = -z^T Q z + D(t) \quad (42)$$

where

$$D(t) = \sum_{i=2}^n \left(z_i \frac{\partial \alpha_{i-1}}{\partial y} (-\varepsilon_2 - \frac{nl^2}{3} z_i \frac{\partial \alpha_{i-1}}{\partial y}) + \right.$$

$$\begin{aligned}
& z_i \frac{\partial \alpha_{i-1}}{\partial y} (-d_1 - \kappa^2 z_i \frac{\partial \alpha_{i-1}}{\partial y}) \\
& + z_1 (d_1 - \kappa^2 z_1) + z_1 (\varepsilon_2 - \frac{n l^2}{3} z_1) + \\
& \frac{1}{l^2} (-\varepsilon^T \varepsilon + d^T P_0 \varepsilon + \varepsilon^T P_0 d). \quad (43)
\end{aligned}$$

It is possible to establish that $\|D(\cdot)\|_{L^\infty} < p$. Then

$$\int_0^\tau z(t)^T Q z(t) dt = \int_0^\tau -\dot{V} dt + \int_0^\tau D(t) dt. \quad (44)$$

The first term is easily bounded:

$$\int_0^\tau -\dot{V} dt = V(0) - V(\tau) \leq V(0) - \eta^2. \quad (45)$$

Then a bound on $\int_0^\tau D(t) dt$ is given by:

$$\int_0^\tau |D(t)| dt \leq \tau \|D(\cdot)\|_{L^\infty} \leq \frac{p(V(0) - \eta^2)}{(2\lambda(Q)\eta^2 - p)} \quad (46)$$

since $\|D(\cdot)\|_{L^\infty} < p$ similarly to inequality 37. This, together with inequality 45 and equation 44 completes the estimate of the state performance measure. The control performance effort is bounded as follows. Consider $t \in [0, \tau]$ and define $r = V(z(t), \varepsilon(t), \hat{\Theta}'(t), \tilde{\Theta}(t))$, then $(y(t), \nu(t), \zeta(t), \omega(t), \hat{\Theta}'(t), \tilde{\Theta}(t))$ lies in the set:

$$\begin{aligned}
& \left\{ (y, \nu, \zeta, \omega, \hat{\Theta}', \tilde{\Theta}) \mid \zeta_j^T P_0 \zeta_j \leq 4\bar{\lambda}(P_0)^3 \sup_{y^2 \leq 2V_0} \|\Phi'_j(y)\|; \right. \\
& \omega^T P_0 \omega \leq 4\bar{\lambda}(P_0)^3 \sup_{y^2 \leq 2V_0} \|\gamma y + f^0(y)\|; \\
& (y, \nu_2, \dots, \nu_n) = z; \\
& \nu_1 = y - \omega_1 - \sum_{j=1}^m \theta'_j \zeta_{j,1} - \varepsilon_1; \\
& \left. \eta^2 \leq V(z, 0, \hat{\Theta}, \tilde{\Theta}) + \frac{1}{l^2} \varepsilon_1^2 \leq r \right\}. \quad (47)
\end{aligned}$$

Then we have the inequality:

$$u^2(t) \leq \tilde{u}^2 \left(V(z(t), \varepsilon(t), \hat{\Theta}'(t), \tilde{\Theta}(t)) \right), \quad (48)$$

and hence by the change of variables $v = V(z(t), \varepsilon(t), \hat{\Theta}'(t), \tilde{\Theta}(t))$ we have the bound:

$$\begin{aligned}
\int_0^\tau u^2(t) dt & \leq \int_0^\tau \tilde{u}^2(V(z(t), \varepsilon(t), \hat{\Theta}'(t), \tilde{\Theta}(t))) dt, \\
& = \int_{v(0)}^{v(\tau)} \tilde{u}^2(v) \frac{1}{-\dot{V}} dv, \\
& \leq \frac{1}{2\lambda(Q)\eta^2 - p} \int_{\eta^2}^{V_0} \tilde{u}^2(v) dv, \quad (49) \\
& \quad (50)
\end{aligned}$$

since $\dot{V} \leq -(2\lambda(Q)\eta^2 - p)$ for all $t \in [0, \tau]$. ■

We can now give the main theorems. Firstly we give an admissibility definition:

Definition 3 A model $\Phi = (\phi_1, \dots, \phi_n)$ is $(Q, \Omega, \Omega_0, \kappa, l)$ admissible if: s, g, q_i for $1 \leq i \leq n$ are finite, and $2\lambda(Q)\eta^2 > \frac{ns^2}{4\kappa^2} + \frac{g^2}{l^2}$.

Then:

Theorem 4 Let $\Omega \subset \mathcal{X}$ be a fixed closed set. Consider the system Σ_Δ given by equation 1 with functional uncertainty $\Delta \subset \Delta(L^2(\Omega; w), f^0, \delta)$ and initial condition $x_0 \in \mathcal{X}$, $y(0)^2 > 2\eta^2$. Consider the performance measure $\mathcal{P} = \mathcal{P}(Q, k, \Omega_0)$ for positive diagonal Q , and $k > 0$, $\eta > 0$. Let $G = (G_1, \dots, G_n)$ where G_i $1 \leq i \leq n$ are positive definite adaptive structure matrices. Implement the controller $\Xi(G, Q, \alpha, \Phi, \kappa, l, \gamma)$ where Φ is a finite dimensional model and $\alpha > 0$. Suppose the filter gain γ is admissible, and $\kappa > 0$, $l > 0$. Suppose Φ is $(Q, \Omega, \Omega_0, \kappa, l)$ admissible, and let

$$\begin{aligned}
W_\alpha &= \frac{1}{2} z_0^T z_0 + \frac{1}{l^2} x_0^T P_0 x_0 + \frac{n}{\alpha} \frac{(\delta_1 + q_1)^2}{\lambda(G_1)\lambda(R_1)} \\
&+ \frac{n}{2\alpha} \sum_{i=2}^n \frac{(\delta_i + q_i)^2}{\lambda(G_i)\lambda(R_i)}, \quad (51)
\end{aligned}$$

where $R_i = \langle \phi_i, \phi_j \rangle_{L^2(\Omega; w)}$ is the Gram matrix of the model component ϕ_i and P_0 is defined as in 2.

Then for all adaption gains $\alpha > 0$ and state estimation control gains $l > 0$ such that:

$$[-\sqrt{2W_\alpha}, \sqrt{2W_\alpha}] \subset \Omega^\circ \quad (52)$$

then there exists $\tau < \infty$ such that 1. (Σ_Δ, Ξ) is well posed on $[0, \tau]$, 2. $y(\tau) = \pm\sqrt{2}\eta$ and 3.

$$\begin{aligned}
\mathcal{P}(Q, k, \eta) &\leq \frac{2\lambda(Q)\eta^2}{2\lambda(Q)\eta^2 - p} (W_\alpha - \eta^2) + \\
&+ \frac{1}{2\lambda(Q)\eta^2 - p} \int_{\eta^2}^{W_\alpha} \tilde{u}^2(v) dv \quad (53)
\end{aligned}$$

where $\tilde{u}: \mathbb{R} \rightarrow \mathbb{R}$ is defined

$$\begin{aligned}
\tilde{u}(r) &= \sup \left\{ |u(y, \nu, \zeta, \omega, \hat{\Theta}', \tilde{\Theta})| \in \mathbb{R} \mid \right. \\
&\zeta_j^T P_0 \zeta_j \leq 4\bar{\lambda}(P_0)^3 \sup_{y^2 \leq 2W_\alpha} \|\Phi'_j(y)\|; \\
&\omega^T P_0 \omega \leq 4\bar{\lambda}(P_0)^3 \sup_{y^2 \leq 2W_\alpha} \|\gamma y + f^0(y)\|; \\
&(y, \nu_2, \dots, \nu_n) = z; \\
&\nu_1 = y - \omega_1 - \sum_{j=1}^m \theta'_j \zeta_{j,1} - \varepsilon_1; \\
&\left. \eta^2 \leq V(z, \varepsilon, \hat{\Theta}', \tilde{\Theta}) \leq r \right\}.
\end{aligned}$$

and

$$p = \frac{ns^2}{4\kappa^2} + \frac{g^2}{l^2}. \quad (54)$$

Proof: (Sketch). This result follows from Proposition 2, once we have shown that $V_0 \leq W_\alpha$. ■

The following result applies to systems with global uncertainties, in particular it applies to systems where the uncertainty level (δ) is unknown to the control design. Notably we do not require knowledge of the uncertainty level, but it is necessary to know the uncertainty growth as specified by a L^2 weighted space. Because the model is required to have a global domain, it is necessary to consider infinite dimensional models, but to ensure physical realisable models we restrict ourselves to the class of semiglobally finite dimensional (SFD) models, namely model $\phi: \mathcal{O} \rightarrow \mathcal{X}$ is said to be SFD if for any compact $\Omega \subset \mathcal{O}$ only a finite number of basis functions of ϕ have supports intersecting Ω .

We ensure the adaptive controller adapts fast enough by the following construction of suitable weighted adaption structure matrices $\tilde{G}_i^*$ for the model component ϕ_i within w_i weighted Gram matrix R_i for $1 \leq i \leq n$ is as follows. Partition ϕ_b by $[\phi_b^1 | \phi_b^2 | \phi_b^3 | \dots]$, and partition θ equivalently: $\theta_b = [\theta_b^1 | \theta_b^2 | \theta_b^3 | \dots]$, where for each i, b θ_b^i is a finite-dimensional vector. Let R_b^i be the global $L^2(\mathbb{R}; w_b)$ weighted Gram matrix for the model ϕ_b^i , and define:

$$H_b^i = \frac{a_i}{\lambda(R_b^i)} I_{\dim(\theta_b^i) \times \dim(\theta_b^i)}, \quad (55)$$

where $\{a_i\}$ is any positive sequence for which $\sum_{i=1}^{\infty} 1/a_i = 1$. Let G_i^* $1 \leq i \leq n$ be the block diagonal matrices:

$$\tilde{G}_i^* = \text{diag}[H_i^1, H_i^2, \dots]. \quad (56)$$

Theorem 5 Consider the system Σ_Δ given by equation 1 with functional uncertainty $\Delta \subset \Delta(L^2(\mathcal{X}; w), f^0, \delta)$ where is $\delta > 0$ unknown and the state initial condition is $x_0 \in \mathcal{X}$, $y(0)^2 > 2\eta^2$. Consider the performance measure $\mathcal{P}(Q, k, \eta)$ where Q is a positive diagonal matrix, and $k > 0$, $\eta > 0$. Suppose Φ is globally $(Q, \Omega, \Omega_0, \kappa, l)$ admissible. Let $G^* = (G_1^*, \dots, G_n^*)$, be given by 56, and implement the controller $\Xi(G^*, Q, \alpha, \Phi, \kappa, l, \gamma)$ where $\alpha > 0$, $\kappa > 0$, $l > 0$, γ is admissible and Φ is an SFD model.

Then there exists $\tau < \infty$ such that 1. (Σ_Δ, Ξ) is well posed on $[0, \tau]$, 2. $y(\tau) = \pm\sqrt{2}\eta$, 3. Only a finite number of adaptive estimates and filters are non-zero. Then define $\Omega = [-\sqrt{2}W_\alpha, \sqrt{2}W_\alpha]$ where

$$W_\alpha = \frac{1}{2} z_0^T z_0 + \frac{1}{l^2} x_0^T P_0 x_0 + \frac{n}{\alpha} (\delta_1 + q_1)^2 + \frac{n}{2\alpha} \sum_{i=1}^n (\delta_i + q_i)^2 \quad (57)$$

where P_0 is as in Proposition 2. Then additionally, 4.

$$\mathcal{P}(Q, k, \eta) \leq \frac{2\lambda(Q)\eta^2}{2\lambda(Q)\eta^2 - p} (W_\alpha - \eta^2)$$

$$+ \frac{1}{2\lambda(Q)\eta^2 - p} \int_{\eta^2}^{W_\alpha} \tilde{u}^2(v) dv \quad (58)$$

where $\tilde{u}: \mathbb{R} \rightarrow \mathbb{R}$ is defined as in Theorem 4 for the restricted model $\Phi|_\Omega$, and

$$p = \frac{ns^2}{4\kappa^2} + \frac{g^2}{l^2}. \quad (59)$$

Proof: (Sketch). Similarly to the previous theorem, this result follows from Proposition 2 by showing that $V_0 \leq W_\alpha$. The fact that only a finite number of basis parameters are adapted follows from the fact that the output is bounded, and that a parameter is adapted only when the output lies within the support of the corresponding basis function, the fact that the model is SFD completes the argument. ■

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