Surface charge and surface current densities at material boundaries

Richard Marchand
University of Alberta

October 2, 2023

Abstract

In electromagnetism, materials with a polarization density $\vec{P}$ or a magnetization density $\vec{M}$ are known to exhibit a bound surface charge density $\sigma_b = \vec{P} \cdot \hat{n}$ or a surface current density $\vec{\kappa}_b = \vec{M} \times \hat{n}$ respectively, where $\hat{n}$ is the unit vector perpendicular to the material boundary surface, directed outward. These expressions can be obtained from volume integrations for the electric potential V, or the magnetic vector potential $\vec{A}$, in which the integrals are restricted to the material volumes delimited by their respective boundaries. In that case, applying the divergence theorem leads to surface integrals on material boundaries, and to the above mentioned surface quantities. In this paper a simple derivation is presented, showing that both σ_b and $\vec{\kappa}_b$ are included in the expressions for the volume charge or current densities, provided that the divergence and curl operators are evaluated at the boundary so as to account for discontinuities at interfaces.

1 Introduction

A general derivation of boundary conditions for electromagnetic fields and associated bound surface charge and current densities has been presented by Namias (1988)[1] for materials in motion. Yet, the physics and mathematics leading to these boundary conditions is generally given short shrift in introductory E&M textbooks. In this article, a simple derivation is presented for bound surface charge and current densities at a material boundary, using mathematical concepts which should be familiar to intermediate undergraduate physics students. Referring to standard textbooks on electromagnetism, the expression for the quasi-static electric potential from material objects with electric polarization vector $\vec{P}(\vec{r}\,')$ is

$$\begin{aligned} V(\vec{r}) &= \frac{1}{4\pi\epsilon_0}\int d^3r'\vec{P}(\vec{r}\,')\cdot\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} = \frac{1}{4\pi\epsilon_0}\int d^3r'\vec{P}(\vec{r}\,')\cdot\nabla'\frac{1}{|\vec{r}-\vec{r}\,'|} \\ &= \frac{1}{4\pi\epsilon_0}\int d^3r'\nabla'\cdot\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\vec{P}(\vec{r}\,')\right) - \frac{1}{4\pi\epsilon_0}\int d^3r'\frac{\nabla'\cdot\vec{P}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \\ &= \frac{1}{4\pi\epsilon_0}\oiint da'\frac{\vec{P}(\vec{r}\,')\cdot\hat{n}}{|\vec{r}-\vec{r}\,'|} + \frac{1}{4\pi\epsilon_0}\int d^3r'\frac{-\nabla'\cdot\vec{P}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}, \end{aligned} \tag{1}$$

where $\nabla'\cdot$ is the divergence operator in the $\vec{r}\,'$ coordinates. In the top line of this equation, use is made of the expression for the ratio $(\vec{r}-\vec{r}\,')/|\vec{r}-\vec{r}\,'|^3$ in terms of the gradient $\nabla'|\vec{r}-\vec{r}\,'|^{-1}$. The

middle line uses the product rule for the divergence of a vector field times a scalar function. Finally in the bottom line, the divergence theorem is used to write the first integral in the previous line as a surface integral. The second term is readily recognized as the expression for the electric potential associated with a volume charge density $\rho_b = -\nabla' \cdot \vec{P}(\vec{r}')$. Similarly the numerator in the first term is identified as the bound surface charge density σ_b, under the (often tacit) assumption that the volume integrations in the top two lines are restricted to volumes containing materials, delimited by their respective surfaces, thus excluding space between materials and beyond. The procedure is similar for the bound current density where the magnetic vector potential resulting from materials with magnetization vector $\vec{M}$ is given by (Griffiths[2] pp. 274-275)

$$\begin{aligned}
\vec{A}(\vec{r}) &= \frac{\mu_0}{4\pi}\int d^3r' \frac{\vec{M}\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} \\
&= \frac{\mu_0}{4\pi}\int d^3r' \vec{M}\times\nabla'\frac{1}{|\vec{r}-\vec{r}'|} = -\frac{\mu_0}{4\pi}\int d^3r'\nabla'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)\times\vec{M} \\
&= -\frac{\mu_0}{4\pi}\int d^3r'\left[\nabla'\times\left(\frac{\vec{M}}{|\vec{r}-\vec{r}'|}\right) - \frac{\nabla'\times\vec{M}}{|\vec{r}-\vec{r}'|}\right] \\
&= \frac{\mu_0}{4\pi}\oiint da' \frac{\vec{M}\times\hat{n}}{|\vec{r}-\vec{r}'|} + \frac{\mu_0}{4\pi}\int d^3r'\frac{\nabla'\times\vec{M}}{|\vec{r}-\vec{r}'|},
\end{aligned} \tag{2}$$

from which we recognize the bound surface and volume current densities $\vec{\kappa}_b = \vec{M}\times\hat{n}$ and $\nabla'\times\vec{M}$ respectively. This procedure leads to the correct expressions for the bound charge and current densities, within material volumes and at their surfaces, but it also raises questions:

1. *Why restrict the integration to volumes delimited by material boundaries? Isn't the divergence theorem applicable to any volume, including all space extending to infinity, where surface integrals would vanish?*

2. *Where exactly "at" the material boundary is the surface integral calculated? Should it be infinitesimally close to the boundary from the inside, from the outside, or somehow, in the "middle" of the discontinuous drop from material to vacuum?*

3. *In view of question* 2*, are we missing any contribution to or from the surface charge and current densities? In particular, in the definitions of the displacement vector and the auxiliary magnetic fields, resulting respectively from*

$$\nabla\cdot\vec{E} = \frac{1}{\epsilon_0}\left(\rho_{free} - \nabla\cdot\vec{P}\right), \tag{3}$$

and

$$\nabla\times\vec{B} = \mu_0\left(\vec{J}_{free} + \nabla\times\vec{M}\right), \tag{4}$$

where ρ_{free} and $\vec{J}_{free}$ are the free charge density, and the free current density respectively, why is there no mention of the bound surface charge density in 3 and none of the bound surface current density in 4?

These questions don't arise, for example, when calculating the electric potential due to a volume charge density ρ, for which the result is clearly the same whether the volume integration

is limited to the space occupied by material objects, or it extends to all space, between and beyond these materials. The answer is not so straightforward when integrating derivatives of volume densities, and it is instructive to consider how the integrals are carried out *at* or *across* the boundaries. In the following we show that $-\nabla \cdot \vec{P}$ and $\nabla \times \vec{M}$ give both the volume and surface density of bound electric charges and currents, provided that they are evaluated so as to account for possible discontinuities at material boundaries. This in turn provides insight in the use of surface integrals, and in particular it answers questions 2 and 3 above.

2 Surface charge and current densities in $-\nabla \cdot \vec{P}$ and $\nabla \times \vec{M}$

Our first task here is to represent a discontinuity in the electric polarization and magnetization vectors so that we can formally derive an expression for their derivatives at a boundary. In order to do this, we need to use some basic properties of the Dirac delta function $\delta(x)$ and the Heaviside function $H(x)$. We recall the definitions

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}, \tag{5}$$

and assuming $a < b$,

$$\int_a^b \delta(x - x_0) f(x) dx = \begin{cases} 0 & \text{if } \quad x_0 < a \text{ or } b < x_0 \\ f(x_0) & \text{if } \quad a < x_0 < b \end{cases} \tag{6}$$

for an arbitrary continuous function $f(x)$. From this, it follows that

$$\int_{-\infty}^{x} \delta(x - x_0) dx = H(x - x_0), \tag{7}$$

and formally, using the chain rule for the derivative of H with respect to x,

$$\frac{d}{dx} H(x - x_0) = H'(x - x_0) = \delta(x - x_0), \tag{8}$$

where the prime on H stands for the derivative of H with respect to its argument. Of course, mathematically the derivative of a Heaviside function is not defined at the discontinuity, but it is if we consider $\delta(x)$ as an approximation to a strongly localized distribution, such as the normal distribution

$$f_N(x - x_0) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x - x_0)^2}{2\sigma^2} \right), \tag{9}$$

in the limit $\sigma \to 0$ or for a very small σ, and the Heaviside function as the integral of this distribution, as the standard deviation σ approaches zero. Thus, in what follows, we use derivatives of the Heaviside function with the understanding that in practice, H can be thought of as an approximation of a differentiable function albeit with a very steep transition from zero to unity. We are now ready to evaluate $-\nabla \cdot \vec{P}$ and $\nabla \times \vec{M}$ at a material boundary. For simplicity and without loss of generality we assume that a material ends abruptly at its boundary outside of which there is vacuum where $\vec{P}$ and $\vec{M}$ are zero. That is, the boundary is between a material and vacuum, and not between two materials with different physical properties. The

formalism presented below could readily be extended to boundaries separating two different materials by following the same approach, or simply by using the linear superposition principle. Macroscopically, the boundaries are assumed to be smooth so that, for a sufficiently small (but still macroscopic) piece, it is approximately flat. In that case, the transition from "inside" to "outside" can be represented using the Heaviside function:

$$\vec{P} = H\left((\vec{s}-\vec{r})\cdot\hat{n}\right)\vec{P} \tag{10}$$

for the polarization vector, and similarly

$$\vec{M} = H\left((\vec{s}-\vec{r})\cdot\hat{n}\right)\vec{M} \tag{11}$$

for the magnetization vector where, as illustrated in Fig. 1, $\hat{n}$ is a unit vector perpendicular to the surface, pointing outward, and $\vec{s}$ is an arbitrary point on the surface near the location where derivatives are evaluated. Note that, since we are considering a region "very close to

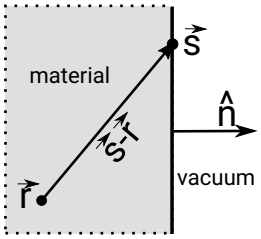

Figure 1: Illustration of a material-vacuum interface

the boundary", vector fields $\vec{P}$ and $\vec{M}$ could be replaced with constant values, say, $\vec{P}_0$ and $\vec{M}_0$, and the only dependence on position would be in the H function used to represent the discontinuity. A possible $\vec{r}$ dependence in $\vec{P}$ and $\vec{M}$ is nonetheless retained in Eqs. 10 and 11, as it captures both bound volume charge and current densities. Using Eq. 10, the product rule, and the chain rule, it follows that

$$\begin{aligned}
-\nabla\cdot\vec{P} &= H\left(-\nabla\cdot\vec{P}\right) - \nabla H\cdot\vec{P} \\
&= H\left(-\nabla\cdot\vec{P}\right) - H'\vec{P}\cdot\nabla\left((\vec{s}-\vec{r})\cdot\hat{n}\right) \\
&= H\left(-\nabla\cdot\vec{P}\right) + \delta\left((\vec{s}-\vec{r})\cdot\hat{n}\right)\vec{P}\cdot\hat{n}.
\end{aligned} \tag{12}$$

The first term in Eq. 12 is the usual bound volume charge density with H representing the discontinuity at the boundary, and the second term is the (singular / strongly localized) surface charge density. The bound current density from $\nabla\times\vec{M}$ is obtained in the same way. Using Eq. 11 for $\vec{M}$ near the boundary we find

$$\begin{aligned}
\nabla\times\vec{M} &= H\left(\nabla\times\vec{M}\right) + \nabla H\times\vec{M} \\
&= H\left(\nabla\times\vec{M}\right) + H'\nabla\left((\vec{s}-\vec{r})\cdot\hat{n}\right)\times\vec{M} \\
&= H\left(\nabla\times\vec{M}\right) - \delta\left((\vec{s}-\vec{r})\cdot\hat{n}\right)\hat{n}\times\vec{M} \\
&= H\left(\nabla\times\vec{M}\right) + \delta\left((\vec{s}-\vec{r})\cdot\hat{n}\right)\vec{M}\times\hat{n}.
\end{aligned} \tag{13}$$

The two terms in the right side of the last equation are the bound volume and surface current densities respectively. From Eqs. 12 and 13, we see that the bound volume and surface charge densities are both contained in $-\nabla \cdot \vec{P}$, and likewise, the bound volume and surface current densities are in $\nabla \times \vec{M}$.

3 Summary and conclusion

The bound surface charge and current densities are shown to be contained respectively in $-\nabla \cdot \vec{P}$ and $\nabla \times \vec{M}$, generally presented as exclusively volume charge and current densities in introductory textbooks, while bound surface and current densities are written separately as different expressions. These expressions are usually obtained by applying the divergence theorem in the integrals for the electric potential and the magnetic vector potential, and limiting volume integrals to the space occupied by material objects, excluding regions between them and beyond. The same expressions for bound surface charge and current densities are obtained more generally by integrating over all space, without restriction. In this case the surface integral resulting from the divergence theorem vanishes, and the correct expressions for the bound surface charge and current densities are obtained from the divergence and curl operators provided that they are evaluated so as to account for the discontinuity at material boundaries. While this analysis reproduces known results, it provides additional insight as it shows that $-\nabla \cdot \vec{P}$ and $\nabla \times \vec{M}$, also account for surface densities for bound charges and currents in the constitutive equations 3 and 4. Coming back to questions 2 and 3 above, we find that in the standard procedure with volume integrations restricted material volumes, integration surfaces must be formally assumed to be infinitesimally close to the boundaries *inside* materials, and that no physics is omitted by doing so. In the presentation above a magnetization vector $\vec{M}$ is assumed without mention of the contributions at microscopic scales. Interestingly, several contributions to $\vec{M}$ have been considered (Garg[3] p. 295), and shown to lead to the same expressions for the bound volume and surface current densities as presented above. Finally, while for solid materials with macroscopically sharp boundaries, surface charge and current densities are readily obtained from the standard procedure, with materials such as fluids with initially sharp but progressively diffuse boundaries, using the expressions for the charge and current densities presented here should be selected as a preferred approach.

References

[1] V. Namias, "Discontinuity of the electromagnetic fields, potentials, and currents at fixed and moving boundaries," *American Journal of Physics*, vol. 56, no. 10, pp. 898–904, 1988.

[2] D. J. Griffiths, *Introduction to Electrodynamics*. Pearson, 2013.

[3] A. Garg, *Classical electromagnetism in a nutshell*, vol. 13. Princeton University Press, 2012.