
A Simpler Approach to Accelerated Stochastic Optimization: Iterative Averaging Meets Optimism

Pooria Joulani^{*1} Anant Raj^{*2} András György¹ Csaba Szepesvári¹³

Abstract

Recently there have been several attempts to extend Nesterov’s accelerated algorithm to smooth stochastic and variance-reduced optimization. In this paper, we show that there is a simpler approach to acceleration: applying *optimistic* online learning algorithms and querying the gradient oracle at the *online average* of the intermediate optimization iterates. In particular, we tighten a recent result of Cutkosky (2019) to demonstrate theoretically that online iterate averaging results in a reduced optimization gap, independently of the algorithm involved. We show that carefully combining this technique with existing generic optimistic online learning algorithms yields the optimal accelerated rates for optimizing strongly-convex and non-strongly-convex, possibly composite objectives, with deterministic as well as stochastic first-order oracles. We further extend this idea to variance-reduced optimization. Finally, we also provide “universal” algorithms that achieve the optimal rate for smooth and non-smooth composite objectives simultaneously without further tuning, generalizing the results of Kavis et al. (2019) and solving a number of their open problems.

1. Introduction

Our goal in this paper is to obtain algorithms with optimal convergence rates for the following problem:

$$\text{find } x^* = \operatorname{argmin}_{x \in \mathcal{X}} \ell(x) = f(x) + \phi(x), \quad (1)$$

where $\mathcal{X}$ is a convex constraint set in the d -dimensional Euclidean space, f is convex and smooth, and ϕ is a (possi-

bly non-smooth) convex function. When $\phi = 0$, and given access to (noise-free) gradients of f , Nesterov’s accelerated gradient algorithms (Nesterov, 2018) achieve optimal rates of convergence for Problem (1). Several recent papers, summarized in Table 1, have attempted to obtain similarly accelerated rates that improve upon the sub-optimal rates of Stochastic Gradient Descent (SGD) when the gradients of f are corrupted by noise and/or when $\phi \neq 0$.

Despite the major effort to obtain these extensions, existing results suffer from several limitations such as: (a) inhibiting noise in the gradient (Allen-Zhu and Orecchia, 2017; Wang and Abernethy, 2018); (b) potentially querying the gradient oracle outside the constraint set (Levy et al., 2018; Cutkosky, 2019) (c) not providing optimal rates for strongly-convex objectives (Cutkosky, 2019); (d) extra logarithmic terms appearing in the error bounds (Levy et al., 2018; Cutkosky, 2019); (e) not handling proximal updates when $\phi \neq 0$ (Levy et al., 2018; Kavis et al., 2019; Cutkosky, 2019) or (f) relying on prior knowledge of problem parameters (Tseng, 2008; Beck and Teboulle, 2009; Hu et al., 2009; Xiao, 2010; Lan, 2012; Chen et al., 2012).

In this paper, we demonstrate a simple direct approach to deriving accelerated rates: following Cutkosky (2019), we propose running an online learning algorithm and feeding it with (possibly noisy) first-order information obtained at the *weighted average* of its iterates. Then, building on the recent simple, tight modular analysis techniques of generic optimistic online learning algorithms (Joulani et al., 2017; 2020), we are able to alleviate all the aforementioned limitations, design new accelerated algorithms with straightforward convergence analyses, and solve a number of problems left open in previous work.

1.1. Contributions and Related Work

Our main contributions can be summarized as follows:

- We provide a direct, simple template for deriving and analyzing accelerated algorithms for stochastic and deterministic convex optimization with composite objectives. We further extend the above framework to variance-reduced stochastic non-strongly-convex optimization.

^{*}Equal contribution ¹DeepMind, London, UK ²Max-Planck Institute for Intelligent Systems, Tübingen, Germany; work done during an internship at Deepmind ³University of Alberta, Edmonton, AB, Canada. Correspondence to: Pooria Joulani <pjoulani@google.com>.

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	$\mathcal{X}$	f	ϕ	Oracle	Universal	Notes
Tseng (2008)	Any	Non-SC	✓	D	-	
Beck and Teboulle (2009)	$\mathbb{R}^d$	Non-SC	✓	D	-	
Hu et al. (2009)	$\mathbb{R}^d$	SC / Non-SC	✓	S+D	-	Assumes bounded trajectory
Xiao (2010)	Any	Non-SC	✓	S + D	-	Dual-Averaging
Lan (2012)	Any	Non-SC	✓	S+D	-	Not utilizing prox-map of ϕ
Chen et al. (2012)	Any	SC / Non-SC	✓	S+D	-	Exponential noise-free rate
	Any	SC	✓	S+D	-	
Allen-Zhu and Orecchia (2017)	Any	Non-SC	-	D	-	Linear-coupling
	Any	SC	-	D	-	Exponential rate
Wang and Abernethy (2018)	Any	Non-SC	✓	D	-	Primal-dual view
	Any	SC	-	D	-	Exponential rate
This paper (Corollary 4)	Any	SC / Non-SC	✓	S + D	-	Exponential rate
	Any	SC	✓	D	-	
Cutkosky (2019)	Compact	Non-SC	-	S+D	✓	Accessing f outside $\mathcal{X}$
Levy et al. (2018)	Compact	Non-SC	-	S+D	✓	Accessing f outside $\mathcal{X}$
Kavis et al. (2019)	Compact	Non-SC	-	S + D	✓	
This paper (Theorem 5)	Compact	Non-SC	✓	S + D	✓	

Table 1. Summary of previous work obtaining accelerated rates of convergence. Cutkosky (2019) analyses strongly-convex optimization as well, but the rates are sub-optimal (i.e., non-accelerated). Here, “Non-SC” means non-strongly convex (that is, strong-convexity of f is not required), “SC” means strongly convex, “D” and “S” stand for deterministic and stochastic oracles, respectively. Universality means the algorithm achieves the smooth and non-smooth rates simultaneously without requiring the knowledge of the problem’s smoothness and noise level. The “bounded-trajectory” assumption means that the error bound scales with the maximum distance of the iterates from the optimum x^* , but the algorithm does not enforce this to be bounded (e.g., through projection to a compact set). See also the survey by Bubeck (2015) and the recent book of Nesterov (2018).

- For composite non-strongly-convex objectives, we provide a new *universal* algorithm (in the sense of Nesterov, 2015): given only access to the proximal projection oracle of ϕ onto the constraint set, without prior knowledge of the smoothness or noise level, the new algorithm simultaneously achieves the optimal rate of convergence for smooth and non-smooth f . This, together with the fact that the algorithm uses coordinate-wise adaptive step-sizes, resolves two problems left open by Kavis et al. (2019).

In particular, in Lemma 1 and Corollary 2, we tighten the recent analysis of online iterate averaging by Cutkosky (2019). Compared to their Theorem 1, Corollary 2 exposes additional terms that reduce the optimization gap. These terms, whose absence prevented Cutkosky (2019, Theorem 3) from getting the optimal accelerated rates, are similar to what Wang and Abernethy (2018) obtained through an indirect formulation of acceleration as a two-player game.

Next, we show how to utilize the aforementioned reduction to obtain accelerated rates. This is achieved by using properly-tuned *optimistic online learning* algorithms

(Rakhlin and Sridharan, 2013a;b; Mohri and Yang, 2016) as the underlying optimization machinery. Importantly, this tuning can be done somewhat independently of the assumptions on the objective (such as the presence of noise, strong convexity, or a non-zero ϕ) or the algorithmic techniques (such as proximal updates or adaptive learning rates), thanks to the recent modular analyses of online learning algorithms by Joulani et al. (2017; 2020). This results in a simple, straightforward acceleration framework.

Furthermore, we extend the analysis to variance-reduced optimization for smooth non-strongly-convex functions. We show that incorporating negative momentum (common in the accelerated SVRG literature, see, e.g., Allen-Zhu, 2017; Lan et al., 2019) in our framework introduces an additional reduction in the optimization gap, enabling us to obtain the optimal convergence rate. We also analyze a simpler version of the variance reduced algorithm without negative momentum, which enjoys a variance-reduced, though still sub-optimal rate of convergence for the last iterate.

Finally, we provide a universal algorithm for non-strongly-convex composite optimization, extending the works of

Levy et al. (2018); Cutkosky (2019); Kavis et al. (2019) to the case when $\phi \neq 0$. The new algorithm features proximal updates and coordinate-wise adaptive step-sizes, thus solving two problems left open by Kavis et al. (2019). Unlike Levy et al. (2018) and Cutkosky (2019), the new algorithm does not query the optimization oracle outside the constraint set $\mathcal{X}$, and does not suffer from extra log terms in the bound. Unlike the algorithm of Kavis et al. (2019) (which is based on mirror-descent) our algorithm is based on dual-averaging, which is better suited to sparse learning with a proximal ℓ_1 penalty (Xiao, 2009; McMahan, 2011).

Notation. $\mathbb{R}$ denotes the set of real numbers. For any positive integer n , $[n] = \{1, \dots, n\}$. Let $h : \mathcal{D} \rightarrow \mathbb{R}$ where $\mathcal{D} \subset \mathbb{R}^d$ for some positive integer d . The gradient or a subgradient of h is denoted by h' . When h is convex, the Bregman-divergence $B_h : \mathcal{D} \times \mathcal{D}^o \rightarrow \mathbb{R}$ is defined as $B_h(x, y) = h(x) - h(y) - \langle h'(y), x - y \rangle$, where $\mathcal{D}^o$ denotes the interior of $\mathcal{D}$. We say that h is μ -strongly convex with respect to (w.r.t.) a norm $\|\cdot\|$ if for all $x \in \mathcal{D}, y \in \mathcal{D}^o$, $\frac{\mu}{2}\|x - y\|^2 \leq B_h(x, y)$, and it is μ -strongly convex w.r.t. a function $n : \mathcal{D} \times \mathcal{D}^o \rightarrow [0, \infty)$ if $\mu \cdot n(x, y) \leq B_h(x, y)$ for all $x \in \mathcal{D}, y \in \mathcal{D}^o$ (note that h is μ -strongly convex w.r.t. a norm $\|\cdot\|$ if it is μ -strongly convex w.r.t. the function $\|\cdot\|^2/2$). For non-negative integers a, b and a sequence of numbers or vectors $x_0, x_1, \dots$, we let $x_{a:b} = \sum_{s=a}^b x_s$ if $a \leq b$ and 0 otherwise. With a slight abuse of notation, for a vector $x \in \mathbb{R}^d$, we denote its coordinates as $x = (x_1, \dots, x_d)$; whether the subscript refers to a coordinate or a time index is usually clear from the context (to reduce the possible ambiguity, we normally use i and j to index coordinates of x , and t and s to index time, i.e., typically $x_j \in \mathbb{R}$ denotes a real coordinate and $x_t \in \mathbb{R}^d$ denotes an iterate computed at time step t). For an event E , $\mathbb{I}\{E\}$ denotes its indicator function, that is $\mathbb{I}\{E\} = 1$ if E is true, otherwise $\mathbb{I}\{E\} = 0$.

2. Preliminaries

For simplicity, we assume that an optimizer $x^* \in \mathcal{X}$ of Problem (1) exists, i.e., $\ell^* := \ell(x^*) \leq \ell(x)$ for all $x \in \mathcal{X}$.¹

Smoothness of functions. When f is differentiable over $\mathbb{R}^d$, given a norm $\|\cdot\|$, the following are equivalent definitions of smoothness of f (Nesterov, 2018, Theorem 2.1.5): f is L -smooth if

- (i) for all $x, y \in \mathbb{R}^d$, $B_f(x, y) \leq \frac{L}{2}\|x - y\|^2$;
- (ii) for all $x, y \in \mathbb{R}^d$, $\|f'(x) - f'(y)\|_* \leq L\|x - y\|$;
- (iii) for all $x, y \in \mathbb{R}^d$,

$$\|f'(x) - f'(y)\|_*^2 \leq (2L)B_f(x, y). \quad (2)$$

¹We do not require $\mathcal{X}$ to be closed or compact, which are normally assumed to ensure x^* exists.

Throughout the paper, we only require² that f is differentiable over $\mathcal{X}$, and use (2), holding for all $x, y \in \mathcal{X}$, as the notion of smoothness under which accelerated rates are obtained. Alternatively, assuming (ii) holds only for $x, y \in \mathcal{X}$, one can still obtain the same rates with a very similar analysis as we provide, at the expense of an additional gradient oracle call per step of the algorithms; we leave the details for an extended version of the paper.³

Iterative optimization. We consider first-order sequential optimization procedures with access to a stochastic gradient oracle that returns unbiased estimates of f' . We assume that such an oracle is associated with a map $F' : \mathcal{X} \times [0, 1] \rightarrow \mathbb{R}^d$ and a distribution P_ζ over $[0, 1]$ such that for all $x \in \mathcal{X}$ and for ζ drawn from P_ζ , $\mathbb{E}[F'(x, \zeta)] = f'(x)$.

A sequential optimization method then, in iteration t , queries the oracle at a point $y_t \in \mathcal{X}$, receives a gradient estimate g_t such that $\mathbb{E}[g_t | \mathcal{H}_t] = f'(y_t)$ where $\mathcal{H}_t = \sigma\left((g_s)_{s=1}^{t-1}, (y_s)_{s=1}^t\right)$ is the sigma-algebra generated by all the information used by the algorithm before making the query at y_t to the gradient oracle. In case ϕ is not zero, we also assume that the optimization method has access to the prox-function of ϕ (cf. Eq. 6). After T iterations, the algorithm produces an estimate $\bar{x}_T$ of x^* , based on all the information it has seen, where the quality of the estimate is measured by the error $\mathbb{E}[\ell(\bar{x}_T)] - \ell^*$.

Online linear optimization. One way to design and analyze iterative optimization methods is through *online linear optimization* (OLO) algorithms. An OLO algorithm sequentially comes up, at each time step $t \in [T]$, with a prediction x_t , then receives a linear loss function $\langle \alpha_t u_t, \cdot \rangle$, with the aim of maintaining a small cumulative composite loss $\sum_{t=1}^T \alpha_t (\langle u_t, x_t - x \rangle + \phi(x_t) - \phi(x))$ a.k.a. its *regret*, compared to a competitor point x . Here $u_t \in \mathbb{R}^d$ is unknown to the algorithm before selecting x_t , but the non-negative weights α_t are known ahead of time.⁴ One can convert an OLO algorithm to an iterative optimization algorithm by using $y_t = x_t$ to query the oracle, using $u_t = g_t$ in the linear loss to the OLO algorithm, and employing the average $\bar{x}_T = \sum_{t=1}^T \frac{\alpha_t}{\alpha_{1:T}} x_t$ as the final estimate of x^* .

The appeal of this “vanilla online-to-batch” approach, shown in Algorithm 1, is that it reduces the convergence analysis of $\bar{x}_T$ for convex f and ϕ to the regret analysis of the underlying OLO algorithm. In particular, by Jensen’s

²Extensions when f is non-differentiable at boundary points are straightforward.

³Assuming only that (i) holds for all $x, y \in \mathcal{X}$ does not imply (ii) or (2) in general (even for $x, y \in \mathcal{X}$). E.g., $f'(x)$ can grow arbitrarily in the directions orthogonal to $\mathcal{X}$ while (i) holds.

⁴In fact, it is enough to assume that, for all t , α_t is known before selecting x_t .

Algorithm 1 Vanilla Online-to-Batch

- 1: **Input:** Stochastic gradient oracle F' , non-negative weights $(\alpha_t)_{t=1}^T$ with $\alpha_1 > 0$, online linear optimization algorithm $\mathcal{A}$
- 2: Get the initial point $x_1 \in \mathcal{X}$ from $\mathcal{A}$
- 3: **for** $t = 1$ **to** $T - 1$ **do**
- 4: Get the next stochastic gradient g_t at the *current* iterate x_t
- 5: Send $\langle \alpha_t g_t, \cdot \rangle$ as the next linear loss to $\mathcal{A}$
- 6: Let x_{t+1} be the next iterate from $\mathcal{A}$
- 7: **end for**
- 8: **return** the average iterate $\frac{\sum_{t=1}^T \alpha_t x_t}{\alpha_{1:T}}$.

Algorithm 2 Anytime Online-to-Batch (Cutkosky, 2019)

- 1: **Input:** Stochastic gradient oracle F' , non-negative weights $(\alpha_t)_{t=1}^T$ with $\alpha_1 > 0$, online linear optimization algorithm $\mathcal{A}$
- 2: Get the initial point $x_1 \in \mathcal{X}$ from $\mathcal{A}$ and let $\bar{x}_1 \leftarrow x_1$
- 3: **for** $t = 1$ **to** $T - 1$ **do**
- 4: Get the next stochastic gradient g_t at the *average* iterate $\bar{x}_t$
- 5: Send $\langle \alpha_t g_t, \cdot \rangle$ as the next linear loss to $\mathcal{A}$
- 6: Let x_{t+1} be the next iterate from $\mathcal{A}$
- 7: Let $\bar{x}_{t+1} \leftarrow \frac{\sum_{s=1}^{t+1} \alpha_s x_s}{\alpha_{1:t+1}}$
- 8: **end for**
- 9: **return** the average iterate $\bar{x}_T$

inequality,

$$\begin{aligned}
 & \mathbb{E}[\ell(\bar{x}_T)] - \ell^* \\
 & \leq \sum_{t=1}^T \mathbb{E} \left[\frac{\alpha_t (\langle f'(x_t), x_t - x^* \rangle + \phi(x_t) - \phi(x^*))}{\alpha_{1:T}} \right] \\
 & = \mathbb{E} \left[\frac{\sum_{t=1}^T \alpha_t (\langle g_t, x_t - x^* \rangle + \phi(x_t) - \phi(x^*))}{\alpha_{1:T}} \right] \\
 & \leq \mathbb{E} \left[\frac{\mathcal{R}_T(x^*)}{\alpha_{1:T}} \right], \tag{3}
 \end{aligned}$$

where $\mathcal{R}_T(x^*)$ is an upper-bound for the regret of the OLO algorithm. Thus, to analyze the convergence of $\bar{x}_T$, one can simply plug-in an off-the-shelf regret bound (reviewed at the end of this section) for the underlying OLO algorithm.

Anytime online-to-batch. An alternative, elegant online-to-batch conversion (Algorithm 2) was recently proposed by Cutkosky (2019), which uses the “online” average $\bar{x}_t = \sum_{s=1}^t \frac{\alpha_s}{\alpha_{1:t}} x_s$ as the query point, i.e., $y_t = \bar{x}_t$. Cutkosky (2019, Theorem 1) showed (with $\phi = 0$) that (3) holds under this conversion scheme as well. In the next section, we show that in fact Algorithm 2 enjoys a tighter version of (3) that enables us to prove accelerated rates.

Generic regret bound. Next, we recall the regret bound for a general family of OLO algorithms known as “adaptive optimistic follow the regularized leader” or AO-FTRL (Rakhlin and Sridharan, 2013a;b; Mohri and Yang, 2016). At time t , AO-FTRL makes its t -th prediction as

$$x_t = \operatorname{argmin}_{x \in \mathcal{X}} \left\langle \sum_{s=1}^{t-1} \alpha_s g_s + \alpha_t \tilde{g}_t, x \right\rangle + \alpha_{1:t} \phi(x) + r_{0:t-1}(x), \tag{4}$$

where, the $r_t : \mathcal{X} \rightarrow \mathbb{R}$ are convex *regularizer* functions, and for every t , $\tilde{g}_t$, the *optimistic* part of the update, is interpreted as a prediction of g_t before it is received.

It is straightforward to see that AO-FTRL captures a wide range of algorithms used in optimization (Xiao, 2009; McMahan, 2017). For example, the dual-averaging algorithm of Xiao (2009) corresponds to the case when $\phi = 0$ and $r_{0:t-1} = \frac{\eta_t}{2} \|\cdot\|_2^2$ for $\eta_t > 0$, in which case it is easy to verify that

$$x_t = \Pi_{\mathcal{X}} \left(-\frac{\sum_{s=1}^{t-1} \alpha_s g_s + \alpha_t \tilde{g}_t}{\eta_t} \right), \tag{5}$$

where $\Pi_{\mathcal{X}}$ denotes Euclidean projection onto set $\mathcal{X}$. More generally, allowing coordinatewise step sizes $\eta_t \in [0, \infty)^d$ and a possibly non-zero ϕ , with $r_{0:t-1}(x) = \frac{1}{2} \sum_{j=1}^d \eta_{t,j} x_j^2$ we recover the proximal (a.k.a. “composite-objective” or “regularized”) dual-averaging update (Xiao, 2009):

$$\begin{aligned}
 x_t & = \operatorname{prox}_{\alpha_{1:t} \phi, \eta_t} \left(-\sum_{s=1}^{t-1} \alpha_s g_s - \alpha_t \tilde{g}_t \right) \\
 & = \operatorname{argmin}_{x \in \mathcal{X}} \alpha_{1:t} \phi(x) + \frac{1}{2} \sum_{j=1}^d \eta_{t,j} \left(x_j - \frac{z_{t-1,j}}{\eta_{t,j}} \right)^2, \tag{6}
 \end{aligned}$$

where $\operatorname{prox}_{\alpha_{1:t} \phi, \eta_t}$ is the prox-function of $\alpha_{1:t} \phi$ with coordinatewise step sizes $\eta_{t,j}$ and $z_{t-1} = -\left(\sum_{s=1}^{t-1} \alpha_s g_s + \alpha_t \tilde{g}_t\right)$. Note that AdaGrad-style updates (Duchi et al., 2011) can be recovered by setting η_t based on the past gradient estimates $g_s, \tilde{g}_s$ (for $s < t$).

If $r_t \geq 0$, the cumulative regularizer $\alpha_{1:t} \phi + r_{0:t-1}$ is 1-strongly convex w.r.t. a norm $\|\cdot\|_{(t)}$, and the AO-FTRL update is well-defined, that is, the minimizer $x_t \in \mathcal{X}$ exists and $\left\langle \sum_{s=1}^{t-1} \alpha_s g_s + \alpha_t \tilde{g}_t, x_t \right\rangle + \alpha_{1:t} \phi(x_t) + r_{0:t-1}(x_t)$ is finite, then Theorem 6 of Joulani et al. (2020) gives the following regret bound (see Appendix E):

$$\mathcal{R}_T(x^*) = r_{0:T-1}(x^*) + \sum_{t=1}^T \frac{1}{2} \alpha_t^2 \|g_t - \tilde{g}_t\|_{(t)^*}^2. \tag{7}$$

3. Acceleration with Anytime Online-to-Batch

First, we present a lemma that generalizes the regret decomposition of Joulani et al. (2017) to work with the averaging

scheme of Cutkosky (2019). Crucially, the decomposition keeps track of some negative Bregman-divergence terms, which are instrumental in reducing the contribution of the OLO regret to the error of $\bar{x}_T$.

Lemma 1. *For $t = 1, 2, \dots, T$, let $\alpha_t > 0$ and $x_t \in \mathbb{R}^d$, and define $\bar{x}_t = (\sum_{s=1}^t \alpha_s x_s) / \alpha_{1:t}$, $B_t = \alpha_t B_f(x^*, \bar{x}_t)$, and $\bar{B}_t^f = \alpha_{1:t-1} B_f(\bar{x}_{t-1}, \bar{x}_t)$, $t > 1$. Then, if ϕ is convex,*

$$\begin{aligned} & \alpha_{1:T} (\ell(\bar{x}_T) - \ell^*) \\ & \leq \sum_{t=1}^T \alpha_t (\langle f'(\bar{x}_t), x_t - x^* \rangle + \phi(x_t) - \phi(x^*)) \\ & \quad - B_{1:T} - \bar{B}_{2:T}^f. \end{aligned} \quad (8)$$

The lemma immediately gives rise to the following generic error bound, which improves upon Theorem 1 of Cutkosky (2019) by keeping around the aforementioned $-\bar{B}_t^f$ and $-B_t$ terms. While the B_t are the usual Bregman-divergence terms (also appearing in the vanilla online-to-batch) that are utilized to get fast rates for strongly convex functions (and can be dropped in general as long as the function is star-convex; see Joulani et al., 2020), the important new terms here are the $-\bar{B}_t^f$ terms, which allow us to prove accelerated rates for online averaging.

Corollary 2 (Generic Error Bound). *Under the assumptions of Lemma 1, if for all $t = 1, 2, \dots, T$, $g_t \in \mathbb{R}^d$ satisfies $\mathbb{E}[g_t | \bar{x}_t] = f'(\bar{x}_t)$ and we have*

$$\sum_{t=1}^T \alpha_t (\langle g_t, x_t - x^* \rangle + \phi(x_t) - \phi(x^*)) \leq \mathcal{R}_T(x^*) \quad (9)$$

for some upper-bound $\mathcal{R}_T(x^*)$, then

$$\mathbb{E}[\ell(\bar{x}_T) - \ell(x^*)] \leq \mathbb{E} \left[\frac{\mathcal{R}_T(x^*) - B_{1:T} - \bar{B}_{2:T}^f}{\alpha_{1:T}} \right]. \quad (10)$$

The corollary follows since g_t is a conditionally unbiased estimate of $f'(\bar{x}_t)$, so the first term on the r.h.s. of (8) is, in expectation, equal to the term on the l.h.s. of (9), and hence upper-bounded by $\mathbb{E}[\mathcal{R}_T(x^*)]$. Next, we prove the lemma.

Proof of Lemma 1. Writing $f(\bar{x}_T)$ as a telescoping sum,

$$\begin{aligned} & f(\bar{x}_T) - f(x^*) \\ & = -f(x^*) + \frac{\alpha_1 f(\bar{x}_1)}{\alpha_{1:T}} + \sum_{t=2}^T \frac{\alpha_{1:t} f(\bar{x}_t) - \alpha_{1:t-1} f(\bar{x}_{t-1})}{\alpha_{1:T}} \\ & = \sum_{t=1}^T \frac{\alpha_t (f(\bar{x}_t) - f(x^*))}{\alpha_{1:T}} + \sum_{t=2}^T \frac{\alpha_{1:t-1} (f(\bar{x}_t) - f(\bar{x}_{t-1}))}{\alpha_{1:T}} \\ & = \sum_{t=1}^T \frac{\alpha_t \langle f'(\bar{x}_t), \bar{x}_t - x^* \rangle - B_t}{\alpha_{1:T}} \end{aligned}$$

$$\begin{aligned} & + \sum_{t=2}^T \frac{\alpha_{1:t-1} \langle f'(\bar{x}_t), \bar{x}_t - \bar{x}_{t-1} \rangle - \bar{B}_t^f}{\alpha_{1:T}} \\ & = \sum_{t=1}^T \frac{\alpha_t \langle f'(\bar{x}_t), \bar{x}_t - x^* \rangle - B_t}{\alpha_{1:T}} \\ & \quad + \sum_{t=2}^T \frac{\alpha_t \langle f'(\bar{x}_t), x_t - \bar{x}_t \rangle - \bar{B}_t^f}{\alpha_{1:T}} \\ & = \frac{\sum_{t=1}^T \alpha_t \langle f'(\bar{x}_t), x_t - x^* \rangle - B_{1:T} - \bar{B}_{2:T}^f}{\alpha_{1:T}}, \end{aligned}$$

where the third step follows since by the definition of Bregman divergence, $f(z) - f(y) = \langle f'(z), z - y \rangle - B_f(y, z)$, the fourth step follows since by the definition of $\bar{x}_t$, for $t = 2, 3, \dots, T$ we have $\alpha_t(\bar{x}_t - x_t) = \alpha_{1:t-1}(\bar{x}_{t-1} - \bar{x}_t)$, and the last step uses $\bar{x}_1 = x_1$. The proof is completed by $\phi(\bar{x}_T) - \phi(x^*) \leq \sum_{t=1}^T \frac{\alpha_t}{\alpha_{1:T}} (\phi(x_t) - \phi(x^*))$, which holds by Jensen's inequality. $\square$

Acceleration. The main idea behind deriving accelerated rates is combining (7) with (10), and selecting $\alpha_t, \tilde{g}_t$ appropriately so that the negative terms $-\bar{B}_t^f$ in (10) offset the contribution of the terms $\frac{\alpha_t^2}{2} \|g_t - \tilde{g}_t\|_{(t)^*}^2$ in (7) to the final error bound of $\bar{x}_T$. For example, let f be L -smooth over $\mathbb{R}^d$ or assume otherwise that (2) holds with the norm $\|\cdot\| = \|\cdot\|_2$. Suppose the optimization algorithm uses the dual averaging update (5) with $\alpha_t = t$, $\eta_t = \eta = 2L$, deterministic gradients $g_t = f'(\bar{x}_t)$, and $\tilde{g}_t = g_{t-1}$. Then, $r_{0:t-1}$ is 1-strongly convex w.r.t. the norm $L\|\cdot\|_2^2$, and the norm terms $\frac{\alpha_t^2}{2} \|g_t - \tilde{g}_t\|_{(t)^*}^2$ in (7) can be bounded as

$$\begin{aligned} \frac{\alpha_t^2}{2} \|g_t - \tilde{g}_t\|_{(t)^*}^2 & = \alpha_t^2 \frac{1}{4L} \|f'(\bar{x}_t) - f'(\bar{x}_{t-1})\|_2^2 \\ & \leq \frac{\alpha_t^2}{2\alpha_{1:t-1}} \bar{B}_t^f = \frac{t^2}{t(t-1)} \bar{B}_t^f \leq \bar{B}_t^f, \end{aligned}$$

where the first inequality follows using (2). Hence, $\mathcal{R}_T(x^*) - \bar{B}_{2:T}^f \leq L\|x^*\|_2^2 + \frac{1}{4L}\|f'(x_1)\|_2^2$. Noticing that $\alpha_{1:T} = \Omega(T^2)$ gives the well-known accelerated $\mathcal{O}(1/T^2)$ rate for the error of $\bar{x}_T$. The next theorem, proved in Appendix A, makes this argument precise for the general setting with noise, non-zero ϕ and generic AO-FTRL.

Theorem 3. *In Algorithm 2, let the base method $\mathcal{A}$ generate its iterates by the AO-FTRL update (4), using $\tilde{g}_t = g_{t-1}$ as the optimistic prediction of g_t for $t > 1$ and arbitrary $\tilde{g}_1$. Suppose that f and ϕ are convex, and there exists a norm $\|\cdot\|$ such that either that f is 1-smooth w.r.t. $\|\cdot\|$ over $\mathbb{R}^d$ or otherwise (2) holds with $L = 1$ for all $x, y \in \mathcal{X}$. Further suppose that for all $t \in [T]$, $r_{t-1} \geq 0$ is convex, the AO-FTRL update (4) is well-defined with finite value at the optimum x_t , and there exist $\beta_t > 0$ and a norm $\|\cdot\|_{(t)}$ such that $\alpha_{1:t}\phi + r_{0:t-1}$ is 1-strongly-convex w.r.t. $\frac{\beta_t}{2}\|\cdot\|^2 + \frac{1}{2}\|\cdot\|_{(t)}^2$. Then, if $\alpha_t^2\beta_t^{-1} \leq \alpha_{1:t-1}$ for all $t > 1$,*

we have

$$\begin{aligned} \mathbb{E}[\ell(\bar{x}_T) - \ell^*] &\leq \sum_{t=1}^T \mathbb{E} \left[\frac{r_{t-1}(x^*) - r_{t-1}(x_t) - B_t}{\alpha_{1:T}} \right] \\ &\quad + \sum_{t=1}^T \mathbb{E} \left[\frac{\alpha_t^2 \|\sigma_t - \sigma_{t-1}\|_{(t)^*}^2}{2\alpha_{1:T}} \right] \\ &\quad + \mathbb{E} \left[\frac{\alpha_1^2 \|f'(\bar{x}_1) - \tilde{g}_1\|_*^2}{2\beta_1 \alpha_{1:T}} \right], \end{aligned} \quad (11)$$

where $\sigma_t = g_t - f'(\bar{x}_t)$, $t \in [T]$, and $\sigma_0 = 0$.

4. Applications

In this section we use the framework of the previous section, and Theorem 3 in particular, to obtain accelerated convergence rates with proximal updates, noisy gradients, and universal algorithms.

4.1. Accelerated Proximal Dual-Averaging

First, we show that with appropriately setting α_t and η_t , one can obtain the optimal accelerated rates for the proximal dual averaging update (6). In particular, we consider the case of a single step size for all coordinates (with a slight abuse of notation, $\eta_{t,i} = \eta_t$ for all i) and $r_{0:t-1} = \frac{\eta_t}{2} \|\cdot\|_2^2$. Then, under the conditions of Theorem 3, we have

$$\begin{aligned} \mathbb{E}[\ell(\bar{x}_T) - \ell^*] &\leq \sum_{t=1}^T \mathbb{E} \left[\frac{\alpha_t^2 \|\sigma_t - \sigma_{t-1}\|_{(t)^*}^2}{2\alpha_{1:T}} \right] \\ &\quad + \mathbb{E} \left[\frac{\eta_T \|x^*\|_2^2 - \sum_{t=1}^T (\eta_t - \eta_{t-1}) \|x_t\|_2^2 - 2B_t}{2\alpha_{1:T}} \right] \\ &\quad + \mathbb{E} \left[\frac{\alpha_1^2}{2\beta_1 \alpha_{1:T}} \|f'(\bar{x}_1) - \tilde{g}_1\|_*^2 \right]. \end{aligned} \quad (12)$$

Thus, the optimal rates follow immediately by properly setting η_t and α_t , as captured by the following corollary.

Corollary 4 (Accelerated Proximal Dual-Averaging). *Let f and ϕ be convex and assume that either f is L -smooth over $\mathbb{R}^d$ or otherwise (2) holds for all $x, y \in \mathcal{X}$. Consider the online-averaged (stochastic) proximal dual averaging algorithm, given by Algorithm 2 with update (6) using $\tilde{g}_t = g_{t-1}$ as the optimistic prediction of g_t for $t > 1$, and $\tilde{g}_1 = 0$, where the gradient estimates g_t are unbiased, that is, $\mathbb{E}[g_t | \bar{x}_t] = f'(\bar{x}_t)$. Let $\sigma_*^2 = \max_{t=1}^T \mathbb{E}[\|\sigma_t\|_2^2]$, where $\sigma_t = g_t - f'(\bar{x}_t)$, and let $D = \max\{\|x^*\|_2, \|x_1 - x_f^*\|_2\}$, where x_f^* is the minimizer of f over $\mathbb{R}^d$. Then we have the following error bounds:*

(i) If $\eta_t = 4L + \eta\alpha_t\sqrt{t}$ for some $\eta > 0$ and $\alpha_t = t$, we have

$$\mathbb{E}[\ell(\bar{x}_T) - \ell^*] \leq \frac{\left(4L + \frac{L}{4} + \eta T \sqrt{T}\right) D^2 + \frac{4\sigma_*^2}{\eta} T \sqrt{T}}{T(T+1)}$$

$$= \mathcal{O} \left(\frac{LD^2}{T^2} + \frac{\eta D^2 + \eta^{-1} \sigma_*^2}{\sqrt{T}} \right).$$

(ii) If ϕ_t is μ -strongly-convex then using $\eta_t = 4L$ and $\alpha_t = t$, we have

$$\begin{aligned} \mathbb{E}[\ell(\bar{x}_T) - \ell^*] &\leq \frac{(4L + \frac{L}{4}) D^2 + \frac{8\sigma_*^2 T}{\mu}}{T(T+1)} \\ &= \mathcal{O} \left(\frac{LD^2}{T^2} + \frac{\sigma_*^2}{\mu T} \right). \end{aligned}$$

(iii) If $g_t = f'(x_t)$ (i.e., the noiseless case) and ϕ is μ -strongly-convex, then for $\eta_t = 0$ and any sequence of $\alpha_t > 0$, $t \in [T]$ satisfying

$$\sqrt{c\kappa} \geq \frac{\alpha_{1:t}}{\alpha_t} \geq \sqrt{2\kappa} \quad t > 1, \quad (13)$$

for some $c \geq 2$ where $\kappa = (L + \mu)/\mu$ denotes the condition number, we have

$$\ell(\bar{x}_T) - \ell^* \leq \frac{\|f'(x_1)\|^2 \left(1 - \frac{1}{\sqrt{c\kappa}}\right)^{T-1}}{2\mu}. \quad (14)$$

Remark 1. The above rates of $\mathcal{O}(1/T^2)$ for a non-strongly-convex f are optimal in T when there is no noise ($\sigma_t = 0$), and the bound (14) also almost matches the optimal $\mathcal{O}((1 - 1/\sqrt{\kappa})^T)$ rate for the noiseless strongly-convex case. When there is noise, the worst-case rate of $\mathcal{O}(1/\sqrt{T})$ (for non-strongly-convex f) and $\mathcal{O}(1/T)$ (for strongly-convex f) are unavoidable, according to the lower-bounds of Nemirovsky and Yudin (1983): when the noise dominates, there is no hope of exploiting the smoothness in the signal (i.e., the gradient). Therefore, similarly to our paper, all previous work obtain only a lower-order improvement, e.g., from $1/T + \sigma/\sqrt{T}$ (of smooth non-strongly-convex SGD) to $1/T^2 + \sigma/\sqrt{T}$. If the noise is small, the latter rate is closer to the noise-free optimal rate of $1/T^2$, and determines the convergence speed of the algorithm in the initial stages of optimization. In contrast, the former bound (for SGD) is sub-optimal in the noise-free case. The possible improvements are lower-order in case of noisy strongly-convex optimization as well.

Proof of Corollary 4. First, notice that with any step size $\eta_t = 4L + \gamma_t$, the algorithm is equivalent to Algorithm 2 with AO-FTRL as the base algorithm, using regularizers $r_{0:t-1} = \frac{4L+\gamma_t}{2} \|\cdot\|_2^2$, which satisfy the conditions of Theorem 3 with $\beta_t = 4$, $\|\cdot\|^2 = L\|\cdot\|_2^2$, and $\|\cdot\|_{(t)}^2 = (\gamma_t + \alpha_{1:t}\mu)\|\cdot\|_2^2$, where μ is the strong-convexity parameter of ϕ (i.e., $\mu = 0$ in part (i), and $\mu > 0$ in parts (ii) and (iii)). Hence, starting from (12), with $\alpha_t = t$ we have

$$\mathbb{E}[\ell(\bar{x}_T) - \ell(x^*)]$$

$$\leq \sum_{t=1}^T \frac{t^2 \mathbb{E} [\|\sigma_t - \sigma_{t-1}\|_2^2]}{(\gamma_t + \alpha_{1:t}\mu)T(T+1)} + \frac{(4L + \gamma_T) \|x^*\|_2^2}{T(T+1)} \\ - \frac{\sum_{t=1}^T (\gamma_t - \gamma_{t-1}) \mathbb{E} [\|x_t\|_2^2]}{T(T+1)} + \frac{\mathbb{E} [\|f'(\bar{x}_1)\|_2^2]}{4T(T+1)L}.$$

In the above, $\mathbb{E} [\|\sigma_t - \sigma_{t-1}\|_2^2] \leq 4\sigma_*^2$. In addition, since f is convex and satisfies (2), we have $\frac{1}{2L} \|f'(\bar{x}_1)\|_2^2 \leq B_f(x_1, x_f^*) \leq \frac{L}{2} \|x_1 - x_f^*\|^2$ where x_f^* is the minimizer of f over $\mathbb{R}^d$. Then, plugging in $\gamma_t = \eta\alpha_t\sqrt{t}$ (respectively, $\gamma_t = 0$) and dropping the non-positive terms $-(\gamma_t - \gamma_{t-1})\|x_t\|_2^2$ immediately gives part (i) (respectively, part (ii)) of the corollary.

To prove part (iii), first recall that for $\eta_{t,j} > 0$, (6) is equivalent to the AO-FTRL update (4) with $r_{0:t-1}(x) = \frac{1}{2} \sum_{j=1}^d \eta_{t,j} x_j^2$. For $\eta_t = 0$, we define the update to be AO-FTRL with $r_{0:t-1} = 0$, hence the update will be of the form $x_t = \operatorname{argmin}_{x \in \mathcal{X}} \langle z_{t-1}, x \rangle + \alpha_{1:t} \phi(x)$ (recall that $z_{t-1} = -\sum_{s=1}^{t-1} \alpha_s g_s - \alpha_t \tilde{g}_t$). Then, since ϕ is strongly-convex, despite having $r_s = 0$ for all s , we have that $\alpha_{1:t} \phi + r_{0:t-1}$ is strongly-convex w.r.t. $\beta_t \frac{L}{2} \|\cdot\|_2^2$ with $\beta_t = \alpha_{1:t} \frac{\mu}{L}$. Hence, by Theorem 3,⁵ we have

$$\alpha_{1:T} (\ell(\bar{x}_T) - \ell(x^*)) \leq \frac{\alpha_1}{2\mu} \|f'(x_1) - \tilde{g}_1\|^2, \quad (15)$$

as long as for all $t > 1$, the assumption $\alpha_t^2 \beta_t^{-1} \leq \alpha_{1:t-1}$ of Theorem 3 is satisfied: that is, we have

$$\frac{\alpha_t^2}{\alpha_{1:t} \alpha_{1:t-1}} \leq \frac{\mu}{L} = \frac{1}{\kappa - 1}. \quad (16)$$

It remains to show that (16) is satisfied, and simplify the bound (15). To that end, note that on the one hand, by (13) we have $\alpha_{1:t-1}/\alpha_t \geq \sqrt{2\kappa} - 1$, which in turn implies $\frac{\alpha_{1:t} \alpha_{1:t-1}}{\alpha_t^2} \geq 2\kappa - \sqrt{2\kappa} \geq \kappa - 1$, proving (16). On the other hand, (13) implies $\alpha_{1:t-1} \leq (1 - \frac{1}{\sqrt{c\kappa}}) \alpha_{1:t}$ for all $t > 1$; therefore $\alpha_1 \leq \alpha_{1:T} \left(1 - \frac{1}{\sqrt{c\kappa}}\right)^{T-1}$. Putting this back into (15) finishes the proof. $\square$

4.2. A Proximal Adaptive Universal Algorithm

Next, we present the universal convergence of Algorithm 2 with AdaGrad-style step sizes, proved in Appendix B.

Theorem 5. Suppose that the iterates x_t are given by AO-FTRL with AdaGrad step sizes, i.e., using (4) with $r_0 = 0$,

$$r_t(x) = \gamma \sum_{j=1}^d \frac{\eta_{t,j} - \eta_{t-1,j}}{2} (x_j - x_{t,j})^2, \quad t \geq 1,$$

⁵Note that instead of using a norm, here we set $\|\cdot\|_{(t)}$ in Theorem 3 to be zero. While this is not a valid choice, an inspection of the proof of the theorem verifies that the theorem still holds in this case when $\sigma_t = 0$ for all t and the dual norm is also redefined to be zero.

where $\gamma > 0$, $\eta_{t,j} = \sqrt{\sum_{s=1}^t \alpha_s^2 (g_{s,j} - \tilde{g}_{s,j})^2}$, $t > 0$ and $\eta_0 = 0$. Further suppose that g_t are unbiased estimates of $f'(\bar{x}_t)$, and we use $\tilde{g}_t = g_{t-1}$, $t > 1$ and $\tilde{g}_1 = 0$. Let R be an upper-bound on $|x_j^* - x_{t,j}|^2$. Then the following hold:

(i) If $\mathbb{E} [g_{t,j}^2] \leq G_j^2$ for all $t \in [T]$, then

$$\mathbb{E} [\ell(\bar{x}_T) - \ell^*] \leq \sum_{j=1}^d \mathbb{E} \left[\frac{\left(\frac{\gamma R^2}{2} + \frac{2}{\gamma}\right)}{\alpha_{1:T}} \sqrt{\sum_{t=1}^T \alpha_t^2 G_{t,j}^2} \right] \\ = \mathcal{O} \left(\frac{R \sum_{j=1}^d G_j}{\sqrt{T}} \right),$$

for $\gamma = 2/R$, where $G_{t,j} := (g_{t,j} - \tilde{g}_{t,j})$.

(ii) If f is L -smooth over $\mathbb{R}^d$ or otherwise (2) holds for all $x, y \in \mathcal{X}$, and $\mathbb{E} [\sigma_{t,j}^2] \leq \sigma_j^2$ for all $t \in [T]$ (recall that $\sigma_t = g_t - f'(\bar{x}_t)$), then

$$\mathbb{E} [\ell(\bar{x}_T) - \ell^*] \leq \frac{1}{\alpha_{1:T}} \sum_{j=1}^d \mathbb{E} \left[6L \left(\frac{\gamma R^2}{2} + \frac{2}{\gamma} \right)^2 \right] \\ + \frac{1}{\alpha_{1:T}} \left(\frac{\gamma R^2}{2} + \frac{2}{\gamma} \right) \left(2\Delta + \sum_{j=1}^d \sqrt{\sum_{t=1}^T 6\alpha_t^2 \sigma_j^2} \right) \\ = \mathcal{O} \left(\frac{LdR^2 + \Delta R}{T^2} + \frac{\sigma R}{\sqrt{T}} \right),$$

for $\gamma = 2/R$, where $\Delta = \sum_{j=1}^d \mathbb{E} [\|f'(x_{1,j})\|]$.

Remark 2. Note that both bounds are achieved by the same algorithm, without further prior knowledge about f (including knowledge of L or σ). The first bound above is a data-adaptive bound that holds even if f is not smooth, and is optimal when the data is sparse (Duchi et al., 2013). The second bound is of the optimal rate $\mathcal{O}(1/T^2 + \sigma/\sqrt{T})$ when f is smooth.

Remark 3. The bound R required by the theorem is enforced, e.g., when $\mathcal{X}$ is compact. This implies that in the unconstrained optimization setting, similarly to Levy et al. (2018), we assume that we are still given a compact set $\mathcal{X}$ containing x^* and project to that set in the algorithm.

5. Accelerated Variance-Reduced Methods

In this section, we apply our framework to the variance reduced setting. At any instance t , let the algorithm have access to an unbiased stochastic gradient oracle F' . Then the gradient estimate g_t at $\bar{x}_t$ is calculated using F' in combination with evaluating the gradient exactly (without noise) from time to time:

$$g_t = F'(\bar{x}_t, \zeta_t) - F'(\tilde{x}_t, \zeta_t) + f'(\tilde{x}_t) \quad (17)$$

where $\tilde{x}_t$ is a snapshot point where the most recent exact (non-stochastic) gradient was computed.⁶ To ensure g_t is unbiased given the history $\mathcal{H}_t$, we make the following assumption:

$$\mathbb{E}[F'(x, \zeta_t) | \mathcal{H}_t] = f'(x) \text{ for any } x \text{ that is } \mathcal{H}_t \text{ measurable.} \quad (18)$$

in which case we have $\mathbb{E}[g_t | \mathcal{H}_t] = f'(\bar{x}_t)$. The underlying operational assumption in computing g_t is that given $f'(\tilde{x}_t)$, $F'(\tilde{x}_t, \zeta)$ can be determined easily. For example, finite sum minimization problems satisfy this assumption. In particular, in such problems, $f = \sum_{i=1}^n f_i$ for some functions $f_i : \mathcal{X} \rightarrow \mathbb{R}$, $F'(x, i) = f'_i(x)$ for all $i \in [n]$, and ζ_t has a uniform distribution over $[n]$. It is easy to check that our assumptions hold for this setup. In this case, the computational (iteration) complexity of an algorithm is measured by the number of times the gradient of any of the f_i is computed, e.g., computing a full gradient f' requires n gradient evaluations.

We consider two settings: averaging with and without negative momentum: here, instead of computing our average $\bar{x}_t$ as the average of the previous outputs of the optimization algorithm $\mathcal{A}$, we calculate it as

$$\bar{x}_t = \frac{\alpha_{1:t-1} \bar{x}_{t-1} + \alpha_t x_t + p_t \tilde{x}_t}{\alpha_{1:t} + p_t} \quad (19)$$

where the $\alpha_t > 0$ are the averaging weights as before, and $p_t \geq 0$ introduces a negative momentum towards the most recent snapshot point $\tilde{x}_t$. If $p_t = 0$, the formula reduces to the earlier formula of $\bar{x}_t = \sum_{s=1}^t \alpha_s x_s / \alpha_{1:t}$.

The resulting algorithm, presented in Algorithm 3, extends Algorithm 1 of Joulani et al. (2020) to the anytime averaging scheme with negative momentum. Algorithm 3 operates in epochs (the outer loop in the algorithm goes over the epochs): At the beginning of epoch s , the gradient snapshot is calculated. Then, in the s th run of the inner loop, from time $T_{1:s-1} + 1$ to $T_{1:s}$, an optimization algorithm $\mathcal{A}$ is run for T_s steps with the variance reduced gradient estimates (17) and averaging (19). Finally, the snapshot point is updated at the end of the epoch (the exact form of the update is given later for the different variants we consider).

5.1. Warm-up: A Simple Variance-Reduced Algorithm

We first consider a version of our accelerated variance-reduced method without any negative momentum ($p_t = 0$ for all t) with the snapshot point being the first iterate of each epoch (calculated at the end of the last iteration of

⁶The main application of the variance-reduced setting is the case when f is a finite sum, and ζ_t determines which elements of the sum are used in computing the stochastic gradient estimate. In this setting the exact (non-stochastic) gradient is often referred to as the *full gradient*, or *gradient snapshot*.

Algorithm 3 Variance-Reduced Anytime Online-to-Batch with Negative Momentum

- 1: **Input:** Stochastic gradient oracle F' , non-negative weights $(\alpha_t)_{t=1}^T$ with $\alpha_1 > 0$, epoch lengths $T_1, T_2, \dots, T_S$, online linear optimization algorithm $\mathcal{A}$
 - 2: Get the initial point $x_1 \in \mathcal{X}$ from $\mathcal{A}$
 - 3: $\tilde{x} \leftarrow x_1, \bar{x}_1 \leftarrow x_1$
 - 4: **for** $s = 1$ **to** S **do**
 - 5: Calculate and store the full (deterministic) gradient $f'(\tilde{x})$ ▷ Denote $\tilde{x}_t = \tilde{x}$ to be the most recent snapshot
 - 6: **for** $t = T_{1:s-1} + 1$ **to** $T_{1:s}$ **do**
 - 7: Get the next *variance-reduced* stochastic gradient g_t at the *average* iterate $\bar{x}_t$:
 - 8: $g_t \leftarrow F'(\bar{x}_t, \zeta_t) - F'(\tilde{x}_t, \zeta_t) + f'(\tilde{x}_t)$
 - 9: Send $\langle \alpha_t g_t, \cdot \rangle$ as the next linear loss to $\mathcal{A}$
 - 10: Let x_{t+1} be the next iterate from $\mathcal{A}$
 - 11: Let $\bar{x}_{t+1} \leftarrow \frac{\alpha_{1:t} \bar{x}_t + \alpha_{t+1} x_{t+1} + p_{t+1} \tilde{x}_t}{\alpha_{1:t+1} + p_{t+1}}$
 - 12: **end for**
 - 13: Update the snapshot point $\tilde{x}$.
 - 14: **end for**
 - 15: **return** the average iterate $\bar{x}_T$ and / or the latest snapshot $\tilde{x}$.
-

the previous epoch): In line 13, $\tilde{x} = \bar{x}_{t+1}$, that is, in every epoch $s \in [S]$, $\tilde{x}_t = \bar{x}_{T_{1:s-1}+1}$ for all $t \in [T_{1:s-1} + 1, T_{1:s}]$.

To get our accelerated algorithm, we use AO-FTRL with regularizer $r_{1:t-1} = \frac{\eta_t}{2} \|\cdot\|_2^2$ as $\mathcal{A}$ with the snapshot used as the optimistic gradient estimate $\alpha_t \tilde{g}_t = \alpha_t f'(\tilde{x}_t)$. Under mild smoothness assumptions, we have the following bound on the performance of this algorithm:

Theorem 6. *Suppose that f is convex, and assume that f , as well as $F(\cdot, \zeta)$ for all ζ , are either L -smooth w.r.t. $\|\cdot\|_2$ over $\mathbb{R}^d$ or otherwise satisfy (2) for all $x, y \in \mathcal{X}$. Further suppose that $F(\cdot, \zeta)$ satisfies (18). Assume that Algorithm 3 is run with epoch lengths $T_s = \min\{\tau, 2^{s-1}\}$ for some maximum epoch length τ , snapshot update $\tilde{x} = \bar{x}_{t+1}$ in line 13, $\alpha_t = t$, and $\mathcal{A}$ selected as AO-FTRL with regularizer $r_{1:t-1} = \frac{\eta_t}{2} \|\cdot\|_2^2$ for $\eta_t = 16L\tau^2$ and optimistic gradient estimates $\alpha_t \tilde{g}_t = \alpha_t f'(\tilde{x}_t)$. Then, for any $T > \tau$,*

$$\mathbb{E}[\ell(\bar{x}_T)] - \ell(x^*) \leq \frac{16L\tau^2 \|x^*\|_2^2 + \frac{\|f'(\bar{x}_1)\|_2^2}{16L\tau^2}}{T(T+1)}.$$

Proof. By Theorem 17 in Appendix E,

$$\mathcal{R}_T(x^*) = \frac{\eta_T}{2} \|x^*\|_2^2 + \sum_{t=1}^T \frac{\alpha_t^2}{2\eta_t} \|g_t - \tilde{g}_t\|_2^2$$

bounds the linearized composite objective regret of $\mathcal{A}$ in our case (recall that the gradients are scaled with α_t). Thus, combining with Corollary 2 and using that $B_{1:T} \geq 0$, we have

$$\mathbb{E}[\ell(\bar{x}_T)] - \ell(x^*)$$

$$\leq \frac{1}{\alpha_{1:T}} \mathbb{E} \left[\frac{\eta_T}{2} \|x^*\|_2^2 + \sum_{t=1}^T \frac{\alpha_t^2}{2\eta_t} \|g_t - \tilde{g}_t\|_2^2 - \bar{B}_{2:T}^f \right]$$

Lemma 9 in Appendix C shows that

$$\sum_{t=2}^T \alpha_t^2 \|g_t - \tilde{g}_t\|_2^2 \leq 32L\tau^2 \bar{B}_{2:T}^f,$$

which then can be used to cancel all terms but $\|g_1 - \tilde{g}_1\|_2^2/(2\eta_1)$ from the summation above. Using that $g_1 = f'(\bar{x}_1)$ and $\tilde{g}_1 = 0$, and substituting η_t finishes the proof of the theorem. $\square$

In the finite sum optimization setting, by selecting $\tau = n$, our algorithm achieves ε error after $\mathcal{O}\left(n \log n + n\sqrt{\frac{L}{\varepsilon}}\right)$ individual gradient evaluations, via a simple direct approach. In contrast, Catalyst (Lin et al., 2015), RPDG (Lan and Zhou, 2018), Katyusha (Allen-Zhu, 2017) and related papers use an indirect approach (as termed by Allen-Zhu (2017)), where non-strongly-convex functions are optimized by adding strongly-convex perturbations. These more complicated methods can achieve iteration complexity $\mathcal{O}\left(n \log \frac{1}{\varepsilon} + \sqrt{\frac{nL}{\varepsilon}}\right)$, which has a better dependence on n in the dominant second term. Nevertheless, these algorithms also do not achieve the optimal rate obtained by Lan et al. (2019), which is achieved using negative momentum and epoch averaging.

5.2. Improved Variance-Reduced Acceleration

In this section we combine our averaging framework with negative momentum, first introduced by Allen-Zhu (2017), to achieve the optimal accelerated rate for variance-reduced algorithms. For the sake of simplicity, we assume $\phi = 0$, which essentially makes $\ell(x) = f(x)$ for all x .

To incorporate the negative momentum, we compute $\bar{x}_t$ via (19) with $p_t > 0$. A direct consequence of this expression is that for all $t = 1, 2, \dots, T$:

$$\alpha_t(\bar{x}_t - x_t) = \alpha_{1:t-1}(\bar{x}_{t-1} - \bar{x}_t) + p_t(\tilde{x}_t - \bar{x}_t) \quad (20)$$

The analysis of the algorithm is based on the following regret decomposition, proved in Appendix D:

Lemma 7 (Regret Decomposition). *For $t \in [T]$, let $\alpha_t, p_t > 0$, $x_t, \tilde{x}_t \in \mathbb{R}^d$, and define $\bar{x}_t$ as in Equation (19), $B_t = \alpha_t B_f(x^*, \bar{x}_t)$ and $\bar{B}_t^f = \alpha_{1:t-1} B_f(\bar{x}_{t-1}, \bar{x}_t)$. Then, for all $x^* \in \mathbb{R}^d$,*

$$f(\bar{x}_T) - f^* = \frac{1}{\alpha_{1:T}} \left[\sum_{t=1}^T \langle \alpha_t f'(\bar{x}_t), x_t - x^* \rangle - B_{1:T} - \sum_{t=2}^T \bar{B}_t^f + \sum_{t=1}^T p_t (f(\tilde{x}_t) - f(\bar{x}_t) - B_f(\tilde{x}_t, \bar{x}_t)) \right] \quad (21)$$

The above error decomposition is similar to that of Lemma 1, but has an extra term due to the negative momentum. However, the rest of the terms can be bounded as it was done previously. Unlike in Theorem 6, for any outer loop iteration s the snapshot point is now defined as an average:

$$\tilde{x}_{s+1} = \frac{1}{\sum_{t=T_{1:s-1}+1}^{T_{1:s}} p_t} \sum_{t=T_{1:s-1}+1}^{T_{1:s}} p_t \bar{x}_t, \quad (22)$$

where $p_t > 0$ for all t . Next we show the main result of this section, namely that the resulting algorithm achieves the optimal accelerated rate for variance-reduced algorithms. In the next theorem, proved in Appendix D, $s(t)$ denotes the epoch number corresponding to iteration t and $\tau_{s(t)}$ denotes the length of that epoch.

Theorem 8. *Suppose that f is convex, and assume that f , as well as $F(\cdot, \zeta)$ for all ζ , are either L -smooth w.r.t. $\|\cdot\|_2$ over $\mathbb{R}^d$ or otherwise satisfy (2) for all $x, y \in \mathcal{X}$. Further assume that F and F' are unbiased estimates of f and f' , that is, (18) holds and we also have $\mathbb{E}[F(x, \zeta_t) | \mathcal{H}_t] = f(x)$ for any $\mathcal{H}_t$ -measurable x . Suppose that Algorithm 3 is run with epoch lengths $T_s = \min\{\tau, 2^{s-1}\}$ for some maximum epoch length τ , snapshot update $\tilde{x}$ defined in (22), $\alpha_t = t$, and $\mathcal{A}$ selected as AO-FTRL with regularizer $r_{1:t-1} = \frac{\eta_t}{2} \|\cdot\|_2^2$ for $\eta_t = 28L\tau_{s(t)} \log 2t$ and optimistic gradient estimates $\alpha_t \tilde{g}_t = \alpha_t f'(\tilde{x}_t)$. Then, for any T ,*

$$\begin{aligned} & \mathbb{E}[\ell(\bar{x}_T)] - \ell(x^*) \\ & \leq \frac{56L\tau_{s(T)} \|x^*\|_2^2 + \frac{\|f'(\bar{x}_1)\|_2^2}{28L \log 2} + 4(f(\tilde{x}_1) - f^*)}{T(T+1)} \log 2T. \end{aligned}$$

The rate provided in Theorem 8 is optimal up to a logarithmic factor. In particular, for the finite sum setting, with $\tau = n$, the algorithm needs $\tilde{\mathcal{O}}\left(n \log n + \sqrt{\frac{nL}{\varepsilon}}\right)$ individual gradient evaluations to reach ε error, matching the rate recently obtained by Lan et al. (2019). Unlike in previous work, our convergence guarantee holds for the last iterate instead of a snapshot point or the average of the last epoch.

6. Conclusions

We demonstrated that online iterate averaging combined with optimistic online learning can lead to accelerated rates in several scenarios. The resulting algorithms and their analyses are surprisingly simple and often yield the optimal rates. Exploring the full power of this method is left for future work. In particular, it would be interesting to extend this approach to obtain accelerated exponential rates for variance-reduced optimization of strongly-convex objectives, and remove the extra logarithmic terms in the non-strongly-convex variance-reduction case.

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A. Proof of Theorem 3

Proof of Theorem 3. First, we bound the linear composite regret $\sum_{t=1}^T \alpha_t (\langle g_t, x_t - x^* \rangle + \phi(x_t) - \phi(x^*))$ by a bound $\mathcal{R}_T$ that results in slightly better constants compared to (7). For $t \in [T]$, let u_t, v_t be any two vectors such that $g_t - \tilde{g}_t = u_t + v_t$, and define $D_t = r_{t-1}(x^*) - r_{t-1}(x_t)$. Then, Theorem 17 bounds the regret of the AO-FTRL updates made by $\mathcal{A}$ as follows:

$$\begin{aligned}
 & \sum_{t=1}^T (\langle \alpha_t g_t, x_t - x^* \rangle + \alpha_t \phi(x_t) - \alpha_t \phi(x^*)) - D_{1:T} \\
 & \leq \sum_{t=1}^T (\alpha_t \langle g_t - \tilde{g}_t, x_t - x_{t+1} \rangle - B_{\alpha_{1:t}\phi + r_{0:t-1}}(x_{t+1}, x_t)) \\
 & \leq \sum_{t=1}^T \left(\langle \alpha_t u_t + \alpha_t v_t, x_t - x_{t+1} \rangle - \frac{\beta_t}{2} \|x_t - x_{t+1}\|^2 - \frac{1}{2} \|x_t - x_{t+1}\|_{(t)}^2 \right) \\
 & \leq \sum_{t=1}^T \frac{\alpha_t^2 \|v_t\|_{(t)*}^2}{2} + \sum_{t=1}^T \frac{\alpha_t^2 \|u_t\|_*^2}{2\beta_t}, \tag{23}
 \end{aligned}$$

where the second step uses the strong-convexity of $\alpha_{1:t}\phi + r_{0:t-1}$, and the third step follows by the Fenchel-Young inequality $\langle z, x \rangle - \frac{\beta}{2} \|x\|^2 \leq \frac{1}{2\beta} \|z\|_*^2$. Now, let $u_1 = f'(\bar{x}_1) - \tilde{g}_1$, $u_t = f'(\bar{x}_t) - f'(\bar{x}_{t-1})$, $t = 2, \dots, T$, and $v_t = \sigma_t - \sigma_{t-1}$, $t \in [T]$, and notice that for all $t = 2, \dots, T$,

$$\begin{aligned}
 \frac{\alpha_t^2 \|u_t\|_*^2}{2\beta_t} & \leq \frac{\alpha_{1:t-1} \|u_t\|_*^2}{2} = \frac{\alpha_{1:t-1}}{2} \|f'(\bar{x}_t) - f'(\bar{x}_{t-1})\|_*^2 \\
 & \leq \alpha_{1:t-1} B_f(\bar{x}_{t-1}, \bar{x}_t) = \bar{B}_t^f, \tag{24}
 \end{aligned}$$

using the assumption of $\alpha_t^2 \beta_t^{-1} \leq \alpha_{1:t-1}$ and (2) with $L = 1$. Plugging the definitions of u_t and v_t into (23) and using (24), we obtain

$$\begin{aligned}
 \sum_{t=1}^T \langle \alpha_t g_t, x_t - x^* \rangle & \leq \sum_{t=1}^T (D_t + \alpha_t \phi(x_t) - \alpha_t \phi(x^*)) \\
 & + \sum_{t=1}^T \frac{\alpha_t^2 \|\sigma_t - \sigma_{t-1}\|_{(t)*}^2}{2} + \frac{\alpha_1^2 \|f'(\bar{x}_1) - \tilde{g}_1\|_*^2}{2\beta_1} + \bar{B}_{2:T}^f. \tag{25}
 \end{aligned}$$

Applying Corollary 2, combining with (25), and cancelling the matching $\bar{B}_{2:T}^f$ terms concludes the proof. $\square$

B. Proof of Theorem 5

Proof of Theorem 5. Starting from Lemma 1, we plug-in the bound of composite AO-FTRL from Mohri and Yang (2016, Theorem 3) (using $f_t \leftarrow \langle \alpha_t g_t, \cdot \rangle$):

$$\begin{aligned}
 \sum_{t=1}^T \alpha_t (\langle g_t, x_t - x^* \rangle + \phi(x_t) - \phi(x^*)) & \leq r_{0:T}(x^*) + \sum_{t=1}^T \alpha_t^2 \|g_t - \tilde{g}_t\|_{(t)*}^2 \\
 & \leq \frac{\gamma R^2}{2} \sum_{j=1}^d \sqrt{\sum_{t=1}^T \alpha_t^2 (g_{t,j} - \tilde{g}_{t,j})^2} + \frac{1}{\gamma} \sum_{j=1}^d \sum_{t=1}^T \frac{\alpha_t^2 (g_{t,j} - \tilde{g}_{t,j})^2}{\sqrt{\sum_{s=1}^t \alpha_s^2 (g_{s,j} - \tilde{g}_{s,j})^2}} \\
 & \leq \left(\frac{\gamma R^2}{2} + \frac{2}{\gamma} \right) \sum_{j=1}^d \sqrt{\sum_{t=1}^T \alpha_t^2 (g_{t,j} - \tilde{g}_{t,j})^2},
 \end{aligned}$$

where the first inequality follows from Mohri and Yang (2016, Theorem 3) with the norm $\|y\|_{(t)*} = \sum_{j=1}^d \frac{1}{\gamma \eta_{t,j}} (y_j)^2$, the second from the definitions using $r_{0:T} \leq \sum_{t=1}^T \sum_{j=1}^d \frac{\gamma R^2}{2} (\eta_{t,j} - \eta_{t-1,j}) = \sum_{j=1}^d \frac{\gamma R^2}{2} \eta_{T,j}$, and the third from the standard inequality that $\sum_{t=1}^T a_t / \sqrt{a_{1:t}} \leq 2\sqrt{a_{1:T}}$.

Putting back into Corollary 2, we obtain

$$\mathbb{E}[\ell(\bar{x}_T) - \ell^*] \leq \frac{1}{\alpha_{1:T}} \mathbb{E} \left[\left(\frac{\gamma R^2}{2} + \frac{2}{\gamma} \right) \sum_{j=1}^d \sqrt{\sum_{t=1}^T \alpha_t^2 (g_{t,j} - \tilde{g}_{t,j})^2} - \bar{B}_{2:T}^f \right]. \quad (26)$$

Dropping the negative terms $\bar{B}_{2:T}^f$, using Jensen's inequality to take the expectation under the square-root, using $\alpha_t = t$ and applying the bound G_j completes the proof of the first part of the theorem.

To prove the second part, observe that for $t > 1$, because $\tilde{g}_t = g_{t-1}$, we have

$$|g_{t,j} - \tilde{g}_{t,j}| = |f'_j(\bar{x}_t) + \sigma_{t,j} - f'_j(\bar{x}_{t-1}) - \sigma_{t-1,j}| \leq |\sigma_{t,j}| + |\sigma_{t-1,j}| + |f'_j(\bar{x}_t) - f'_j(\bar{x}_{t-1})|.$$

Now, denote $\Delta_{t,j} = |f'_j(\bar{x}_t) - f'_j(\bar{x}_{t-1})|$. By Jensen's inequality, $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$ for any real numbers a, b, c . In addition, for $t > 1$, $\sum_{j=1}^d \Delta_{t,j}^2 = \|f'(\bar{x}_t) - f'(\bar{x}_{t-1})\|_2^2 \leq 2LB_f(\bar{x}_{t-1}, \bar{x}_t)$ by (2). Hence,

$$\sum_{t=2}^T \alpha_t^2 (g_{t,j} - \tilde{g}_{t,j})^2 \leq \sum_{t=2}^T 3\alpha_t^2 [|\sigma_{t,j}|^2 + |\sigma_{t-1,j}|^2 + \Delta_{t,j}^2],$$

on the one hand, and on the other hand

$$-\bar{B}_{2:T}^f \leq -\sum_{j=1}^d \sum_{t=2}^T \frac{\alpha_{1:t-1}}{2L} \Delta_{t,j}^2.$$

Putting back into (26) and using the tower rule on the terms $|\sigma_{t,j}|$, we have

$$\begin{aligned} \mathbb{E}[\ell(\bar{x}_T) - \ell^*] &\leq \frac{1}{\alpha_{1:T}} \mathbb{E} \left[\left(\frac{\gamma R^2}{2} + \frac{2}{\gamma} \right) \sum_{j=1}^d \sqrt{\sum_{t=1}^T \alpha_t^2 (g_{t,j} - \tilde{g}_{t,j})^2} - \bar{B}_{2:T}^f \right] \\ &\leq \frac{1}{\alpha_{1:T}} \sum_{j=1}^d \mathbb{E} \left[\left(\frac{\gamma R^2}{2} + \frac{2}{\gamma} \right) \sqrt{\sum_{t=1}^T 6\alpha_t^2 \sigma_j^2 + 3\Delta_{1,j}^2 + \sum_{t=2}^T 3\alpha_t^2 \Delta_{t,j}^2} - \sum_{t=2}^T \frac{\alpha_{1:t-1}}{2L} \Delta_{t,j}^2 \right] \\ &\leq \frac{1}{\alpha_{1:T}} \sum_{j=1}^d \mathbb{E} \left[\left(\frac{\gamma R^2}{2} + \frac{2}{\gamma} \right) \sqrt{\sum_{t=1}^T 6\alpha_t^2 \sigma_j^2 + 3\Delta_{1,j}^2 + \sum_{t=2}^T 6L \frac{\alpha_t^2}{\alpha_{1:t-1}} \left(\frac{\alpha_{1:t-1}}{2L} \Delta_{t,j}^2 \right)} - \sum_{t=2}^T \frac{\alpha_{1:t-1}}{2L} \Delta_{t,j}^2 \right] \\ &\leq \frac{1}{\alpha_{1:T}} \sum_{j=1}^d \mathbb{E} \left[\left(\frac{\gamma R^2}{2} + \frac{2}{\gamma} \right) \sqrt{\sum_{t=1}^T 6\alpha_t^2 \sigma_j^2 + 3\Delta_{1,j}^2} \right. \\ &\quad \left. + \frac{1}{\alpha_{1:T}} \sum_{j=1}^d \mathbb{E} \left[\left(\frac{\gamma R^2}{2} + \frac{2}{\gamma} \right) \sqrt{24L \sum_{t=2}^T \frac{\alpha_{1:t-1}}{2L} \Delta_{t,j}^2} - \sum_{t=2}^T \frac{\alpha_{1:t-1}}{2L} \Delta_{t,j}^2 \right], \right] \end{aligned}$$

using in the last step $\alpha_t = t$ and the fact that for $t > 1$, $\frac{\alpha_t^2}{\alpha_{1:t-1}} \leq 4$. Next, we note that for $a, b \geq 0$, $2\sqrt{ab} - b \leq a$. Therefore, using that $\alpha_t = t$, we have

$$\begin{aligned} \mathbb{E}[\ell(\bar{x}_T) - \ell^*] &\leq \frac{1}{\alpha_{1:T}} \sum_{j=1}^d \mathbb{E} \left[\left(\frac{\gamma R^2}{2} + \frac{2}{\gamma} \right) \sqrt{\sum_{t=1}^T 6\alpha_t^2 \sigma_j^2 + 3\Delta_{1,j}^2} \right] \\ &\quad + \frac{1}{\alpha_{1:T}} \sum_{j=1}^d \mathbb{E} \left[6L \left(\frac{\gamma R^2}{2} + \frac{2}{\gamma} \right)^2 \right]. \end{aligned}$$

Separating the first square-root and using $\Delta = \sum_{j=1}^d \Delta_{1,j}$ completes the proof of the second part. $\square$

C. Variance reduction for smooth functions

Lemma 9. *Suppose the assumptions of Theorem 6 hold. Then*

$$\sum_{t=1}^T \alpha_t^2 \mathbb{E} [\|g_t - \tilde{g}_t\|_2^2] \leq 32L\tau^2 \bar{B}_t^f.$$

Proof. For any t in epoch s , let $\tilde{\tau}_t = t - (T_{1:s-1} + 1)$ denote the number of time steps elapsed since the beginning of the epoch. That is, the last snapshot taken is $\bar{x}_t = \bar{x}_{t-\tilde{\tau}_t}$. Therefore, for any $t \geq 2$,

$$\begin{aligned} \mathbb{E} [\|g_t - \tilde{g}_t\|_2^2 | \mathcal{H}_t] &= \mathbb{E} [\|F'(\bar{x}_t, \zeta_t) - F'(\bar{x}_{t-\tilde{\tau}_t}, \zeta_t)\|_2^2 | \mathcal{H}_t] \\ &\leq \mathbb{E} [\|F'(\bar{x}_t, \zeta_t) - F'(\bar{x}_{t-\tilde{\tau}_t}, \zeta_t)\|_2^2 | \mathcal{H}_t] \\ &= \mathbb{E} \left[\left\| \sum_{i=1}^{\tilde{\tau}_t} (F'(\bar{x}_{t-i+1}, \zeta_t) - F'(\bar{x}_{t-i}, \zeta_t)) \right\|_2^2 \middle| \mathcal{H}_t \right] \\ &\leq \tilde{\tau}_t \sum_{i=1}^{\tilde{\tau}_t} \mathbb{E} [\|F'(\bar{x}_{t-i+1}, \zeta_t) - F'(\bar{x}_{t-i}, \zeta_t)\|_2^2 | \mathcal{H}_t] \\ &\leq 2\tilde{\tau}_t L \sum_{i=1}^{\tilde{\tau}_t} \mathbb{E} [B_f(\bar{x}_{t-i}, \bar{x}_{t-i+1}) | \mathcal{H}_t], \end{aligned}$$

where the last inequality follows from (2). Multiplying by α_t^2 and summing up for all t , we get

$$\sum_{t=2}^T \alpha_t^2 \tilde{\tau}_t \sum_{i=1}^{\tilde{\tau}_t} \mathbb{E} [B_f(\bar{x}_{t-i}, \bar{x}_{t-i+1})] = \sum_{i=2}^T \mathbb{E} [B_f(\bar{x}_{i-1}, \bar{x}_i)] \sum_{t=1}^T \alpha_t^2 \tilde{\tau}_t \mathbb{I} \{t - \tilde{\tau}_t + 1 \leq i \leq t\}.$$

To bound the inner sum, notice that since the length of the s th epoch is $T_s = \min(\tau, 2^{s-1})$, if i is the starting iteration index of the s th epoch, then $\tau_s \leq i$. Hence, if i and t belong to the same outer loop, $t \leq i + \tau_t \leq 2i$. Thus, using that $\alpha_t = t$ by definition,

$$\sum_{t=1}^T \alpha_t^2 \tilde{\tau}_t \mathbb{I} \{t - \tilde{\tau}_t + 1 \leq i \leq t\} = \sum_{t=1}^T \alpha_t^2 \tilde{\tau}_t \mathbb{I} \{i \leq t \leq i + \tilde{\tau}_t - 1\} \leq \sum_{t=i}^{i+\tau-1} \tau \alpha_{2i}^2 \leq 4\tau^2 i^2 \leq 8\tau^2 i(i-1) = 16\tau^2 \alpha_{1:i-1}.$$

Combining the results we get

$$\sum_{t=2}^T \alpha_t^2 \mathbb{E} [\|g_t - \tilde{g}_t\|_2^2] \leq 32L\tau^2 \sum_{t=2}^T \alpha_{1:t-1} B_f(\bar{x}_{t-1}, \bar{x}_t) = 32L\tau^2 \bar{B}_t^f,$$

finishing the proof. $\square$

D. Improved variance-reduced rate for smooth functions

Proof of Lemma 7. The proof follows similar steps as the proof of Lemma 1, with the difference that we need to handle the negative momentum term as well:

$$\begin{aligned} f(\bar{x}_T) - f^* &= \frac{1}{\alpha_{1:T}} \sum_{t=1}^T \alpha_t f(\bar{x}_t) - f^* + \overbrace{f(\bar{x}_T) - \frac{1}{\alpha_{1:T}} \sum_{t=1}^T \alpha_t f(\bar{x}_t)}^{:= \varepsilon_T} \\ &= \frac{1}{\alpha_{1:T}} \sum_{t=1}^T \alpha_t (f(\bar{x}_t) - f^*) + \varepsilon_T \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\alpha_{1:T}} \left[\sum_{t=1}^T \langle \alpha_t f'(\bar{x}_t), \bar{x}_t - x^* \rangle - \sum_{t=1}^T \overbrace{\alpha_t B_f(x^*, \bar{x}_t)}^{:=B_t} \right] + \varepsilon_T \\
 &= \frac{1}{\alpha_{1:T}} \left[\sum_{t=1}^T \langle \alpha_t f'(\bar{x}_t), x_t - x^* \rangle + \sum_{t=1}^T \langle \alpha_t f'(\bar{x}_t), \bar{x}_t - x_t \rangle - B_{1:T} \right] + \varepsilon_T \\
 &= \frac{1}{\alpha_{1:T}} \left[\sum_{t=1}^T \langle \alpha_t f'(\bar{x}_t), x_t - x^* \rangle - B_{1:T} \right] + \overbrace{\frac{1}{\alpha_{1:T}} \sum_{t=1}^T \langle \alpha_t f'(\bar{x}_t), \bar{x}_t - x_t \rangle}^{:=\tilde{\Delta}_T} + \varepsilon_T
 \end{aligned} \tag{27}$$

Before evaluating $\tilde{\Delta}_T$, we consider the term,

$$\begin{aligned}
 \Delta_T &= \frac{1}{\alpha_{1:T}} \left[\sum_{t=2}^T \langle f'(\bar{x}_t), \alpha_{1:t-1}(\bar{x}_{t-1} - \bar{x}_t) \rangle \right] + \varepsilon_T \\
 &= \frac{1}{\alpha_{1:T}} \left[\sum_{t=2}^T \alpha_{1:t-1} (f(\bar{x}_{t-1}) - f(\bar{x}_t) - B_f(\bar{x}_{t-1}, \bar{x}_t)) \right] + \varepsilon_T \\
 &= \frac{1}{\alpha_{1:T}} \sum_{t=2}^T \alpha_{1:t-1} (f(\bar{x}_{t-1}) - f(\bar{x}_t)) - \frac{1}{\alpha_{1:T}} \sum_{t=2}^T \alpha_{1:t-1} B_f(\bar{x}_{t-1}, \bar{x}_t) + \varepsilon_T \\
 &= \underbrace{\frac{1}{\alpha_{1:T}} \sum_{t=1}^T \alpha_t f(\bar{x}_t) - f(\bar{x}_T) + \varepsilon_T}_{:=0} - \underbrace{\frac{1}{\alpha_{1:T}} \sum_{t=2}^T \alpha_{1:t-1} B_f(\bar{x}_{t-1}, \bar{x}_t)}_{:=\tilde{B}_t^f}
 \end{aligned} \tag{28}$$

Let us now evaluate $\tilde{\Delta}_T$. We have :

$$\begin{aligned}
 \tilde{\Delta}_T &= \frac{1}{\alpha_{1:T}} \sum_{t=1}^T \langle \alpha_t f'(\bar{x}_t), \bar{x}_t - x_t \rangle + \varepsilon_T \\
 &\stackrel{(20)}{=} \frac{1}{\alpha_{1:T}} \sum_{t=1}^T \langle f'(\bar{x}_t), \alpha_{1:t-1}(\bar{x}_{t-1} - \bar{x}_t) + p_t(\tilde{x}_t - \bar{x}_t) \rangle + \varepsilon_T \\
 &= \underbrace{\frac{1}{\alpha_{1:T}} \sum_{t=1}^T \langle f'(\bar{x}_t), \alpha_{1:t-1}(\bar{x}_{t-1} - \bar{x}_t) \rangle + \varepsilon_T}_{:=\Delta_T} + \frac{1}{\alpha_{1:T}} \sum_{t=1}^T p_t \langle f'(\bar{x}_t), \tilde{x}_t - \bar{x}_t \rangle \\
 &= \frac{1}{\alpha_{1:T}} \sum_{t=1}^T p_t \langle f'(\bar{x}_t), \tilde{x}_t - \bar{x}_t \rangle - \frac{1}{\alpha_{1:T}} \sum_{t=2}^T \bar{B}_t^f \\
 &= \frac{1}{\alpha_{1:T}} \sum_{t=1}^T p_t (f(\tilde{x}_t) - f(\bar{x}_t) - B_f(\tilde{x}_t, \bar{x}_t)) - \frac{1}{\alpha_{1:T}} \sum_{t=2}^T \bar{B}_t^f
 \end{aligned} \tag{29}$$

The second last equation comes directly from Equation 28. Hence, finally we have:

$$f(\bar{x}_T) - f^* = \frac{1}{\alpha_{1:T}} \left[\sum_{t=1}^T \langle \alpha_t f'(\bar{x}_t), x_t - x^* \rangle - B_{1:T} - \sum_{t=2}^T \bar{B}_t^f + \sum_{t=1}^T p_t (f(\tilde{x}_t) - f(\bar{x}_t) - B_f(\tilde{x}_t, \bar{x}_t)) \right] \tag{30}$$

□

The following variance bound is standard in the literature (e.g., Allen-Zhu 2017, Lemma 2.4). For completeness, we provide a proof using our assumptions and notation.

Lemma 10. Fix $t \geq 1$ and assume that

$$\mathbb{E}[F(\bar{x}_t, \zeta_t) | \bar{x}_t, \tilde{x}_t] = f(\bar{x}_t), \quad (31a)$$

$$\mathbb{E}[F(\tilde{x}_t, \zeta_t) | \bar{x}_t, \tilde{x}_t] = f(\tilde{x}_t), \quad (31b)$$

$$\mathbb{E}[F'(\bar{x}_t, \zeta_t) | \bar{x}_t, \tilde{x}_t] = f'(\bar{x}_t), \quad (31c)$$

$$\mathbb{E}[F'(\tilde{x}_t, \zeta_t) | \bar{x}_t, \tilde{x}_t] = f'(\tilde{x}_t). \quad (31d)$$

Let

$$g_t = F'(\bar{x}_t, \zeta_t) - F'(\tilde{x}_t, \zeta_t) + f'(\tilde{x}_t).$$

Further assume that for all ζ , $F(\cdot, \zeta)$ is L -smooth w.r.t. the 2-norm $\|\cdot\|$ over $\mathbb{R}^d$ or otherwise satisfies (2) for all $x, y \in \mathcal{X}$. Then,

$$\mathbb{E}[\|g_t - f'(\bar{x}_t)\|^2] \leq \mathbb{E}\|F'(\bar{x}_t, \zeta_t) - F'(\tilde{x}_t, \zeta_t)\| \leq 2L\mathbb{E}[B_f(\bar{x}_t, \tilde{x}_t)].$$

Proof. By assumption, from (2) we have

$$\|F'(x, \zeta) - F'(x', \zeta)\|^2 \leq 2L(F(x, \zeta) - F(x', \zeta) - \langle F'(x', \zeta), x - x' \rangle), \quad (32)$$

for all $x, x' \in \mathcal{X}$. Now, thanks to $\mathbb{E}[\|U - \mathbb{E}[U]\|^2] \leq \mathbb{E}[\|U\|^2]$ which holds for any U random vector,

$$\begin{aligned} \mathbb{E}[\|g_t - f'(\bar{x}_t)\|^2 | \bar{x}_t, \tilde{x}_t] &= \mathbb{E}[\|F'(\bar{x}_t, \zeta_t) - f'(\bar{x}_t) - (F'(\bar{x}_t, \zeta_t) - f'(\tilde{x}_t))\|^2 | \bar{x}_t, \tilde{x}_t] \\ &\leq \mathbb{E}[\|F'(\bar{x}_t, \zeta_t) - F'(\tilde{x}_t, \zeta_t)\|^2 | \bar{x}_t, \tilde{x}_t] \\ &\leq 2L\mathbb{E}[F(\bar{x}_t, \zeta_t) - F(\tilde{x}_t, \zeta_t) - \langle F'(\tilde{x}_t, \zeta_t), \bar{x}_t - \tilde{x}_t \rangle | \bar{x}_t, \tilde{x}_t] \\ &= 2L\{f(\bar{x}_t) - f(\tilde{x}_t) - \langle f'(\tilde{x}_t), \bar{x}_t - \tilde{x}_t \rangle\}. \end{aligned} \quad (\text{by (32)})$$

Taking the expectation of both sides finishes the proof. $\square$

Proof of Theorem 8. Note that $B_f(\tilde{x}_1, \bar{x}_1) = 0$. Also since, $\phi = 0$, $\ell(x) = f(x)$ for all x , hence we will be using f instead of ℓ in the proof. Also $s(t)$ represents the outerloop iteration number corresponding to the total iteration t . We introduce another notation, for any s , $\tau_s = T_s$. From Lemma 7, we have :

$$f(\bar{x}_T) - f^* = \frac{1}{\alpha_{1:T}} \left[\sum_{t=1}^T \langle \alpha_t f'(\bar{x}_t), x_t - x^* \rangle - B_{1:T} - \sum_{t=2}^T \bar{B}_t^f + \sum_{t=1}^T p_t(f(\tilde{x}_t) - f(\bar{x}_t) - B_f(\tilde{x}_t, \bar{x}_t)) \right]$$

Now we have:

$$\frac{1}{\alpha_{1:T}} \left[\sum_{t=1}^T \langle \alpha_t f'(\bar{x}_t), x_t - x^* \rangle \right] = \frac{1}{\alpha_{1:T}} \left[\sum_{t=1}^T \langle \alpha_t f'(\bar{x}_t) - \alpha_t g_t + \alpha_t g_t, x_t - x^* \rangle \right] = \frac{1}{\alpha_{1:T}} \left[\sum_{t=1}^T \langle \alpha_t g_t, x_t - x^* \rangle + \delta_{1:T} \right]$$

where $\delta_t = \alpha_t \langle f'(\bar{x}_t) - g_t, x_t - x^* \rangle$. It is clear that $\mathbb{E}[\delta_t] = 0 \forall t$.

By Theorem 17 in the appendix,

$$\frac{1}{\alpha_{1:T}} \mathbb{E} \left[\sum_{t=1}^T \langle \alpha_t f'(\bar{x}_t), x_t - x^* \rangle \right] \leq \mathcal{R}_T(x^*) = \frac{\eta_T}{2} \|x^*\|_2^2 + \sum_{t=1}^T \frac{\alpha_t^2}{2\eta_t} \mathbb{E}[\|g_t - \tilde{g}_t\|_2^2]$$

bounds the linearized objective regret of $\mathcal{A}$ in our case (recall that the gradients are scaled with α_t). Thus, combining with Lemma 7 and using that $B_{1:T} \geq 0$, we have

$$\mathbb{E}[f(\bar{x}_T)] - f(x^*) \leq \frac{1}{\alpha_{1:T}} \mathbb{E} \left[\frac{\eta_T}{2} \|x^*\|_2^2 + \sum_{t=1}^T \frac{\alpha_t^2}{2\eta_t} \|g_t - \tilde{g}_t\|_2^2 - \bar{B}_{2:T}^f + \sum_{t=1}^T p_t(f(\tilde{x}_t) - f(\bar{x}_t) - B_f(\tilde{x}_t, \bar{x}_t)) \right] \quad (33)$$

Assuming that $\tilde{g}_t = F'(\tilde{x}_t, \zeta_t)$, then we have from Lemma 10,

$$\mathbb{E}[\|g_t - \tilde{g}_t\|_2] \leq 2L\mathbb{E}[B_f(\tilde{x}_t, \bar{x}_t)].$$

Hence, we now choose $p_t \geq \frac{L\alpha_t^2}{\eta_t}$ in the Equation (33) with $\tilde{g}_1 = 0$ and $\bar{B}_{2:T}^f > 0$. Now, we have:

$$\mathbb{E}[f(\bar{x}_T)] - f(x^*) \leq \frac{1}{\alpha_{1:T}} \mathbb{E} \left[\frac{\eta_T}{2} \|x^*\|_2^2 + \frac{\alpha_1^2}{2\eta_1} \|g_1\|^2 + \underbrace{\sum_{t=1}^T p_t (f(\tilde{x}_t) - f(\bar{x}_t))}_{\Gamma_T} \right] \quad (34)$$

In the rest of the proof, we will bound the term Γ_T and use induction to show final convergence rate. Let us denote $\mathbb{E}[f(\bar{x}_t) - f^*]$ with D_t and $\mathbb{E}[f(\tilde{x}_t) - f^*]$ with $\tilde{D}_{s(t)}$ where s in $s(t)$ represents number of outer epochs so far till update t . Let us also represent the snapshot point for epoch s is by $\tilde{x}_s$. Hence, $\tilde{D}(s) = \mathbb{E}[f(\tilde{x}_s)] - f^*$. We also assume that $T_{1:S} \leq T \leq T_{1:S+1}$.

We also denote $D(s)$ as $f(\tilde{x}^s) - f^*$. Then

$$\begin{aligned} \sum_{t=1}^T p_t (f(\tilde{x}_t) - f(\bar{x}_t)) &= \sum_{t=1}^T p_t (\tilde{D}_{s(t)} - D_t) \\ &\leq p_1 \tilde{D}(1) + \sum_{s=2}^S \tilde{D}(s) \left(\sum_{t=T_{1:s-1}+1}^{T_{1:s}} p_t \right) - \sum_{s=1}^S \left(\sum_{t=T_{1:s-1}+1}^{T_{1:s}} p_t D_t \right) \end{aligned} \quad (35)$$

Since, we have

$$\tilde{x}^{s+1} = \frac{1}{\sum_{t=T_{1:s-1}+1}^{T_{1:s}} p_t} \sum_{t=T_{1:s-1}+1}^{T_{1:s}} p_t \tilde{x}_t. \quad (36)$$

From the definition of $\tilde{x}_s$, we can now apply Jensen's inequality above to get

$$\sum_{t=1}^T p_t (f(\tilde{x}_t) - f(\bar{x}_t)) \leq p_1 \tilde{D}(1) + \sum_{s=1}^S \left(\sum_{t=T_{1:s-1}+1}^{T_{1:s}} p_t D_t \right) \underbrace{\left(\frac{\sum_{t=T_{1:s-1}+1}^{T_{1:s}} p_t}{\sum_{t=T_{1:s-1}+1}^{T_{1:s}} p_t} - 1 \right)}_{:=\Theta_s}. \quad (37)$$

Let us assume that for $p_t = \frac{\alpha_t^2}{\gamma_t \tau_{s(t)}}$ $\forall t \in [T_{1:s-1}+1, T_{1:s}]$ where $\alpha_t = t$ and γ_t is an increasing sequence of positive numbers. We rewrite the term Θ_s .

$$\Theta_s = \left(\frac{\sum_{t=T_{1:s-1}+1}^{T_{1:s}} p_t}{\sum_{t=T_{1:s-1}+1}^{T_{1:s}} p_t} - 1 \right) \leq \left(\frac{\tau_s}{\tau_{s+1}} \frac{\sum_{t=T_{1:s-1}+1}^{T_{1:s}} t^2}{\sum_{t=T_{1:s-1}+1}^{T_{1:s}} t^2} - 1 \right)$$

We define s' is the index for which $\tau_s = \tau$ for all $s > s'$. Hence, for $s > s'$ we have following bound for Θ_s :

$$\begin{aligned} \Theta_s &\leq \left(\frac{\tau_s}{\tau_{s+1}} \frac{\sum_{t=T_{1:s-1}+1}^{T_{1:s}} t^2}{\sum_{t=T_{1:s-1}+1}^{T_{1:s}} t^2} - 1 \right) = \left(\frac{\sum_{t=T_{1:s-1}+1}^{T_{1:s}} t^2}{\sum_{t=T_{1:s-1}+1}^{T_{1:s}} t^2} - 1 \right) = \left(\frac{\sum_{t=T_{1:s-1}+1}^{T_{1:s}+\tau} t^2}{\sum_{t=T_{1:s-1}+1}^{T_{1:s-1}+\tau} t^2} - 1 \right) \\ &= \left(\frac{\sum_{t=T_{1:s-1}+1}^{T_{1:s-1}+\tau} (t+\tau)^2}{\sum_{t=T_{1:s-1}+1}^{T_{1:s-1}+\tau} t^2} - 1 \right) = \left(\frac{\sum_{t=T_{1:s-1}+1}^{T_{1:s-1}+\tau} (t+\tau)^2 - t^2}{\sum_{t=T_{1:s-1}+1}^{T_{1:s-1}+\tau} t^2} \right) = \left(\frac{\sum_{t=T_{1:s-1}+1}^{T_{1:s-1}+\tau} \tau(2t+\tau)}{\sum_{t=T_{1:s-1}+1}^{T_{1:s-1}+\tau} t^2} \right) \\ &\leq \frac{3\tau}{T_{1:s-1}} = \frac{3\tau_s}{T_{1:s-1}}. \end{aligned}$$

Now let us consider the case when $s \leq s'$.

$$\Theta_s \leq \left(\frac{\tau_s}{\tau_{s+1}} \frac{\sum_{t=T_{1:s-1}+1}^{T_{1:s}} t^2}{\sum_{t=T_{1:s-1}+1}^{T_{1:s}} t^2} - 1 \right) = \left(\frac{1}{2} \frac{\sum_{t=T_{1:s-1}+1}^{T_{1:s}} t^2}{\sum_{t=T_{1:s-1}+1}^{T_{1:s}} t^2} - 1 \right) \leq 7.$$

Hence,

$$\begin{aligned} D_T &\leq \frac{1}{\alpha_{1:T}} \left[\frac{\eta_T}{2} \|x^*\|^2 + \frac{1}{2\eta_1} \|f'(x_1)\|^2 + p_1 \tilde{D}(1) + \sum_{s=1}^S \Theta_s \sum_{t=T_{1:s-1}+1}^{T_{1:s}} \frac{\alpha_t^2}{\gamma_t \tau_s} D_t \right] \\ &\leq \frac{1}{T^2} \left[\eta_T \|x^*\|^2 + \frac{1}{\eta_1} \|f'(x_1)\|^2 + 2p_1 \tilde{D}(1) + \sum_{s=1}^S \Theta_s \sum_{t=T_{1:s-1}+1}^{T_{1:s}} \frac{2t^2}{\gamma_t \tau_s} D_t \right] \end{aligned} \quad (38)$$

In the above equation, we can choose $\gamma_t = 28 \log 2t$ then we have:

$$D_T \leq \frac{1}{T^2} \left[\eta_T \|x^*\|^2 + \frac{1}{\hat{\eta}} \|f'(x_1)\|^2 + 2p_1 \tilde{D}(1) + \sum_{s=1}^S \Theta_s \sum_{t=T_{1:s-1}+1}^{T_{1:s}} \frac{2t^2}{28 \log 2t \tau_s} D_t \right] \quad (39)$$

Now, it is important to remember that $p_t \geq \frac{L\alpha_t^2}{\eta_t}$. Hence, $\eta_t = 28L\tau_{s(t)} \log 2t$. We now prove the following statement by induction:

$$D_T \leq \frac{\tau_{S(T)} \log 2T}{T^2} \left[56L \|x^*\|^2 + \frac{2}{\eta} \|f'(x_1)\|^2 + 4p_1 \tilde{D}(1) \right] = \frac{C\tau_{S(T)} \log 2T}{T^2}$$

where $C = 112L \|x^*\|^2 + \frac{2}{\eta_1} \|f'(x_1)\|^2 + 4p_1 \tilde{D}(1)$. For $t = 1$, the bound is satisfied trivially. Let us now check the bound for $t = 2$.

$$\begin{aligned} D_2 &\leq \frac{1}{2^2} \left[\eta_2 \|x^*\|^2 + \frac{1}{\eta_1} \|f'(x_1)\|^2 + 2p_1 \tilde{D}(1) + \Theta_1 \frac{2 \cdot 1^2}{28 \log 2 \tau_1} D_1 \right] \\ &= \frac{1}{2^2} \left[\eta_2 \|x^*\|^2 + \frac{1}{\eta_1} \|f'(x_1)\|^2 + 2p_1 \tilde{D}(1) + \frac{1^2}{2 \log 2 \tau_1} D_1 \right] \\ &\leq \frac{\log 4}{2^2} \left[56L \|x^*\|^2 + \frac{2}{\hat{\eta}} \|f'(x_1)\|^2 + 4p_1 \tilde{D}(1) \right] \end{aligned}$$

Now let us consider that for $T - 1$, the given statement is true for all iterations before that, then got the new iteration T we have the following:

$$\begin{aligned} D_T &\leq \frac{1}{T^2} \left[\eta_T \|x^*\|^2 + \frac{1}{\eta_1} \|f'(x_1)\|^2 + 2p_1 \tilde{D}(1) + \sum_{s=1}^S \Theta_s \sum_{t=T_{1:s-1}+1}^{T_{1:s}} \frac{2t^2}{28 \log t \tau_s} D_t \right] \\ &= \frac{1}{T^2} \left[\eta_T \|x^*\|^2 + \frac{1}{\eta_1} \|f'(x_1)\|^2 + 2p_1 \tilde{D}(1) + \sum_{s=1}^{s'} \Theta_s \sum_{t=T_{1:s-1}+1}^{T_{1:s}} \frac{2t^2}{28 \log t \tau_s} D_t \right. \\ &\quad \left. + \sum_{s=s'+1}^S \Theta_s \sum_{t=T_{1:s-1}+1}^{T_{1:s}} \frac{2t^2}{28 \log 2t \tau_s} D_t \right] \\ &\leq \frac{1}{T^2} \left[\eta_T \|x^*\|^2 + \frac{1}{\eta_1} \|f'(x_1)\|^2 + 2p_1 \tilde{D}(1) + \sum_{s=1}^{s'} \Theta_s \sum_{t=T_{1:s-1}+1}^{T_{1:s}} \frac{2t^2}{28 \log 2t \tau_s} D_t \right. \\ &\quad \left. + \sum_{s=s'+1}^S \Theta_s \sum_{t=T_{1:s-1}+1}^{T_{1:s}} \frac{2t^2}{28 \log 2t \tau_s} D_t \right] \\ &\leq \frac{1}{T^2} \left[\eta_T \|x^*\|^2 + \frac{1}{\eta_1} \|f'(x_1)\|^2 + 2p_1 \tilde{D}(1) + \frac{C(\log \tau_S + 1)}{2} + \sum_{s=s'+1}^S 3 \frac{\tau_s}{T_{1:s-1}} \frac{C}{14} \right] \\ &\leq \frac{1}{T^2} \left[\underbrace{\eta_T \|x^*\|^2 + \frac{1}{\eta_1} \|f'(x_1)\|^2 + 2p_1 \tilde{D}(1)}_{\leq \frac{C\tau_{S(T)} \log 2T}{2}} + \underbrace{\frac{C(\log \tau_S + 1)}{2} + \frac{C \log 2T}{2}}_{\leq \frac{C\tau_{S(T)} \log 2T}{2}} \right] \end{aligned}$$

$$\leq \frac{C\tau_{s(T)} \log 2T}{T^2} \leq \frac{C\tau \log 2T}{T^2}.$$

□

E. Regret bounds for online linear optimization

In this section, we provide the conditions and regret-bound $\mathcal{R}_T$ for vanilla AO-FTRL, which is used by Theorems 3 and 5, as well as the SVRG results. The analysis is based on Joulani et al. (2017; 2020).

Theorem 17. *Let $\phi : \mathcal{X} \rightarrow \mathbb{R}$ be a convex function. For $t \in [T]$, let $r_t : \mathcal{X} \rightarrow \mathbb{R}$, $\alpha_t > 0$, $g_t, \tilde{g}_t \in \mathbb{R}^d$, $x_t \in \mathcal{X}$ be such that the AO-FTRL update (4) is well-defined and x_t is given by (4) for all $t \in [T]$. Suppose that for all $t \in [T]$, r_{t-1} is convex and the objective in the AO-FTRL update (4) has finite value at the optimum x_t . Then, for any $x \in \mathcal{X}$,*

$$\begin{aligned} \sum_{t=1}^T \alpha_t (\langle g_t, x_t - x \rangle + \phi(x_t) - \phi(x)) &\leq \sum_{t=1}^T (r_{t-1}(x) - r_{t-1}(x_t) - B_{\alpha_{1:t}\phi + r_{0:t-1}}(x_{t+1}, x_t)) \\ &\quad + \sum_{t=1}^T \alpha_t \langle g_t - \tilde{g}_t, x_t - x_{t+1} \rangle. \end{aligned} \quad (40)$$

If, in addition, for all $t \in [T]$, $B_{\alpha_{1:t}\phi + r_{0:t-1}}(x_{t+1}, x_t) \geq \frac{1}{2} \|\cdot\|_{(t)}^2$ for some norm $\|\cdot\|_{(t)}$, then

$$\sum_{t=1}^T \alpha_t (\langle g_t, x_t - x \rangle + \phi(x_t) - \phi(x)) \leq \sum_{t=1}^T (r_{t-1}(x) - r_{t-1}(x_t)) + \sum_{t=1}^T \frac{\alpha_t^2}{2} \|g_t - \tilde{g}_t\|_{(t)*}^2. \quad (41)$$

Proof. The assumption that ϕ and r_t are real-valued and defined on $\mathcal{X}$ ensures that they are proper, which, together with convexity ensures that $\alpha_{1:t}\phi + r_{0:t-1}$ is directionally differentiable (Bauschke and Combettes, 2011, Prop. 17.2). Together with the assumption that the AO-FTRL objective in (4) is finite-valued, we guarantee Assumption 1 of Joulani et al. (2020), while their Assumption 5 is satisfied given that the combined linear-composite function $\alpha_t(\langle g_t, \cdot \rangle + \phi)$ is convex. The first bound (40) then follows from the intermediate bound C.1 in the proof of Theorem 6 of Joulani et al. (2020), using $x^* \leftarrow x$, $g_t \leftarrow \alpha_t g_t$, $\tilde{g}_{T+1} \leftarrow 0$, $p_t \leftarrow 0$, $t \in [T]$, $\tilde{q}_t \leftarrow r_t + \alpha_{t+1}\phi$, $t \in \{0\} \cup [T-1]$, and $\tilde{q}_T \leftarrow 0$. The second bound (41) follows from the statement of Theorem 6 of Joulani et al. (2020), using using $x^* \leftarrow x$, $p_t \leftarrow 0$, $t \in [T]$, and $\tilde{q}_t \leftarrow r_t + \alpha_{t+1}\phi$, $t \in \{0\} \cup [T-1]$, noting that Assumption 8 of Joulani et al. (2020) is the extra condition we have assumed in the second part of the theorem. □