ADVANCES IN APPLIED MATHEMATICS **17,** 308–336 (1996)
ARTICLE NO. 0013

Perturbation of Polynomial Ideals

Rong-Qing Jia*

Department of Mathematics, University of Alberta, Edmonton, Canada T6G 2G1

Received March 3, 1994; accepted July 3, 1994

1. INTRODUCTION

In this paper we explore the interrelationship between the theory of polynomial ideals and certain branches of analysis including multivariate approximation theory and linear partial differential equations. Specifically, we use the perturbation technique from analysis to study the codimension of a multivariate polynomial ideal.

Let K be a field. In this paper, K will often be the field $\mathbb{C}$ of complex numbers. We denote by $K[Z_1, \ldots, Z_s]$ (resp. $K[[Z_1, \ldots, Z_s]]$) the **ring of polynomials** (resp. the **ring of formal power series**) in s indeterminates over K. Let I be an ideal of $K[Z_1, \ldots, Z_s]$. The **codimension** of I is the dimension of the quotient space $K[Z_1, \ldots, Z_s]/I$ over K. If this dimension is finite, then I is said to be of finite codimension.

The perturbation technique for polynomial ideals was used in the study of polynomial mappings on $\mathbb{C}^s$. Suppose $p_1, \ldots, p_s$ are homogeneous polynomials in $\mathbb{C}[Z_1, \ldots, Z_s]$ such that the ideal I generated by them is of finite codimension. In this case, the origin is the only common zero of $p_1, \ldots, p_s$. Let F be the mapping from $\mathbb{C}^s$ to $\mathbb{C}^s$ given by

$$F(z) = \big(p_1(z), \ldots, p_s(z)\big), \qquad z = (z_1, \ldots, z_s) \in \mathbb{C}^s.$$

The **algebraic multiplicity** of the mapping F is defined to be the codimension of I. Given $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_s) \in \mathbb{C}^s$, we denote by $\mu(\varepsilon)$ the number of common zeros of the polynomials $p_1 - \varepsilon_1, \ldots, p_s - \varepsilon_s$. The **geometric multiplicity** of F is defined to be $\sup\{\mu(\varepsilon): \varepsilon \in \mathbb{C}^s\}$. We may say that the ideal generated by $p_1 - \varepsilon_1, \ldots, p_s - \varepsilon_s$ is a **perturbation** of the ideal generated by $p_1, \ldots, p_s$. A classical result states that the algebraic multi-

* Supported in part by NSERC Canada under Grant OGP 121336.

0196-8858/96 $18.00

plicity of a polynomial mapping on $\mathbb{C}^s$ agrees with its geometric multiplicity. Arnold, Gusein-Zade, and Varchenko in [2, p. 85] indicated that the first detailed proof of this deep result was published in Palamodov's paper [23].

Now let us consider the general case. Given two polynomials p and q in $K[Z_1, \ldots, Z_s]$, we say that q is a **lower-order perturbation** of p if $q - p$ is a polynomial of degree less than $\deg p$. Suppose I is an ideal in $K[Z_1, \ldots, Z_s]$ generated by polynomials $p_1, \ldots, p_m$. We say that an ideal J is a **lower-order perturbation** of I if J is generated by $q_1, \ldots, q_m$, where each q_j is a lower-order perturbation of p_j, $j = 1, \ldots, m$ (see [5]). We are concerned with the relationship between I and J. In particular, we are interested in the relationship between $\operatorname{codim}(I)$ and $\operatorname{codim}(J)$. Here is the idea behind the perturbation technique: The codimension of I might be difficult to compute, but the computation of $\operatorname{codim}(J)$ is easier. Thus one can gain some information about $\operatorname{codim}(I)$ through computing the codimension of the perturbed ideal J. The usefulness of the perturbation technique was demonstrated in the previous paragraph.

How is the theory of polynomial ideals related to approximation theory and linear partial differential equations? To answer this question, we need to consider differentiation on the ring of formal power series. Let $\mathbb{N}$ be the set of positive integers and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. An element $\alpha \in \mathbb{N}_0^s$ is called a **multi-index**, or more precisely, an s-index. If $\alpha = (\alpha_1, \ldots, \alpha_s)$ is a multi-index, then its length is $|\alpha| := \alpha_1 + \cdots + \alpha_s$, and its factorial $\alpha! := \alpha_1! \cdots \alpha_s!$. If $\beta = (\beta_1, \ldots, \beta_s)$ is another multi-index, then $\alpha \le \beta$ means $\alpha_j \le \beta_j$ for all $j = 1, \ldots, s$. Let K be an algebraically closed field of characteristic 0. A formal power series in $K[[Z_1, \ldots, Z_s]]$ is of the form $\sum_{\beta \in \mathbb{N}_0^s} b_\beta Z^\beta$, where $Z^\beta := Z_1^{\beta_1} \cdots Z_s^{\beta_s}$ and $b_\beta \in K$ for all $\beta \in \mathbb{N}_0^s$. Given $\alpha \in \mathbb{N}_0^s$, the differential operator $D^\alpha = D_1^{\alpha_1} \cdots D_s^{\alpha_s}$ on $K[[Z_1, \ldots, Z_s]]$ is defined by

$$D^\alpha \left(\sum_{\beta \in \mathbb{N}_0^s} b_\beta Z^\beta \right) = \sum_{\beta \ge \alpha} b_\beta \frac{\beta!}{(\beta - \alpha)!} Z^{\beta - \alpha}.$$

For a polynomial $p = \sum_\alpha a_\alpha Z^\alpha$, the corresponding differential operator $p(D)$ is defined by $p(D) := \sum_\alpha a_\alpha D^\alpha$. Now let I be an ideal in $K[Z_1, \ldots, Z_s]$. The **kernel** of I is the set

$$I_\perp := \{ f \in K[[Z_1, \ldots, Z_s]] : p(D) f = 0 \text{ for all } p \in I \}$$

(see [5] for the use of the notation $I_\perp$). By the Hilbert basis theorem, any ideal in $K[Z_1, \ldots, Z_s]$ is finitely generated. Suppose I is generated by $p_1, \ldots, p_m \in K[Z_1, \ldots, Z_s]$. Then $f \in I_\perp$ if and only if f satisfies the following system of linear partial differential equations with constant coefficients:

$$p_j(D)f = 0, \qquad j = 1, \dots, m.$$

Obviously $I_\perp$ is a linear subspace of $K[[Z_1, \dots, Z_s]]$ considered as a linear space over K. When K is the complex field $\mathbb{C}$, it was proved by de Boor and Ron in [5] that if I is of finite codimension, then $I_\perp$ is finite dimensional, and

$$\operatorname{codim}(I) = \dim(I_\perp).$$

This result is also valid for any algebraically closed field of characteristic 0 (see Section 2).

Certain systems of linear partial differential and difference equations arise naturally from the study of multivariate approximation. In the following we give a brief description of this phenomenon. Let X be a nonempty finite set. Associate to each $x \in X$ a polynomial $p_x \in \mathbb{C}[Z_1, \dots, Z_s]$. For a subset A of X, define

$$p_A := \prod_{x \in A} p_x. \tag{1.1}$$

Let $\mathscr{B} = \mathscr{B}(X)$ be a collection of subsets of X, each of which has exactly s elements. Let

$$\mathscr{A} = \mathscr{A}(X, \mathscr{B}(X)) := \{A \subseteq X : A \cap B \neq \varnothing \text{ for any } B \in \mathscr{B}(X)\}. \tag{1.2}$$

Consider the ideal $I(X) = I(X, \mathscr{B}(X))$ generated by the polynomials p_A, $A \in \mathscr{A}$. We are interested in the kernel space $I(X)_\perp$. The significance of the kernel space $I(X)_\perp$ lies in the fact that it often determines the approximation power of the integer translates of the corresponding box spline (see [4, 13, 21]).

For a subset Y of X, let

$$\mathscr{B}(Y) := \{B \subseteq Y : B \in \mathscr{B}(X)\}. \tag{1.3}$$

The ideal $I(Y) = I(Y, \mathscr{B}(Y))$ is defined correspondingly. In particular, for each $B \in \mathscr{B}(X)$, $I(B)$ is the ideal generated by p_x, $x \in B$. Dahmen and Micchelli in [11] investigated the relation between the dimension of $I(X)_\perp$ and the dimensions of the "block spaces" $I(B)_\perp$. Their work was motivated by some nontrivial examples in box spline theory (see [4, 9, 10]).

Among other things, Dahmen and Micchelli in [11, Theorem 3.1] proved that the inequality

$$\dim(I(X)_{\perp}) \leq \sum_{B \in \mathscr{B}(X)} \dim(I(B)_{\perp}) \tag{1.4}$$

holds if $(X, \mathscr{B}(X))$ has a matroid structure. They also gave a sufficient condition for equality. In another paper [12], Dahmen and Micchelli conjectured that equality would hold in (1.4), provided each p_x $(x \in X)$ is a homogeneous polynomial. Their conjecture was solved by Shen in [25]. Shen's work was further extended by Jia, Riemenschneider, and Shen in [20]. In particular, they proved the equality

$$\dim(I(X)_{\perp}) = \sum_{B \in \mathscr{B}(X)} \dim(I(B)_{\perp})$$

under the condition that $\mathscr{B}(X)$ is order-closed and each p_x $(x \in X)$ is a polynomial in $K[Z_1, \ldots, Z_s]$ (not necessarily homogeneous).

Along other lines, Ben-Artzi and Ron in [3], and Dyn and Ron in [13] used the idea of perturbation in their study of exponential box splines. Such an idea was refined in the work [5] of de Boor and Ron, who considered the case where every p_x $(x \in X)$ is a polynomial of degree 1. On the other hand, the collection $\mathscr{B}(X)$ could be arbitrary (with no matroid structure). In contrast to (1.4), de Boor and Ron in [5, Theorem 6.6] gave the lower bound

$$\operatorname{codim}(I(X)) \geq \sum_{B \in \mathscr{B}(X)} \operatorname{codim}(I(B)), \tag{1.5}$$

that is,

$$\dim(I(X)_{\perp}) \geq \sum_{B \in \mathscr{B}(X)} \dim(I(B)_{\perp}).$$

They also proved that equality holds for this special case if $\mathscr{B}(X)$ is order-closed.

In this paper our main goal is to establish the lower bound in (1.5) for the general case where each p_x $(x \in X)$ is an *arbitrary* polynomial. In order to achieve this goal, we discuss polynomial ideals of finite codimension in Section 2 and study the perturbation technique in connection with topological degree theory in Section 3. Section 4 is devoted to certain properties of polynomial mappings and algebraic functions. The results in Sections 2–4 are then used to establish the lower bound for the codimension of $I(X)$. Finally, in Section 6, we investigate conditions under which equality holds in (1.5).

The major part of this work was done in 1989 and was reported in the conference "Algebraic and Combinatorial Problems in Multivariate Approximation Theory" organized by W. Dahmen and A. Dress in the Oberwolfach Institute of Mathematics, Germany, October 21–27, 1990. In preparing this paper, I was inspired by the work of C. de Boor and A. Ron, and by that of W. Dahmen and C. A. Micchelli. I take this opportunity to thank all of them.

2. POLYNOMIAL IDEALS OF FINITE CODIMENSION

This section is devoted to some elementary facts concerning polynomial ideals of finite codimension. When the coefficient field is the complex field $\mathbb{C}$, the results in this section were proved in [5] by de Boor and Ron, who in turn attributed these results to Gröbner (see [16, Chap. IV]). Throughout this section, except Theorem 2.4, the coefficient field K is assumed to be an algebraically closed field of characteristic 0. In contrast to the methods used in [5], the proofs given here do not require primary decomposition of ideals.

THEOREM 2.1. *Let K be an algebraically closed field of characteristic zero. If I is an ideal of $K[Z_1, \dots, Z_s]$ with finite codimension, then*

$$\operatorname{codim}(I) = \dim(I_\perp).$$

Before providing a new proof for this theorem, we give two examples to illustrate its significance.

EXAMPLE 2.2. Let $s = 1$ and let I be the ideal in $K[Z]$ generated by one polynomial p of degree $m \geq 1$. Then $\operatorname{codim}(I) = m$, because $\{\overline{Z^j}: j = 0, \dots, m-1\}$ forms a basis for $K[Z]/I$, where $\overline{Z^j}$ denotes the residue class of the monomial Z^j in $K[Z]/I$. On the other hand, from the elementary theory of differential equations we see that the number of solutions $f \in K[[Z]]$ to the equation $p(D)f = 0$ equals m. In other words, $\dim(I_\perp) = m$.

EXAMPLE 2.3. Let $s > 1$. Suppose p_j $(j = 1, \dots, s)$ are polynomials in Z_j of degree m_j. Consider the ideal I generated by $p_1, \dots, p_s$. Then $\operatorname{codim}(I) = m_1 \cdots m_s$, because

$$\left\{\overline{Z^\alpha}: \alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{N}_0^s,\ \alpha_j < m_j \text{ for all } j\right\}$$

forms a basis for $K[Z_1, \dots, Z_s]/I$, where $\overline{Z^\alpha}$ denotes the residue class of the monomial Z^α in $K[Z_1, \dots, Z_s]/I$. On the other hand, $f \in I_\perp$ if and only if f is a linear combination of formal power series of the form $f_1 \cdots f_s$, where each $f_j \in K[[Z_j]]$ is a solution to the equation $p_j(D_j)f_j = 0$, $j = 1, \ldots, s$. The linear span of all such formal power series has dimension $m_1 \cdots m_s$. This shows that $\dim(I_\perp) = m_1 \cdots m_s = \operatorname{codim}(I)$.

Proof of Theorem 2.1. Let I and J be two ideals in $K[Z_1, \ldots, Z_s]$ such that $J \subseteq I$. It follows that $I_\perp \subseteq J_\perp$. Viewing I/J and $J_\perp/I_\perp$ as quotient linear spaces over K, we claim that

$$\dim(J_\perp/I_\perp) \le \dim(I/J). \tag{2.1}$$

If $\dim(I/J) = \infty$, then (2.1) holds automatically; hence we assume that $\dim(I/J) < \infty$ in what follows. Associate to each $f \in J_\perp$ a linear functional f^* on I/J given by

$$f^*(\bar{p}) := p(D)f(0), \qquad p \in I,$$

where $\bar{p}$ denotes the residue class of p in I/J. Since $p(D)f(0) = 0$ for $p \in J$ and $f \in J_\perp$, f^* is well defined. Let $\bar{f}$ denote the residue class of $f \in J_\perp$ in $J_\perp/I_\perp$. If $\overline{f_1} = \overline{f_2}$, then $f_1 - f_2 \in I_\perp$; hence $p(D)(f_1 - f_2)(0) = 0$ for $p \in I$. This shows that $\bar{f} \mapsto f^*$ is a linear mapping from $J_\perp/I_\perp$ to the linear dual of I/J. Thus, the inequality (2.1) will be established if we can show that the mapping $\bar{f} \mapsto f^*$ is injective, i.e., $f^* = 0 \Rightarrow \bar{f} = 0$. To prove this we let $f \in J_\perp$ be such that $f^* = 0$. Then $p(D)f(0) = 0$ for all $p \in I$. Since I is an ideal, for a given $p \in I$ the polynomial $Z^\alpha p$ is also in I for all $\alpha \in \mathbb{N}_0^s$. Consequently, $D^\alpha p(D)f(0) = 0$ for all $\alpha \in \mathbb{N}_0^s$. It follows that $p(D)f = 0$ for all $p \in I$. Therefore $f \in I_\perp$, i.e., $\bar{f} = 0$.

Having established (2.1), we choose $I = K[Z_1, \ldots, Z_s]$ in it. Then (2.1) becomes $\dim(J_\perp) \le \operatorname{codim}(J)$. Replacing J by I, we get

$$\dim(I_\perp) \le \operatorname{codim}(I). \tag{2.2}$$

Moreover, if $\operatorname{codim}(J) < \infty$, the inequality in (2.1) can be written as

$$\dim(J_\perp) - \dim(I_\perp) \le \operatorname{codim}(J) - \operatorname{codim}(I). \tag{2.3}$$

From (2.2) and (2.3) we see that in order to prove Theorem 2.1, it suffices to find an ideal $J \subseteq I$ with finite codimension such that $\operatorname{codim}(J) = \dim(J_\perp)$.

An ideal $J \subseteq I$ with the desired property can be constructed by employing a technique used in [12, Proposition 2.1; 8, Theorem 2.1]. For $\theta = (\theta_1, \ldots, \theta_s) \in K^s$, let e_θ be the formal power series given by

$$\sum_{\alpha \in \mathbb{N}_0^s} \theta^\alpha Z^\alpha / \alpha!,$$

where $\theta^\alpha := \theta_1^{\alpha_1} \cdots \theta_s^{\alpha_s}$ for $\alpha = (\alpha_1, \dots, \alpha_s)$. When $K = \mathbb{C}$, e_θ is the usual exponential function $z \mapsto e^{\theta \cdot z}$, $z = (z_1, \dots, z_s) \in \mathbb{C}^s$, where $\theta \cdot z := \theta_1 z_1 + \cdots + \theta_s z_s$. Let $\mathscr{V}(I)$ denote the **algebraic variety** of the ideal I. Precisely,

$$\mathscr{V}(I) := \{\theta \in K^s : p(\theta) = 0 \text{ for all } p \in I\}.$$

A point in $\mathscr{V}(I)$ is also called a **zero** of I. We observe that $p(D)e_\theta = p(\theta)e_\theta$; hence $\theta \in \mathscr{V}(I)$ if and only if $e_\theta \in I_\perp$. If $\dim(I_\perp) = \infty$, then Theorem 2.1 follows from (2.2) at once. Consider the case $\dim(I_\perp) < \infty$. Since the set $\{e_\theta : \theta \in \mathscr{V}(I)\}$ is linearly independent, the number of points in $\mathscr{V}(I)$ is at most $\dim(I_\perp)$. Let

$$f_j := \prod_{\theta \in \mathscr{V}(I)} (Z_j - \theta_j), \qquad j = 1, \dots, s, \tag{2.4}$$

where θ_j stands for the jth coordinate of θ, $j = 1, \dots, s$. Every f_j is a polynomial vanishing on $\mathscr{V}(I)$. By Hilbert's nullstellensatz, there exists an integer $n > 0$ such that $f_j^n \in I$ for all $j = 1, \dots, s$. Let J be the ideal generated by $f_1^n, \dots, f_s^n$. Then $J \subseteq I$. Moreover, it was shown in Example 2.3 that $\dim(J_\perp) = \operatorname{codim}(J)$. This finishes the proof. ∎

Next, let us consider the multiplicity of an ideal I in $K[Z_1, \dots, Z_s]$ at a point $\theta \in \mathscr{V}(I)$. Here the algebraically closed field K needs not to be of characteristic 0. The set

$$S_\theta := \{g \in K[Z_1, \dots, Z_s] : g(\theta) \neq 0\}$$

is a multiplicative set of $R := K[Z_1, \dots, Z_s]$. Let $\mathscr{O}_\theta := S_\theta^{-1} R$ be the quotient ring of R by S_θ (the **localization** of R at S_θ); i.e.,

$$\mathscr{O}_\theta = \{f/g : f, g \in K[Z_1, \dots, Z_s] \text{ and } g(\theta) \neq 0\}.$$

Thus, $\mathscr{O}_\theta$ is the **local ring** of the point θ on the algebraic variety $\mathscr{V}(I)$. Suppose that θ is an isolated zero of I, i.e., $\{\theta\}$ is one of the irreducible components of $\mathscr{V}(I)$. The (algebraic) **multiplicity** of I at θ, denoted $\mu_\theta(I)$, is defined to be the dimension of the quotient space $\mathscr{O}_\theta / (S_\theta^{-1} I)$ over K. A proof of the following theorem can be found on page 57 of Fulton's book [15].

THEOREM 2.4. *Let I be an ideal of $K[Z_1, \dots, Z_s]$ with finite codimension. Then*

$$\operatorname{codim}(I) = \sum_{\theta \in \mathscr{V}(I)} \mu_\theta(I).$$

When the coefficient field $K = \mathbb{C}$, de Boor and Ron in [5] used another definition of multiplicity. For an isolated zero θ of I, the **multiplicity space** of I at θ, denoted $M_{I,\theta}$, is defined by the rule

$$M_{I,\theta} := \{p \in K[Z_1, \dots, Z_s]: p(D)q(\theta) = 0 \text{ for all } q \in I\}. \quad (2.5)$$

The space $M_{I,\theta}$ is D-invariant, i.e., closed under differentiation (see [5]). The dimension of $M_{I,\theta}$ is called the multiplicity of I at θ. It was proved in [20, Theorem 3.2] that

$$\mu_\theta(I) = \dim(M_{I,\theta}). \quad (2.6)$$

Thus, the two notions of multiplicity agree with each other. This assertion is also true if K is an algebraically closed field of characteristic 0.

Now let us investigate the structure of $I_\perp$. By the Leibniz differentiation formula, one can easily prove that for two polynomials p and q in $K[Z_1, \dots, Z_s]$,

$$p(D)(e_\theta q)(0) = q(D)p(\theta), \qquad \theta \in K^s.$$

From this formula we see that $e_\theta q \in I_\perp$ if and only if $q \in M_{I,\theta}$. This shows that

$$\sum_{\theta \in \mathscr{V}(I)} e_\theta M_{I,\theta} \subseteq I_\perp .$$

Since the sum on the left-hand side of the above inclusion relation is a direct one, its dimension is $\sum_{\theta \in \mathscr{V}(I)} \mu_\theta(I)$, which is $\dim(I_\perp)$ by Theorems 2.1 and 2.4. Thus we arrive at the following conclusion (see [5, Corollary 3.21] for the case $K = \mathbb{C}$).

THEOREM 2.5. *Let I be an ideal of $K[Z_1, \dots, Z_s]$ with finite codimension, where K is an algebraically closed field of characteristic* 0. *Then*

$$I_\perp = \bigoplus_{\theta \in \mathscr{V}(I)} e_\theta M_{I,\theta}.$$

3. PERTURBATION OF POLYNOMIAL IDEALS

From now on the coefficient field K is taken to be the complex field $\mathbb{C}$. Let I be an ideal of $\mathbb{C}[Z_1, \dots, Z_s]$ with finite codimension. Then the algebraic variety $\mathscr{V}(I)$ is a finite set. A zero θ of I is called **simple**, if the multiplicity of I at θ is 1. The ideal I is said to be **simple** if all its zeros are **simple**. A lower-order perturbation J of I is said to be **perfect**, if J is simple and $\operatorname{codim}(J) = \operatorname{codim}(I)$. In this section we investigate the possibility of perfect perturbation of polynomial ideals.

Let us first recall some useful facts about (topological) degree theory from Lloyd's book [22]. Let $\phi = (\phi_1, \ldots, \phi_n)$ be a continuously differentiable mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$. Given $a \in \mathbb{R}^n$, the Jacobian determinant of ϕ at a is denoted by

$$J_\phi(a) := \det\big(D_j \phi_k(a)\big)_{1 \le j, k \le n},$$

where D_j denotes the partial derivative operator with respect to the jth coordinate, $j = 1, \ldots, n$. We say that a is a **critical point** of ϕ if $J_\phi(a) = 0$. A point $b \in \mathbb{R}^n$ is called a **regular value** of ϕ, if $J_\phi(a) \neq 0$ for any $a \in \phi^{-1}(b)$; otherwise, b is called a **critical value** of ϕ. Let Ω be a bounded open subset of $\mathbb{R}^n$. We denote by $\partial\Omega$ and $\overline{\Omega}$ the boundary and the closure of Ω, respectively. Let $b \in \mathbb{R}^n \setminus \phi(\partial\Omega)$. If b is a regular value of $\phi|_\Omega$, then $\Omega \cap \phi^{-1}(b)$ is a finite set, and the **degree** of ϕ at b relative to Ω is defined to be

$$d(\phi, \Omega, b) := \sum_{a \in \phi^{-1}(b) \cap \Omega} \operatorname{sign} J_\phi(a).$$

If b is not a regular value of $\phi|_\Omega$, see [22, Chap. 1] for the definition of $d(\phi, \Omega, b)$.

If a is an isolated zero of ϕ, then the **index** of ϕ at a, denoted $\operatorname{ind}(\phi, a)$, is defined to be $d(\phi, U, 0)$, where U is any open set such that $\overline{U}$ does not contain any other zeros of ϕ. This definition is justified, because two such open sets U_1 and U_2 give the same degree: $d(\phi, U_1, 0) = d(\phi, U_2, 0)$. If Ω is an open set such that $0 \notin \phi(\partial\Omega)$ and ϕ has finitely many zeros in Ω, then

$$d(\phi, \Omega, 0) = \sum_{a \in \phi^{-1}(0) \cap \Omega} \operatorname{ind}(\phi, a). \tag{3.1}$$

We consider now holomorphic mappings on $\mathbb{C}^s$. The norm in $\mathbb{C}^s$ is defined by the rule

$$|z| := \max\big\{|z_j| : j = 1, \ldots, s\big\} \qquad \text{for } z = (z_1, \ldots, z_s) \in \mathbb{C}^s.$$

A holomorphic mapping ϕ from an open set $\Omega \subseteq \mathbb{C}^s$ to $\mathbb{C}^s$ can also be viewed as a mapping from an open subset of $\mathbb{R}^n$ to $\mathbb{R}^n$, where $n = 2s$. In this way $d(\phi, \Omega, b)$ is defined, provided $b \notin \phi(\partial\Omega)$. When ϕ is holomorphic, $d(\phi, \Omega, b)$ is always nonnegative (see [22, p. 145]). Moreover, if a is an isolated zero of ϕ, then the index $\operatorname{ind}(\phi, a)$ is a positive integer. This index is greater than 1 if and only if a is a critical point of ϕ (see Theorem

9.3.3 of [22]). Thus, if 0 is a regular value of ϕ, then (3.1) says that $d(\phi, \Omega, 0)$ equals the number of zeros of ϕ in Ω.

The following important theorem will be used frequently. Its proof can be found in [22, p. 147].

THE GENERALIZED ROUCHÉ THEOREM. *Let Ω be a bounded, open set in $\mathbb{C}^s$. If f and g are holomorphic mappings from a neighborhood of $\overline{\Omega}$ to $\mathbb{C}^s$, and if $|g(z)| < |f(z)|$ for all $z \in \partial\Omega$, then $d(f, \Omega, 0) = d(f+g, \Omega, 0)$.*

Let ϕ be a mapping from $\mathbb{C}^s$ to $\mathbb{C}^s$ given by $\phi(z) = (p_1(z), \dots, p_s(z))$, $z \in \mathbb{C}^s$, where $p_1, \dots, p_s$ are polynomials in $\mathbb{C}[Z_1, \dots, Z_s]$. Such a mapping is called a **polynomial mapping**. Let a be an isolated zero of ϕ. The **algebraic multiplicity** of ϕ at a, denoted $\mu_a(\phi)$, is defined to be $\mu_a(I)$, where I is the ideal generated by $p_1, \dots, p_s$. The index $\operatorname{ind}(\phi, a)$ can be interpreted as the **geometric multiplicity** of ϕ at a. Indeed, since a is an isolated zero of ϕ, there exists $r > 0$ such that ϕ does not vanish on $\overline{B_r(a)} \setminus \{a\}$, where $B_r(a)$ is the ball $\{z \in \mathbb{C}^s: |z-a| < r\}$. Let δ be the minimum of ϕ on the sphere $\{z \in \mathbb{C}^s: |z-a| = r\}$. Then $\delta > 0$. Given $\varepsilon = (\varepsilon_1, \dots, \varepsilon_s) \in B_\delta(0)$, consider the perturbation ϕ^ε of ϕ given by $\phi^\varepsilon(z) = (p_1(z) - \varepsilon_1, \dots, p_s(z) - \varepsilon_s)$ for $z \in \mathbb{C}^s$. By the generalized Rouché theorem,

$$d(\phi^\varepsilon, B_r(a), 0) = d(\phi, B_r(a), 0) = \operatorname{ind}(\phi, a).$$

By Sard's theorem (see [17, p. 205]), almost every $\varepsilon \in B_\delta(0)$ is a regular value of ϕ. Consequently, for almost every $\varepsilon \in B_\delta(0)$, the number of zeros of ϕ^ε in $B_r(a)$ equals $d(\phi^\varepsilon, B_r(a), 0)$, which in turn equals $\operatorname{ind}(\phi, a)$. In particular, if each $p_j = Z_j^{m_j}$ for some positive integer m_j, then for any $\varepsilon \in (\mathbb{C} \setminus \{0\})^s$, 0 is a regular value of ϕ^ε; hence $\operatorname{ind}(\phi, 0) = d(\phi^\varepsilon, B_r(0), 0) = m_1 \cdots m_s$.

A proof of the following important result can be found in [2, Chap. 5]. Here we give another proof based on the perturbation technique.

THEOREM 3.1. *Let ϕ be a polynomial mapping from $\mathbb{C}^s$ to $\mathbb{C}^s$ such that its zero set $\phi^{-1}(0)$ is finite. Then for each $a \in \phi^{-1}(0)$,*

$$\mu_a(\phi) = \operatorname{ind}(\phi, a). \tag{3.2}$$

Proof. It will be proved in Section 5 (as a special case of Lemma 5.2) that

$$\mu_\theta(\phi) \geq \operatorname{ind}(\phi, \theta) \qquad \text{for every } \theta \in \phi^{-1}(0). \tag{3.3}$$

Assuming that (3.3) is true, we prove (3.2) as follows. Suppose we can find another polynomial mapping $\psi: z \mapsto (q_1(z), \dots, q_s(z))$, $z \in \mathbb{C}^s$, such that

ψ and the ideal J generated by $q_1, \dots, q_s$ have the following properties:

$$\operatorname{codim}(J) = d(\psi, B_R(0), 0) < \infty \qquad \text{for some ball } B_R(0) \supset \mathscr{V}(J); \tag{3.4}$$

$$\mu_a(\psi) \geq \mu_a(\phi); \tag{3.5}$$

$$\operatorname{ind}(\psi, a) = \operatorname{ind}(\phi, a). \tag{3.6}$$

Then (3.2) is true. Indeed, Theorem 2.4, (3.4), and (3.1) tell us that

$$\sum_{\theta \in \psi^{-1}(0)} \mu_\theta(\psi) = \operatorname{codim}(J) = d(\psi, B_R(0), 0) = \sum_{\theta \in \psi^{-1}(0)} \operatorname{ind}(\psi, \theta).$$

This in connection with our assumption $\mu_\theta(\psi) \geq \operatorname{ind}(\psi, \theta)$ shows that $\mu_\theta(\psi) = \operatorname{ind}(\psi, \theta)$ for all $\theta \in \psi^{-1}(0)$. In particular, $\mu_a(\psi) = \operatorname{ind}(\psi, a)$. Combining this with (3.3), (3.5), and (3.6), we obtain

$$\operatorname{ind}(\phi, a) = \operatorname{ind}(\psi, a) = \mu_a(\psi) \geq \mu_a(\phi) \geq \operatorname{ind}(\phi, a).$$

It follows that $\mu_a(\phi) = \operatorname{ind}(\phi, a)$, as desired.

Suppose ϕ is given by $\phi(z) = (p_1(z), \dots, p_s(z))$, $z \in \mathbb{C}^s$. Let I be the ideal generated by the polynomials $p_1, \dots, p_s$. In order to find a mapping ψ with the desired properties, we use the polynomials $f_1, \dots, f_s$ given in (2.4). By Hilbert's nullstellensatz, there exists an integer $n > \max_{1 \leq j \leq s} \{\deg p_j\}$ such that $f_j^{n-1} \in I$ for all $j = 1, \dots, s$. Set

$$q_j := f_j^n + p_j, \qquad j = 1, \dots, s.$$

We claim that the mapping $\psi \colon z \mapsto (q_1(z), \dots, q_s(z))$ satisfies all the properties stated in (3.4)–(3.6).

First, we observe that the leading term of f_j is Z_j^m, where $m = \#\mathscr{V}(I)$, the number of elements in $\mathscr{V}(I)$. Let $h_j := Z_j^{mn}$ and let χ be the mapping given by $\chi(z) = (h_1(z), \dots, h_s(z))$, $z \in \mathbb{C}^s$. For any $R > 0$, $d(\chi, B_R(0), 0) = \operatorname{ind}(\chi, 0) = (mn)^s$, as was computed before. Furthermore, for each $j = 1, \dots, s$, $\deg(q_j - h_j) < mn$; hence for sufficiently large $R > 0$, $|\psi(z) - \chi(z)| < |\chi(z)|$ for all $z \in \partial B_R(0)$. Then by the generalized Rouché theorem,

$$d(\psi, B_R(0), 0) = d(\chi, B_R(0), 0) = (mn)^s.$$

On the other hand, the ideal J generated by $q_1, \dots, q_s$ has codimension $(mn)^s$, because

$$\left\{\overline{Z^\alpha} \colon \alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{N}_0^s \text{ with } \alpha_j < mn \text{ for all } j\right\}$$

forms a basis for $\mathbb{C}[Z_1, \dots, Z_s]/J$, where $\overline{Z^\alpha}$ denotes the residue class of Z^α in it. This verifies (3.4).

Second, since $f_j^{n-1} \in I$ for all $j = 1, \dots, s$, we have $J \subseteq I$. Hence $\mu_a(\psi) \geq \mu_a(\phi)$.

Third, each f_j^{n-1} can be written as $\sum_{k=1}^s u_{jk} p_k$, where u_{jk} are polynomials. Consequently, $q_j - p_j = f_j^n = \sum_{k=1}^s v_{jk} p_k$, where $v_{jk} = f_j u_{jk}$. Thus, $v_{jk}(a) = 0$, so that for $r > 0$ sufficiently small, $|\psi(z) - \phi(z)| < |\phi(z)|$ for all $z \in \partial B_r(a)$. By the generalized Rouché theorem, this implies $d(\psi, B_r(a), 0) = d(\phi, B_r(a), 0)$, i.e., $\operatorname{ind}(\psi, a) = \operatorname{ind}(\phi, a)$.

It has been verified that ψ satisfies all the three properties in (3.4)–(3.6), so the proof of Theorem 3.1 is complete. ∎

The following result is an immediate consequence of this theorem and Theorem 2.4.

COROLLARY 3.2. *Let ϕ be a polynomial mapping from $\mathbb{C}^s$ to $\mathbb{C}^s$ such that its zero set $\phi^{-1}(0)$ is finite. If Ω is a bounded open set in $\mathbb{C}^s$ containing $\phi^{-1}(0)$, then*

$$\operatorname{codim}(I) = d(\phi, \Omega, 0).$$

Of particular interest is a special case of Theorem 3.1 in which all the polynomials $p_1, \dots, p_s$ are homogeneous. Given $w = (w_1, \dots, w_s) \in \mathbb{C}^s$, let I^w be the ideal generated by the polynomials $p_1 - w_1, \dots, p_s - w_s$, and let ϕ^w be the mapping given by $\phi^w(z) = (p_1(z) - w_1, \dots, p_s(z) - w_s)$, $z \in \mathbb{C}^s$. If $R > 0$ is sufficiently large, then by the generalized Rouché theorem,

$$d\big(\phi^w, B_R(0), 0\big) = d\big(\phi, B_R(0), 0\big).$$

This together with Corollary 3.2 yields the following.

COROLLARY 3.3. *Let I be an ideal generated by s homogeneous polynomials and let w be any point in $\mathbb{C}^s$. If I has finite codimension, then*

$$\operatorname{codim}(I) = \operatorname{codim}(I^w).$$

It is essential in Corollary 3.3 to assume that I is generated by homogeneous polynomials. For instance, let I be the ideal in $\mathbb{C}[Z_1, Z_2]$ generated by the polynomials $p_1 = Z_1 + Z_2$ and $p_2 = Z_1(Z_1 + Z_2) - 1$. Then $\operatorname{codim}(I) = 0$, but $\operatorname{codim}(I^w) = 1$ for any $w = (w_1, w_2) \in \mathbb{C}^2$ with $w_1 \neq 0$.

Now let us consider the case in which I is generated by homogeneous polynomials $p_1, \dots, p_m$, where m is not necessarily equal to s. It was proved by de Boor and Ron in [5] that

$$\operatorname{codim}(I) \geq \operatorname{codim}(J)$$

for every lower-order perturbation J of I. They conjectured that every homogeneous ideal of polynomials has a perfect lower-order perturbation. The following example supports their conjecture.

EXAMPLE 3.4. Let I be an ideal generated by monomials Z^α, $\alpha \in A$, where A is a finite subset of $\mathbb{N}_0^s$. For each $j = 1, \ldots, s$, let $(c_{jn})_{n=0,1,\ldots}$ be a sequence of distinct complex numbers. Define

$$q_\alpha := \prod_{j=1}^{s} \prod_{k=0}^{\alpha_j - 1} (Z_j - c_{jk}) \qquad \text{for } \alpha = (\alpha_1, \ldots, \alpha_s) \in A.$$

Then q_α is a lower-order perturbation of Z^α for every $\alpha \in A$. Let J be the ideal generated by $\{q_\alpha : \alpha \in A\}$. It can be easily proved that J is a perfect perturbation of I. In other words, every ideal of finite codimension generated by monomials has a perfect perturbation.

4. THE ALTERNATIVE THEOREM

In this section we return to the study of the ideal $I(X)$ as described in Section 1. Recall that X is a nonempty set. Associated to each element $x \in X$ is a polynomial $p_x \in \mathbb{C}[Z_1, \ldots, Z_s]$. Moreover, $\mathscr{B} = \mathscr{B}(X)$ is a collection of subsets of X, each of which has exactly s elements, and $\mathscr{A} = \mathscr{A}(X, \mathscr{B}(X))$ is defined as in (1.2). The ideal $I(X)$ is generated by the polynomials p_A, $A \in \mathscr{A}$, where each p_A is defined as in (1.1). We assume that

$$\operatorname{codim}(I(B)) < \infty \qquad \text{for every } B \in \mathscr{B}(X).$$

The set X can be labeled so that $X = \{1, \ldots, n\}$. Given $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$, let p_k^w be the polynomial $p_k - w_k$ $(k = 1, \ldots, n)$. For $A \subseteq X$, define

$$p_A^w := \prod_{k \in A} p_k^w.$$

Let $I^w(X) = I^w(X, \mathscr{B}(X))$ be the ideal generated by $\{p_A^w : A \in \mathscr{A}\}$. Correspondingly, for each $B \in \mathscr{B}(X)$, the ideal generated by $\{p_k^w : k \in B\}$ is denoted by $I^w(B)$. The main purpose of this section is to establish the following result.

THEOREM 4.1. *There exists a nontrivial polynomial $q \in \mathbb{C}[Z_1, \ldots, Z_n]$ such that for every $w \in \mathbb{C}^n$ with $q(w) \neq 0$ the following two conditions are satisfied*:

(i) *$I^w(B)$ is simple for every $B \in \mathscr{B}$*;

(ii) *$\mathscr{V}(I^w(B))$ and $\mathscr{V}(I^w(B'))$ are disjoint for $B, B' \in \mathscr{B}$ with $B \neq B'$.*

If q is a nontrivial polynomial in $\mathbb{C}[Z_1, \ldots, Z_n]$, then

$$\mathscr{V}(q) := \{z \in \mathbb{C}^n : q(z) = 0\}$$

is called a **hypersurface**. Note that a finite union of hypersurfaces is also a hypersurface.

The proof of Theorem 4.1 is based on the so-called *alternative theorem* stated as follows.

THEOREM 4.2. *Let $p_1, \ldots, p_r$ be polynomials in $m + n$ variables with complex coefficients, and let W be the set of those points $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$ for which the polynomials $p_1(z, w), \ldots, p_r(z, w)$ have a common zero $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$. Then either W itself or its complement in $\mathbb{C}^n$ is contained in some hypersurface in $\mathbb{C}^n$.*

The proof of Theorem 4.2 is postponed. Assuming that Theorem 4.2 is valid, we prove Theorem 4.1 first.

Proof of Theorem 4.1. Let $B \in \mathscr{B}(X)$. Without loss of generality, we may assume that $B = \{1, \ldots, s\}$. Consider the system of equations in the unknown $z = (z_1, \ldots, z_s) \in \mathbb{C}^s$:

$$\begin{gathered} p_1(z) - w_1 = 0 \\ \vdots \\ p_s(z) - w_s = 0 \\ J(z) = 0, \end{gathered}$$

where $J(z)$ denotes the Jacobian determinant of $p_1, \ldots, p_s$ at z. Let W be the set of those points $w = (w_1, \ldots, w_s) \in \mathbb{C}^s$ for which the above system of equations has a solution for z. Then, for each $w \notin W$, $J(z) \neq 0$ for any $z \in \mathscr{V}(I^w(B))$; hence the zeros of $I^w(B)$ are all simple. By the alternative theorem, W itself or its complement is contained in a hypersurface in $\mathbb{C}^s$. But the second alternative cannot happen. To see this, let ϕ be the mapping $z \mapsto (p_1(z), \ldots, p_s(z))$, $z \in \mathbb{C}^s$. Note that $J(z) = 0$ if and only if z is a critical point of ϕ. Thus, $w \in W$ if and only if w is a critical value of ϕ. By the well-known Sard theorem (see, e.g., [17, p. 40]), W has measure zero. Therefore, only the first alternative can happen, i.e., W is contained in a hypersurface in $\mathbb{C}^s$.

Let E be a subset of X of cardinality $s + 1$. Without loss of generality, we may assume that $E = \{1, \ldots, s + 1\}$. Consider the system of equations in the unknown $z \in \mathbb{C}^s$:

$$p_k(z) - w_k = 0, \qquad k = 1, \ldots, s + 1.$$

Let ψ be the mapping $z \mapsto (p_1(z), \ldots, p_{s+1}(z))$, $z \in \mathbb{C}^s$. By the mini-Sard theorem (see [17, p. 205]), the image of ψ has measure zero in the space $\mathbb{C}^{s+1}$. Let W be the set of those points $w = (w_1, \ldots, w_{s+1}) \in \mathbb{C}^{s+1}$ for which the above system of equations has a solution for z. By the same reasoning as before, we see that W is contained in a hypersurface in $\mathbb{C}^{s+1}$.

To summarize, we have proved that to each $B \in \mathscr{B}(X)$ there corresponds a nontrivial polynomial q_B in s variables $\{w_j: j \in B\}$ such that the ideal $I^w(B)$ is simple for every w with $q_B(w) \neq 0$. Moreover, to each subset E of X of cardinality $s+1$ there coresponds a nontrivial polynomial q_E in $s+1$ variables $\{w_k: k \in E\}$ such that the polynomials p_k^w $(k \in E)$ do not have any common zero, provided $q_E(w) \neq 0$. View q_B and q_E as polynomials in $w_1, \ldots, w_n$. Let

$$q := \prod_{\substack{E \subseteq X \\ \#E = s+1}} q_E \prod_{B \in \mathscr{B}(X)} q_B .$$

Then q is a nontrivial polynomial in n variables. If $q(w) \neq 0$ then for every $B \in \mathscr{B}$, $q_B(w) \neq 0$, and, hence, $I^w(B)$ is simple. Moreover, if $q(w) \neq 0$ and $B \neq B'$, then $\mathscr{V}(I^w(B))$ and $\mathscr{V}(I^w(B'))$ are disjoint, for otherwise $z \in \mathscr{V}(I^w(B)) \cap \mathscr{V}(I^w(B'))$ would imply $p_k(z) = 0$ for all $k \in B \cup B'$, whence $\#(B \cup B') \geq s+1$. ∎

In order to prove Theorem 4.2 we need some results from the theory of resultant systems (see [18, Chap. 4]).

THEOREM 4.3. *Let K be an algebraically closed field and u be an indeterminate over K. Given an r-tuple $(d_1, \ldots, d_r)$ of nonnegative integers, let $K[\ldots, v_{kj}, \ldots]$ denote the polynomial ring over K in the indeterminates v_{kj} $(j = 0, \ldots, d_k;\ k = 1, \ldots, r)$. Then there exist finitely many polynomials $R_1, \ldots, R_t \in K[\ldots, v_{kj}, \ldots]$ such that for polynomials $f_1, \ldots, f_r$ in $K[u]$ given by*

$$f_k = \sum_{j=0}^{d_k} c_{kj} u^{d_k - j}, \qquad k = 1, \ldots, r,$$

the following statements are true:

(i) *A necessary condition that $f_1, \ldots, f_r$ have a common zero is $R_l(\ldots, c_{kj}, \ldots) = 0$ for all $l = 1, \ldots, t$, where $R_l(\ldots, c_{kj}, \ldots)$ denotes the result obtained by evaluating R_l at $v_{kj} = c_{kj}$ $(j = 0, \ldots, d_k;\ k = 1, \ldots, r)$.*

(ii) *If at least one of the leading coefficients $c_{10}, \ldots, c_{r0}$ is not zero, then this condition is also sufficient.*

Note that the polynomials $R_1, \ldots, R_t$ are determined by the r-tuple $(d_1, \ldots, d_r)$. Thus, we may call the set $\{R_1, \ldots, R_t\}$ a **resultant system** for $(d_1, \ldots, d_r)$.

Proof of Theorem 4.2. The proof proceeds with induction on m. Consider the case $m = 1$ first. Suppose for $k = 1, \ldots, r$,

$$p_k(z, w) = \sum_{j=0}^{d_k} c_{kj}(w) z^{d_k - j}, \qquad z \in \mathbb{C}, w \in \mathbb{C}^n,$$

where each c_{kj} is a polynomial in $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$ and, as a polynomial in w, $c_{k0} \neq 0$ for every k. Let $\{R_1, \ldots, R_t\}$ be a resultant system for $(d_1, \ldots, d_r)$. For $l = 1, \ldots, t$, let

$$Q_l(w) = R_l\big(\ldots, c_{kj}(w), \ldots\big), \qquad w \in \mathbb{C}^n.$$

Then each Q_l is a polynomial of w. If $w \in W$, then $p_1(z, w), \ldots, p_r(z, w)$ have a common zero $z \in \mathbb{C}$; hence by Theorem 4.3 we have $Q_l(w) = 0$ for all $l = 1, \ldots, t$. If at least one of these polynomials, say Q_1, is nontrivial, then W is contained in the hypersurface $\{w \in \mathbb{C}^n : Q_1(w) = 0\}$. Otherwise, all the polynomials $Q_1, \ldots, Q_t$ are identically zero. In the latter case, Theorem 4.3 tells us that $c_{10}(w) \neq 0$ implies $w \in W$. Hence $\mathbb{C}^n \setminus W$ is contained in the hypersurface $\{w \in \mathbb{C}^n : c_{10}(w) = 0\}$.

Next consider the case $m > 1$ and assume that the theorem is valid for $m - 1$. Suppose $p_1, \ldots, p_r$ are polynomials in $(z, w) \in \mathbb{C}^m \times \mathbb{C}^n$ given by

$$p_k = \sum_{|\alpha| \leq d_k} c_{k,\alpha}(w) z^{\alpha}, \qquad k = 1, \ldots, r,$$

where d_k is the degree of p_k with respect to z. We observe that W is invariant under an invertible linear transform of $(z_1, \ldots, z_m) \in \mathbb{C}^m$. Let $(a_{\lambda\nu})_{1 \leq \lambda, \nu \leq m}$ be an invertible complex matrix and consider the linear transform given by $z_\lambda = \sum_{\nu=1}^m a_{\lambda\nu} \tilde{z}_\nu$, $\lambda = 1, \ldots, m$. After this transform we have

$$p_k = \sum_{|\alpha| \leq d_k} \tilde{c}_{k,\alpha}(w) \tilde{z}^{\alpha}, \qquad k = 1, \ldots, r,$$

where $\tilde{c}_{1,\alpha}(w)$ for $\alpha = (0, \ldots, 0, d_1) \in \mathbb{N}_0^m$ can be computed as

$$\tilde{c}_{1,(0,\ldots,0,d_1)}(w) = \sum_{|\beta| = d_1} a_{1m}^{\beta_1} \cdots a_{mm}^{\beta_m} c_{1\beta}(w).$$

There exist a point $w \in \mathbb{C}^n$ and a multi-index $\beta \in \mathbb{N}_0^m$ with $|\beta| = d_1$ such that $c_{1,\beta}(w) \neq 0$; hence $\tilde{c}_{1,(0,\ldots,0,d_1)}(w) \neq 0$ for an appropriate choice of the matrix $(a_{\lambda\nu})_{1 \leq \lambda, \nu \leq m}$. Thus, we may and we do assume from the beginning that $c_{1,(0,\ldots,0,d_1)}(w) \neq 0$ for some $w \in \mathbb{C}^n$.

Now we write

$$p_k = \sum_{j=0}^{d_k} b_{kj}(z', w) z_m^{d_k - j}, \qquad k = 1, \ldots, r,$$

where $z' = (z_1, \ldots, z_{m-1})$. Note that $b_{1,0}(z', w)$ is independent of z' and is a nontrivial polynomial in w; hence, we may write $b_{1,0}(w)$ instead of $b_{1,0}(z', w)$. Let $R_1, \ldots, R_t$ be a resultant system for $\{d_1, \ldots, d_r\}$, and let

$$Q_l(z', w) := R_l\big(\ldots, b_{kj}(z', w), \ldots\big), \qquad l = 1, \ldots, r.$$

Let W_1 be the set of those $w \in \mathbb{C}^n$ for which $Q_1(z', w), \ldots, Q_t(z', w)$ have a common zero $z' \in \mathbb{C}^{m-1}$. By the induction hypothesis, either W_1 or its complement is contained in a hypersurface in $\mathbb{C}^n$. By Theorem 4.3, $w \in W$ implies $w \in W_1$. Thus, if W_1 is contained in a hypersurface, so is W. Let V be the set $\{w \in \mathbb{C}^n \colon b_{1,0}(w) = 0\}$. By our assumption about $b_{1,0}$, V is a hypersurface in $\mathbb{C}^n$. By Theorem 4.3, $W_1 \setminus V \subseteq W$. It follows that

$$\mathbb{C}^n \setminus W \subseteq (\mathbb{C}^n \setminus W_1) \cup V.$$

Thus, if $\mathbb{C}^n \setminus W_1$ is contained in a hypersurface, so is $\mathbb{C}^n \setminus W$. This finishes the induction procedure. ∎

In the rest of this section we discuss an interesting application of the alternative theorem to the study of algebraic functions, which will be needed later. Let us recall some elementary facts about algebraic functions from [1, pp. 283–306]. In the sequel, by a **region** we mean a connected open set. An analytic function f defined on a region $\Omega \subseteq \mathbb{C}$ constitutes a **function element**, denoted (f, Ω). Two function elements (f_1, Ω_1) and (f_2, Ω_2) are said to be equivalent if (f_2, Ω_2) is an analytic continuation of (f_1, Ω_1). The equivalence classes are called **global analytic functions**. The global analytic function determined by a function element (f, Ω) will be denoted by $\mathbf{f}$, and (f, Ω) is also referred to as a **branch** of f. A global analytic function $\mathbf{f}$ is called an **algebraic function** if all its function elements (f, Ω) satisfy a relation $P(f(z), z) = 0$ in Ω, where P is a nontrivial polynomial in two complex variables. Because of the permanence of functional relations (see [1, p. 288]), in order that $\mathbf{f}$ be an algebraic function it is sufficient to assume that one of its branches satisfies the above relation.

LEMMA 4.4. *Let ϕ be a holomorphic mapping from $\mathbb{C}^s$ to $\mathbb{C}^s$, and let ψ be a holomorphic mapping from a region Ω in $\mathbb{C}$ to $\mathbb{C}^s$. If $\psi(\zeta)$ is a regular value of ϕ for every $\zeta \in \Omega$, and if Δ is a simply connected region contained in Ω, then for every choice of points $\zeta_0 \in \Delta$ and $z_0 \in \mathbb{C}^s$ with $\psi(\zeta_0) = \phi(z_0)$*

there exists precisely one holomorphic mapping $\chi : \Delta \to \mathbb{C}^s$ *such that* $\psi = \phi \circ \chi$ *and* $\chi(\zeta_0) = z_0$. *If, in addition,* ϕ *and* ψ *are polynomial mappings, then in the representation*

$$\chi(\zeta) = \big(f_1(\zeta), \ldots, f_s(\zeta)\big), \qquad \zeta \in \Delta, \tag{4.1}$$

each (f_j, Δ) $(j = 1, \ldots, s)$ *is a branch of an algebraic function with no singularities in* Ω.

Proof. The first statement follows from Theorem 4.17 of [14]. Let V be the set of all regular values of ϕ, and let $U = \phi^{-1}(V)$. Then both U and V are open sets. Moreover, the Jacobian determinant $J_\phi(z) \neq 0$ for any $z \in U$. By assumption we have $\psi(\Delta) \subseteq V$. It is easily seen that the mapping $\phi|_U$ from U to V is a covering mapping. Suppose $\zeta_0 \in \Delta$ and $z_0 \in \mathbb{C}^s$ satisfy $\psi(\zeta_0) = \phi(z_0)$. Since Δ is a simply connected region in $\mathbb{C}$, Theorem 4.17 of [14] is applicable, so we conclude that there exists precisely one mapping $\chi : \Delta \to \mathbb{C}^s$ such that $\psi = \phi \circ \chi$ and $\chi(\zeta_0) = z_0$. The mapping χ must be holomorphic (cf. Theorem 4.9 of [14]). Furthermore, it follows from Theorem 4.14 of [14] that the covering mapping $\phi|_U : U \to V$ has the curve lifting property. Since $\psi(\Omega) \subseteq V$, we conclude that (χ, Δ) can be analytically continued along any curve inside Ω.

Now suppose ϕ and ψ are polynomial mappings given by $\phi(z) = (p_1(z), \ldots, p_s(z))$, $z \in \mathbb{C}^s$, and $\psi(\zeta) = (g_1(\zeta), \ldots, g_s(\zeta))$, $\zeta \in \mathbb{C}$. In order to prove the second statement, we consider the system of equations

$$p_j(z_1, z_2, \ldots, z_s) - g_j(\zeta) = 0, \qquad j = 1, \ldots, s. \tag{4.2}$$

Let W denote the set of those pairs $(z_1, \zeta) \in \mathbb{C}^2$ for which the above system of equations have solutions for $(z_2, \ldots, z_s) \in \mathbb{C}^{s-1}$. By the alternative theorem, either W or its complement is contained in a hypersurface in $\mathbb{C}^2$, i.e., an algebraic curve in $\mathbb{C}^2$. We claim that the second alternative cannot happen. Indeed, there is an open ball O in $\mathbb{C}^s$ containing $\psi(\zeta_0)$ such that $\phi^{-1}(O) = \bigcup_{j=1}^m U_j$, where $U_1, \ldots, U_m$ are disjoint open sets, and $\phi|_{U_j}$ is a homeomorphism from U_j to O for each j. Denote by σ_j the inverse mapping of $\phi|_{U_j}$ $(j = 1, \ldots, m)$. There exists an open disk G in $\mathbb{C}$ such that $\zeta_0 \in G \subseteq \psi^{-1}(O)$. If $\zeta \in G$ and $z = (z_1, \ldots, z_s) \in \mathbb{C}^s$ satisfy the system of equations in (4.2), then $z \in \phi^{-1}(\psi(G))$; hence, $z = \sigma_j(\psi(\zeta))$ for some $\zeta \in G$ and $j \in \{1, \ldots, m\}$. Suppose for $j = 1, \ldots, m$,

$$\sigma_j(\psi(\zeta)) = \big(h_{j1}(\zeta), \ldots, h_{js}(\zeta)\big), \qquad \zeta \in G.$$

Then h_{jk} $(j = 1, \ldots, m;\ k = 1, \ldots, s)$ are holomorphic functions on G. Thus, we have

$$W \cap \{(z_1, \zeta) : \zeta \in G\} = \bigcup_{j=1}^{m} \big\{h_{j1}(\zeta) : \zeta \in G\big\}.$$

This shows that there is no algebraic curve in $\mathbb{C}^2$ containing $\mathbb{C}^2 \setminus W$. Therefore, W is contained in an algebraic curve in $\mathbb{C}^2$. In other words, there exists a nontrivial polynomial P in two complex variables such that $P(z_1, \zeta) = 0$ for all $(z_1, \zeta) \in W$. Since $\phi \circ \chi = \psi$, the representation of χ given in (4.1) tells us that $(f_1(\zeta), \zeta) \in W$ for all $\zeta \in \Delta$; hence $P(f_1(\zeta), \zeta) = 0$ for $\zeta \in \Delta$. This shows that (f_1, Δ) is a branch of an algebraic function. The same reasoning shows that this is also the case for (f_j, Δ) $(j = 2, \ldots, s)$. Finally, it was proved before that each (f_j, Δ) can be analytically continued along any curve inside Ω, so the algebraic function determined by it has no singularities in Ω. ∎

5. LOWER BOUNDS FOR THE DIMENSION

In this section we prove the main result of this paper.

THEOREM 5.1. *Let X be a nonempty finite set and $\mathscr{B}$ a collection of subsets of X, each of which has exactly s elements. Suppose there corresponds a polynomial $p_x \in \mathbb{C}[Z_1, \ldots, Z_s]$ to each $x \in X$. Let $I(X)$ be the ideal generated by $\{p_A: A \in \mathscr{A}\}$, where*

$$\mathscr{A} := \{A \subseteq X: A \cap B \neq \emptyset \quad \text{for all } B \in \mathscr{B}\},$$

and $p_A := \prod_{x \in A} p_x$. Then

$$\operatorname{codim}(I(X)) \geq \sum_{B \in \mathscr{B}} \operatorname{codim}(I(B)), \tag{5.1}$$

where $I(B)$ denotes the ideal generated by $\{p_x: x \in B\}$.

If $\operatorname{codim}(I(X)) = \infty$, then there is nothing to prove; we therefore assume that $\operatorname{codim}(I(X)) < \infty$. It follows that $\operatorname{codim}(I(B)) < \infty$ for all $B \in \mathscr{B}$. In order to prove (5.1), by Theorem 2.4, it suffices to show that for every $\theta \in \mathscr{V}(I(X))$,

$$\mu_\theta(I(X)) \geq \sum_{B \in \mathscr{B}} \mu_\theta(I(B)), \tag{5.2}$$

where, as was in Section 2, $\mu_\theta(I)$ denotes the multiplicity of the ideal I at θ.

Label X so that $X = \{1, \ldots, n\}$. For $B = \{k_1, \ldots, k_s\} \in \mathscr{B}$ with $k_1 < \cdots < k_s$, let ϕ_B be the mapping given by

$$\phi_B(z) = \big(p_{k_1}(z), \ldots, p_{k_s}(z)\big), \qquad z \in \mathbb{C}^s.$$

The proof of (5.2) is based on the following lemma.

LEMMA 5.2. *For every* $\theta \in \mathscr{V}(I(X))$,

$$\mu_\theta(I(X)) \geq \sum_{B \in \mathscr{B}} \operatorname{ind}(\phi_B, \theta). \tag{5.3}$$

Indeed, taking X to be B in (5.3) gives

$$\mu_\theta(\phi_B) = \mu_\theta(I(B)) \geq \operatorname{ind}(\phi_B, \theta) \qquad \text{for } \theta \in \mathscr{V}(I(B)).$$

This verifies (3.3). Hence, if Lemma 5.2 is true, then Theorem 3.1 is applicable. Consequently, $\mu_\theta(I(B)) = \operatorname{ind}(\phi_B, \theta)$. This, together with (5.3), implies (5.2), and so Theorem 5.1 is true.

Proof of Lemma 5.2. Let $\theta \in \mathscr{V}(I(X))$, and let $m := \sum_{B \in \mathscr{B}} \operatorname{ind}(\phi_B, \theta)$. We shall use the perturbation technique to prove $\mu_\theta(I(X)) \geq m$. If $m = 1$, there is nothing to prove; we therefore assume that $m > 1$.

Note that $\theta \in \mathscr{V}(I(X))$ implies $\theta \in \mathscr{V}(I(B))$ for some $B \in \mathscr{B}$. Since $\operatorname{codim}(I(B)) < \infty$, θ is an isolated zero of ϕ_B. Choose and fix a neighborhood U of θ such that ϕ_B has no zeros in $\overline{U} \setminus \{\theta\}$ for any $B \in \mathscr{B}$. It follows that

$$\operatorname{ind}(\phi_B, \theta) = d(\phi_B, U, 0) \qquad \text{for all } B \in \mathscr{B}. \tag{5.4}$$

For $w = (w_1, \dots, w_n) \in \mathbb{C}^n$, let $p_k^w = p_k - w_k$ $(k = 1, \dots, n)$, and let $I^w(X)$ be the ideal generated by $\{p_A^w \colon A \in \mathscr{A}\}$, where $p_A^w = \prod_{k \in A} p_k^w$. For $B = \{k_1, \dots, k_s\} \in \mathscr{B}$ with $k_1 < \cdots < k_s$, let ϕ_B^w be the mapping given by

$$\phi_B^w(z) = \left(p_{k_1}^w(z), \dots, p_{k_s}^w(z)\right), \qquad z \in \mathbb{C}^s.$$

Since $0 \notin \phi_B(\partial U)$ for any $B \in \mathscr{B}$, we have

$$\delta := \min_{B \in \mathscr{B}} \min_{z \in \partial U} |\phi_B(z)| > 0.$$

By the generalized Rouché theorem, if $|w| < \delta$ then

$$d(\phi_B^w, U, 0) = d(\phi_B, U, 0) \qquad \text{for all } B \in \mathscr{B}. \tag{5.5}$$

Let $q \in \mathbb{C}[Z_1, \dots, Z_n]$ be a polynomial satisfying the conditions in Theorem 4.1. Fix a point $c = (c_1, \dots, c_n) \in \mathbb{C}^n$ such that $q(c) \neq 0$. Since $q(c) \neq 0$, there exists $\varepsilon > 0$ such that $q(c\zeta) \neq 0$ for $0 < |\zeta| < \varepsilon$. For simplicity, we write $I^\zeta(X)$ instead of $I^{c\zeta}(X)$. Correspondingly we write $I^\zeta(B)$ for $I^{c\zeta}(B)$, and ϕ_B^ζ for $\phi_B^{c\zeta}$, $B \in \mathscr{B}$. By reducing ε if necessary, we may assume that $|c\varepsilon| < \delta$. Since $I^\zeta(B)$ is simple, $d(\phi_B^\zeta, U, 0)$ is just the number of zeros of ϕ_B^ζ in U. Thus, it follows from (5.4) and (5.5) that

$$\operatorname{ind}(\phi_B, \theta) = \#\left(U \cap \mathscr{V}\left(I^\zeta(B)\right)\right), \qquad 0 < |\zeta| < \varepsilon. \tag{5.6}$$

From the above analysis, we see that if U' is an arbitrary neighborhood of θ contained in U, then there is an ε', $0 < \varepsilon' < \varepsilon$, such that

$$\begin{aligned} \#\big(U' \cap \mathscr{V}\big(I^{\zeta}(B)\big)\big) &= \operatorname{ind}(\phi_B, \theta) \\ &= \#\big(U \cap \mathscr{V}\big(I^{\zeta}(B)\big)\big) \qquad \text{for } 0 < |\zeta| < \varepsilon'. \end{aligned}$$

This shows that $U \cap \mathscr{V}(I^{\zeta}(B)) \subseteq U'$ for $0 < |\zeta| < \varepsilon'$. To paraphrase this fact, we say that the family of the sets $U \cap \mathscr{V}(I^{\zeta}(B))$ converges to θ as ζ tends to 0.

For $B = \{k_1, \dots, k_s\} \in \mathscr{B}$ with $k_1 < \cdots < k_s$, consider the mapping ψ_B given by

$$\psi_B(\zeta) = \big(c_{k_1}\zeta, \dots, c_{k_s}\zeta\big), \qquad 0 < |\zeta| < \varepsilon.$$

Since the ideal $I^{\zeta}(B)$ is simple for $0 < |\zeta| < \varepsilon$, $\psi_B(\zeta)$ is a regular value of ϕ_B for $0 < |\zeta| < \varepsilon$. Let ζ_0 be a fixed point in the *simply connected* region

$$\Delta_\varepsilon := \big\{\rho e^{i\eta} : 0 < \rho < \varepsilon,\ -\pi < \eta < \pi\big\}.$$

We deduce from Lemma 4.4 that for a given $t \in \mathscr{V}(I^{\zeta_0}(B))$, there is precisely one holomorphic mapping ω^t from Δ_ε to $\mathbb{C}^s$ such that $\phi_B \circ \omega^t = \psi_B$ and $\omega^t(\zeta_0) = t$. It follows from $\phi_B \circ \omega^t = \psi_B$ that

$$\omega^t(\zeta) \in \mathscr{V}\big(I^{\zeta}(B)\big) \subseteq \mathscr{V}\big(I^{\zeta}(X)\big) \qquad \text{for } \zeta \in \Delta_\varepsilon.$$

Moreover, in the representation

$$\omega^t(\zeta) = \big(f_1^t(\zeta), \dots, f_s^t(\zeta)\big) \qquad \text{for } \zeta \in \Delta_\varepsilon,$$

each $(f_j^t, \Delta_\varepsilon)$ $(j = 1, \dots, s)$ is a branch of an algebraic function with no singularities in the punctuated disk $\{\zeta : 0 < |\zeta| < \varepsilon\}$. Furthermore, since $U \cap \mathscr{V}(I^{\zeta}(B))$ converges to θ as ζ tends to 0, we have $\lim_{\zeta \to 0} \omega^t(\zeta) = \theta$.

Recall that $\mathscr{V}(I^{\zeta}(B))$ and $\mathscr{V}(I^{\zeta}(B'))$ are disjoint for $B \neq B'$ and $0 < |\zeta| < \varepsilon$. This fact, together with (5.6), tells us that $m = \sum_{B \in \mathscr{B}} \operatorname{ind}(\phi_B, \theta)$ is just the number of points in $U \cap (\bigcup_{B \in \mathscr{B}} \mathscr{V}(I^{\zeta_0}(B)))$. Let $t_1, \dots, t_m$ be these points. We denote by ω_k the holomorphic mapping ω^{t_k}, $k = 1, \dots, m$.

On the other hand, (2.6) says that

$$\mu_\theta(I(X)) = \dim\big(M_{I(X), \theta}\big),$$

where $M_{I(X), \theta}$ is the multiplicity space of $I(X)$ at θ as given in (2.5). For simplicity, we write M, instead of $M_{I(X), \theta}$. Without loss of any generality, we may assume that $\theta = 0$ in what follows. For two polynomials h and p

in $\mathbb{C}[Z_1, \dots, Z_s]$, it is easily seen that $h(D)p(0) = 0$ if and only if $p(D)h(0) = 0$. Hence,

$$M = \{h \in \mathbb{C}[Z_1, \dots, Z_s] : p(D)h(0) = 0 \text{ for all } p \in I(X)\}.$$

Lemma 5.2 will be established, if we can find m linearly independent elements $h_1, \dots, h_m$ in M.

For each $\zeta \in \Delta_\varepsilon$, we denote by M^ζ the linear space spanned by the exponential functions $e_{\omega_1(\zeta)}, \dots, e_{\omega_m(\zeta)}$. Since $\omega_k(\zeta) \in \mathscr{V}(I^\zeta(X))$ for $k = 1, \dots, m$, we have

$$p_A^\zeta(D)e_{\omega_k(\zeta)} = 0 \qquad \text{for all } A \in \mathscr{A},$$

where $p_A^\zeta := p_A^{c\zeta}$. It follows that

$$p_A^\zeta(D)h = 0 \qquad \text{for all } h \in M^\zeta,\ A \in \mathscr{A}. \tag{5.7}$$

Our goal is to find m functions $h_1^\zeta, \dots, h_m^\zeta \in M^\zeta$ for each $\zeta \in \Delta_\varepsilon$ so that $\lim_{\zeta \to 0} h_j^\zeta$ exists and equals a polynomial h_j in M $(j = 1, \dots, m)$ and, in addition, $h_1, \dots, h_m$ are linearly independent. The desired functions $h_1^\zeta, \dots, h_m^\zeta$ are chosen by the equation

$$\begin{bmatrix} h_1^\zeta \\ \vdots \\ h_m^\zeta \end{bmatrix} = \begin{bmatrix} \omega_1(\zeta)^{\beta_1}/\beta_1! & \cdots & \omega_1(\zeta)^{\beta_m}/\beta_m! \\ \vdots & \ddots & \vdots \\ \omega_m(\zeta)^{\beta_1}/\beta_1! & \cdots & \omega_m(\zeta)^{\beta_m}/\beta_m! \end{bmatrix} \begin{bmatrix} e_{\omega_1(\zeta)} \\ \vdots \\ e_{\omega_m(\zeta)} \end{bmatrix}, \tag{5.8}$$

where $(\beta_1, \dots, \beta_m) \in (\mathbb{N}_0^s)^m$ is to be determined. Each function h_j^ζ lies in M^ζ, so it can be expanded as a power series:

$$h_j^\zeta(z) = \sum_{\alpha \in \mathbb{N}_0^s} a_{j,\alpha}^\zeta z^\alpha, \qquad z \in \mathbb{C}^s.$$

Since $e_\omega(z) = \sum_{\alpha \in \mathbb{N}_0^s} (\omega^\alpha/\alpha!) z^\alpha$, it follows from (5.8) that

$$\begin{bmatrix} \omega_1(\zeta)^{\beta_1}/\beta_1! & \cdots & \omega_1(\zeta)^{\beta_m}/\beta_m! \\ \vdots & \ddots & \vdots \\ \omega_m(\zeta)^{\beta_1}/\beta_1! & \cdots & \omega_m(\zeta)^{\beta_m}/\beta_m! \end{bmatrix} \begin{bmatrix} a_{1,\alpha}^\zeta \\ \vdots \\ a_{m,\alpha}^\zeta \end{bmatrix} = \begin{bmatrix} \omega_1(\zeta)^\alpha/\alpha! \\ \vdots \\ \omega_m(\zeta)^\alpha/\alpha! \end{bmatrix}.$$

By Cramer's rule, we have

$$a_{j,\alpha}^\zeta = \frac{g_{(\beta_1, \dots, \beta_{j-1}, \alpha, \beta_{j+1}, \dots, \beta_m)}(\zeta)}{g_{(\beta_1, \dots, \beta_m)}(\zeta)}, \qquad \zeta \in \Delta_\varepsilon, \tag{5.9}$$

where $g_{(\beta_1,\dots,\beta_m)}(\zeta)$ denotes the determinant of the matrix $(\omega_j(\zeta)^{\beta_k}/\beta_k!)_{1\le j,k\le m}$. Note that $\omega_j(\zeta_0)=t_j$ for $j=1,\dots,m$. There exists $(\gamma_1,\dots,\gamma_m)\in(\mathbb{N}_0^s)^m$ such that $g_{(\gamma_1,\dots,\gamma_m)}$ does not vanish at ζ_0, for otherwise $e_{t_1},\dots,e_{t_m}$ would be linearly dependent, which contradicts the fact that $t_1,\dots,t_m$ are pairwise distinct. Fix a choice of such $(\gamma_1,\dots,\gamma_m)$.

In order to make an appropriate choice for $(\beta_1,\dots,\beta_m)$, we need the Puiseux expansion of algebraic functions (see [1, p. 304; 14, p. 58]). If (g,Δ_ε) is a branch of some nontrivial algebraic function with no singularities in the punctuated disk $\{\zeta: 0<|\zeta|<\varepsilon\}$, and if $\lim_{\zeta\to 0} g(\zeta)=0$, then g has the Puiseux expansion

$$g(\rho e^{i\eta}) = \sum_{n=\kappa}^{\infty} C_n(\rho e^{i\eta})^{n/\iota}, \qquad 0<\rho<\varepsilon, -\pi<\eta<\pi,$$

where κ,ι are positive integers, $C_\kappa\ne 0$. Denote by $\tau(g)$ the number κ/ι. When $g=0$, we set $\tau(g)$ to be ∞. If h is another function of the same kind, then $\tau(gh)=\tau(g)+\tau(h)$. Moreover, $\lim_{\zeta\to 0} h(\zeta)/g(\zeta)$ exists, provided $\tau(g)\le\tau(h)$. This limit is 0 if and only if $\tau(g)<\tau(h)$.

Each ω_j $(j=1,\dots,m)$ has the representation

$$\omega_j(\zeta) = \left(f_1^{t_j}(\zeta),\dots,f_s^{t_j}(\zeta)\right), \qquad \zeta\in\Delta_\varepsilon,$$

where the function elements $(f_k^{t_j},\Delta_\varepsilon)$ $(k=1,\dots,s)$ are branches of algebraic functions that have no singularities in the punctuated disk $\{\zeta: 0<|\zeta|<\varepsilon\}$. Let

$$\tau_j := \min\left\{\tau\left(f_1^{t_j}\right),\dots,\tau\left(f_s^{t_j}\right)\right\};$$

each τ_j $(j=1,\dots,m)$ is positive or ∞. Using the Laplace expansion of determinants, we see that $g_{(\alpha_1,\dots,\alpha_m)}$ also has a Puiseux expansion in Δ_ε for each $(\alpha_1,\dots,\alpha_m)\in(\mathbb{N}_0^s)^m$. We claim that $\lim_{\zeta\to 0} g_{(\alpha_1,\dots,\alpha_m)}(\zeta)=0$. Indeed, this is true if $\alpha_j\ne 0$ for at least one j. Otherwise, all $\alpha_j=0$ $(j=1,\dots,m)$ implies that $g_{(\alpha_1,\dots,\alpha_m)}$ is identically zero, because $m>1$. Thus, $\tau(g_{(\alpha_1,\dots,\alpha_m)})$ is well defined and is a positive number or ∞. We write $\tau(\alpha_1,\dots,\alpha_m)$ for $\tau(g_{(\alpha_1,\dots,\alpha_m)})$. Recall that $(\gamma_1,\dots,\gamma_m)\in(\mathbb{N}_0^s)^m$ was so chosen that $g_{(\gamma_1,\dots,\gamma_m)}$ does not vanish at ζ_0. Hence $0<\tau(\gamma_1,\dots,\gamma_m)<\infty$. Now choose a sufficiently large integer N such that $N\ge\max\{|\gamma_1|,\dots,|\gamma_m|\}$ and

$$N\min\{\tau_1,\dots,\tau_m\} > \tau(\gamma_1,\dots,\gamma_m). \tag{5.10}$$

Let $\tau_{\min}$ denote the minimum of $\tau(\alpha_1,\dots,\alpha_m)$, where $\alpha_1,\dots,\alpha_m$ run over all possible s-indices of length $\le N$. Then

$$\tau(\gamma_1,\dots,\gamma_m) \ge \tau_{\min} > 0. \tag{5.11}$$

Choose $(\beta_1, \ldots, \beta_m) \in (\mathbb{N}_0^s)^m$ such that

$$\max\{|\beta_1|, \ldots, |\beta_m|\} \le N, \qquad \tau(\beta_1, \ldots, \beta_m) = \tau_{\min}.$$

We claim that with this choice of $(\beta_1, \ldots, \beta_m)$ the functions $h_1^\zeta, \ldots, h_m^\zeta$ obtained from (5.8) possess the desired properties. To verify our claim, we observe that

$$\tau\big(\beta_1, \ldots, \beta_{j-1}, \alpha, \beta_{j+1}, \ldots, \beta_m\big) \ge \tau_{\min} \qquad \text{for } |\alpha| \le N.$$

Moreover, if $\alpha = (\alpha(1), \ldots, \alpha(s)) \in \mathbb{N}_0^s$, then $\omega_j^\alpha = (f_1^{t_j})^{\alpha(1)} \cdots (f_s^{t_j})^{\alpha(s)}$; hence,

$$\tau\big(\omega_j^\alpha\big) = \sum_{k=1}^{s} \alpha(k)\tau\big(f_k^{t_j}\big) \ge |\alpha|\tau_j.$$

This, together with (5.10) and (5.11), implies that for $|\alpha| \ge N$

$$\tau\big(\beta_1, \ldots, \beta_{j-1}, \alpha, \beta_{j+1}, \ldots, \beta_m\big) \ge N \min\{\tau_1, \ldots, \tau_m\} > \tau_{\min}.$$

Noting that $\tau_{\min} = \tau(\beta_1, \ldots, \beta_m)$, we conclude from (5.9) that $\lim_{\zeta \to 0} \alpha_{j,\alpha}^\zeta$ exists for every $\alpha \in \mathbb{N}_0^s$, and this limit is 0 if $|\alpha| > N$. Let $a_{j,\alpha} := \lim_{\zeta \to 0} a_{j,\alpha}^\zeta$, and let h_j be the polynomial given by

$$h_j(z) = \sum_{\alpha \in \mathbb{N}_0^s} a_{j,\alpha} z^\alpha, \qquad z \in \mathbb{C}^s.$$

It remains to prove that $h_1, \ldots, h_m$ are linearly independent elements in M. In order to prove $h_j \in M$, we use the truncation method. For a positive integer r, define $h_{j,r}^\zeta(z) := \sum_{|\alpha| \le r} a_{j,\alpha}^\zeta z^\alpha$, $z \in \mathbb{C}^s$. Let $p \in I(X)$. Then p can be written as $p = \sum_{A \in \mathscr{A}} u_A p_A$, where $u_A \in \mathbb{C}[Z_1, \ldots, Z_s]$ for each $A \in \mathscr{A}$. Choose r sufficiently large such that $r > N$ and $r \ge \deg(u_A p_A)$ for all $A \in \mathscr{A}$. Then we deduce from (5.7) that

$$p_A^\zeta(D) h_{j,r}^\zeta(0) = p_A^\zeta(D) h_j^\zeta(0) = 0.$$

It follows that

$$\begin{aligned} p(D)h_j(0) &= \sum_{A \in \mathscr{A}} u_A(D) p_A(D) h_j(0) \\ &= \lim_{\zeta \to 0} \sum_{A \in \mathscr{A}} u_A(D) p_A^\zeta(D) h_{j,r}^\zeta(0) = 0. \end{aligned}$$

This shows that $h_j \in M$ $(j = 1, \ldots, m)$. Finally, (5.9) yields

$$a^{\zeta}_{j,\beta_k} = \delta_{kj}, \qquad k, j = 1, \ldots, m,$$

where δ_{kj} is the Kronecker sign. Thus we have $a_{j,\beta_k} = \lim_{\zeta \to 0} a^{\zeta}_{j,\beta_k} = \delta_{kj}$, thereby showing that $h_1, \ldots, h_m$ are linearly independent. The proof of Lemma 5.2 is complete. ∎

To conclude this section we mention the following result on simple polynomial ideals.

THEOREM 5.3. *If $I(X)$ is simple, then equality holds in* (5.1). *Moreover, the ideal $I(X)$ is simple if and only if it satisfies the following two conditions*:

(i) *$I(B)$ is simple for each $B \in \mathscr{B}$*;

(ii) *$\mathscr{V}(I(B)) \cap \mathscr{V}(I(B')) = \emptyset$ for $B, B' \in \mathscr{B}$ with $B \neq B'$.*

Proof. It is easily seen that

$$\mathscr{V}(I(X)) \subseteq \bigcup_{B \in \mathscr{B}} \mathscr{V}(I(B)).$$

If $I(X)$ is simple, then

$$\operatorname{codim}(I(X)) = \#\mathscr{V}(I(X)) \leq \sum_{B \in \mathscr{B}} \#\mathscr{V}(I(B)) \leq \sum_{B \in \mathscr{B}} \operatorname{codim}(I(B)). \tag{5.12}$$

This in connection with (5.1) tells us that all the inequalities in (5.12) are actually equalities. This shows that equality holds in (5.1) and the two conditions (i) and (ii) are satisfied.

Conversely, suppose the two conditions (i) and (ii) are satisfied. We wish to show that $I(X)$ is simple. If $I(X)$ is not simple, then there is some $\theta \in \mathscr{V}(I(X))$ such that the multiplicity space $M_{I(X),\theta}$ has dimension > 1. Since $M_{I(X),\theta}$ is D-invariant, one can find a polynomial Q in $\mathbb{C}[Z_1, \ldots, Z_s]$ of degree 1 such that $Q(D)p_A(\theta) = 0$ for all $A \in \mathscr{A}$. The point θ belongs to $\mathscr{V}(I(B_0))$ for some $B_0 \in \mathscr{B}$. By condition (ii), $\theta \notin \mathscr{V}(I(B))$ for any $B \in \mathscr{B} \setminus \{B_0\}$; hence, there exists some $y_B \in B$ such that $p_{y_B}(\theta) \neq 0$. Let $Y := \{y_B : B \in \mathscr{B}$ and $B \neq B_0\}$. Then $Y \cup \{x\} \in \mathscr{A}$ for every $x \in B_0$. Thus, we see from our choice of Q that $Q(D)(p_Y p_x)(\theta) = 0$ for every $x \in B_0$. But $p_x(\theta) = 0$, so we have $p_Y(\theta) Q(D) p_x(\theta) = 0$. Since $p_Y(\theta) \neq 0$, it follows that $Q(D)p_x(\theta) = 0$ for all $x \in B_0$. This implies that $I(B_0)$ is not simple, violating condition (i). ∎

6. UPPER BOUNDS FOR THE DIMENSION

Having established the lower bound for the codimension of $I(X)$, we would like to find a sharp upper bound for it. In the case when $I(X)$ is not simple, de Boor and Ron indicated in Example 6.5 of [5] that equality might not hold in (5.1). However, when $(X, \mathscr{B})$ is a matroid, a sharp upper bound for codim$(I(X))$ is available. This kind of study was initiated by Dahmen and Micchelli in [11].

Let us recall from [26, p. 8] that $(X, \mathscr{B})$ is a matroid if and only if the following base-change property is satisfied: For $B_1, B_2 \in \mathscr{B}$ and $y \in B_1 \setminus B_2$, there exists $x \in B_2 \setminus B_1$ such that $(B_1 \setminus y) \cup x \in \mathscr{B}$. A subset E of X is called a **spanning set** if E includes some $B \in \mathscr{B}$; otherwise, E is called a **nonspanning set**. For $Y \subseteq X$, let $\mathscr{B}(Y)$ be defined as in (1.3). By $\mathscr{H}(Y)$ we denote the collection of all maximally nonspanning subsets of Y.

The dimension problem studied in [11] can be described as follows. Given a linear space S (over some field), we denote by $L(S)$ the set of all linear mappings from S to itself. Associate to each $x \in X$ a linear mapping $l_x \in L(S)$. The linear mappings l_x $(x \in X)$ are assumed to commute with each other:

$$l_x l_y = l_y l_x, \qquad x, y \in X.$$

Thus, the product

$$l_A := \prod_{x \in A} l_x, \qquad A \subseteq X,$$

is well defined. Let $\mathscr{A} = \mathscr{A}(X, \mathscr{B}(X))$ be defined as in (1.2). We are interested in the dimension of the joint kernel

$$K(X) := \bigcap_{A \in \mathscr{A}} \ker l_A.$$

For any subset Y of X, $K(Y)$ is defined accordingly. In particular, for each $B \in \mathscr{B}$ the kernel space $K(B)$ is just $\bigcap_{x \in B} \ker l_x$. The kernel space $I(X)_\perp$ discussed in Section 1 is a special but important example of this general situation. In that case, S is the linear space of all formal power series in s indeterminates, and each l_x is a differential operator $p_x(D)$ induced by a polynomial p_x in s variables. Dahmen and Micchelli in Theorem 3.3 of [11] established the following theorem on the dimension of $K(X)$.

THEOREM 6.1. *If $(X, \mathscr{B})$ is a matroid, then*

$$\dim K(X) \le \sum_{B \in \mathscr{B}} \dim K(B). \tag{6.1}$$

By taking a closer look into their proof, I found that Theorem 6.1 could be extended as follows.

THEOREM 6.2. *The inequality* (6.1) *is valid, provided that for any subset* Y *of* X *with* $\#\mathscr{B}(Y) > 1$ *there exists a* $y \in Y$ *such that*

(i) $Y \setminus y$ *is a spanning set, and*

(ii) *for any* $H_1, H_2 \in \mathscr{H}(Y \setminus y)$ *with* $H_1 \neq H_2$, $(H_1 \cap H_2) \cup y$ *is a nonspanning set.*

Proof. The proof proceeds with induction on $\#X$. If $\#\mathscr{B}(X) \leq 1$, then (6.1) holds trivially. In particular, this is true when $\#X \leq s$. Now assume that $\#X > s$ and $\#\mathscr{B}(X) > 1$. Pick $y \in X$ such that the above conditions (i) and (ii) are satisfied for $Y = X$. Consider the linear mapping T from $K(X)$ to $\prod_{H \in \mathscr{H}(X \setminus y)} K(H \cup y)$ given by

$$T: f \mapsto \left(L_{X \setminus y \setminus H} f\right)_{H \in \mathscr{H}(X \setminus y)}.$$

The kernel of T is $K(X \setminus y)$. Hence we have

$$\dim K(X) \leq \dim K(X \setminus y) + \sum_{H \in \mathscr{H}(X \setminus y)} \dim K(H \cup y).$$

Thus, by the induction hypothesis, it follows that

$$\dim K(X) \leq \sum_{B \in \mathscr{B}(X \setminus y)} K(B) + \sum_{H \in \mathscr{H}(X \setminus y)} \sum_{B \in \mathscr{B}(H \cup y)} \dim K(B). \quad (6.2)$$

If H_1 and H_2 are two different elements of $\mathscr{H}(X \setminus y)$, then by (ii) the set $(H_1 \cup y) \cap (H_2 \cup y) = (H_1 \cap H_2) \cup y$ is a nonspanning one; hence,

$$\mathscr{B}(H_1 \cup y) \cap \mathscr{B}(H_2 \cup y) = \varnothing.$$

This shows that

$$\mathscr{B} = \mathscr{B}(X \setminus y) \cup \bigcup_{H \in \mathscr{H}(X \setminus y)} \mathscr{B}(H \cup y)$$

is a disjoint union. Therefore, the right-hand side of (6.2) equals $\sum_{B \in \mathscr{B}} K(B)$, and the proof of the theorem is complete. ∎

Theorem 6.2 and Theorem 5.1 together yield the following.

THEOREM 6.3. *Under the conditions of Theorem* 6.2,

$$\operatorname{codim}(I(X)) = \sum_{B \in \mathscr{B}} \operatorname{codim}(I(B)).$$

Theorems 6.2 and 6.3 were reported in [19]. It was also pointed out in [19] that Theorem 6.2 is a true generalization of Theorem 6.1.

Condition (ii) in Theorem 6.2 is called the **intersection condition**. In [6], de Boor, Ron, and Shen proved that the intersection condition is equivalent to having y **replaceable** in $\mathscr{B}$ (see [6] for the definition of replaceability). If $(X, \mathscr{B})$ satisfies all the conditions in Theorem 6.2, then $(X, \mathscr{B})$ is called **fair** by them.

Finally, we remark that Theorem 8.10 of [6] gives a better result than Theorem 6.3 of this paper. Also, using the Ext functor from homological algebra, Dahmen, Dress, and Micchelli in [7] studied the dimension problem comprehensively. In both [6, 7], however, some conditions involving $\mathscr{B}$ must be imposed. In contrast to their work, the lower bound for $\operatorname{codim}(I)$ established in Theorem 5.1 of this paper is valid for an arbitrary $\mathscr{B}$ (as long as each $B \in \mathscr{B}$ has s elements).

REFERENCES

1. L. V. Ahlfors, "Complex Analysis," 3rd ed., McGraw–Hill, New York, 1979.
2. V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko, "Singularities of Differential Maps I," Birkhäuser, Boston, 1985.
3. A. Ben-Artzi and A. Ron, Translates of exponential box splines and their related spaces, *Trans. Amer. Math. Soc.* **309** (1988), 683–710.
4. C. de Boor and K. Höllig, B-splines from parallelepipeds, *J. Analyse Math.* **42** (1982/83), 99–115.
5. C. de Boor and A. Ron, On polynomial ideals of finite codimension with applications to box spline theory, *J. Math. Anal. Appl.* **158** (1991), 168–193.
6. C. de Boor, A. Ron, and Z. W. Shen, On ascertaining inductively the dimension of the joint kernel of certain commuting linear operators, *Adv. in Appl. Math.*, **17** (1996), 209–250.
7. W. Dahmen, A. Dress, and C. A. Micchelli, On Multivariate Splines, Matroids, and the Ext-Functor, *Adv. in Appl. Math.*, **17** (1996), 251–307.
8. W. Dahmen, R. Q. Jia, and C. A. Micchelli, On linear dependence relations for integer translates of compactly supported distributions, *Math. Nachr.* **151** (1991), 303–310.
9. W. Dahmen and C. A. Micchelli, On the local linear independence of translates of a box spline, *Studia Math.* **82** (1985), 243–262.
10. W. Dahmen and C. A. Micchelli, On the solution of certain systems of partial difference equations and linear independence of translates of box splines, *Trans. Amer. Math. Soc.* **292** (1985), 305–320.
11. W. Dahmen and C. A. Micchelli, On multivariate E-splines, *Adv. in Math.* **76** (1989), 33–93.
12. W. Dahmen and C. A. Micchelli, Local dimension of piecewise polynomial spaces, syzygies, and solutions of systems of partial differential equations, *Math. Nachr.* **148** (1990), 117–136.
13. N. Dyn and A. Ron, Local approximation by certain spaces of multivariate exponential polynomials, approximation order of exponential box splines and related interpolation problems, *Trans. Amer. Math. Soc.* **319** (1990), 381–404.
14. O. Forster, "Lectures on Riemann Surfaces," Springer-Verlag, New York, 1981.

15. W. Fulton, ''Algebraic Curves,'' Addison–Wesley, New York, 1989.
16. W. Gröbner, ''Algebraische Geometrie II,'' Bibliograph. Inst., Mannheim, 1970.
17. V. Guillemin and A. Pollack, ''Differential Topology,'' Prentice-Hall, Englewood Cliffs, NJ, 1974.
18. W. V. D. Hodge and D. Pedoe, ''Methods of Algebraic Geometry,'' Vol. I, Cambridge Univ. Press, Cambridge, 1947.
19. R. Q. Jia, Perturbation of polynomial ideals and its applications to multivariate approximation theory, ''Algebraic and Combinatorial Problems in Multivariate Approximation Theory, Oberwolfach Institute of Mathematics, Germany, October 21–27, 1990,'' organized by W. Dahmen and A. Dress.
20. R. Q. Jia, S. D. Riemenschneider, and Z. W. Shen, Dimension of kernels of linear operators, *Amer. J. Math.* **114** (1992), 157–184.
21. J. J. Lei and R. Q. Jia, Approximation by piecewise exponentials, *SIAM J. Math. Anal.* **22** (1991), 1776–1789.
22. N. G. Lloyd, ''Degree Theory,'' Cambridge Univ. Press, Cambridge, 1978.
23. V. P. Palamodov, Multiplicity of holomorphic mappings, *Funct. Anal. Appl.* **1** (1967), 218–226.
24. I. R. Shafarevich, ''Basic Algebraic Geometry,'' Springer-Verlag, Berlin, 1974.
25. Z. W. Shen, Dimension of certain kernel spaces of linear operators, *Proc. Amer. Math. Soc.* **112** (1991), 381–390.
26. D. J. A. Welsh, ''Matroid Theory,'' Academic Press, London, 1976.