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Spectral properties of the transition operator associated to a multivariate refinement equation ☆

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Abstract

Given a finitely supported sequence a on $\mathbb{Z}^s$ and an $s \times s$ dilation matrix M, the transition operator T_a is the linear operator defined by $T_a v(\alpha) := \sum_{\beta \in \mathbb{Z}^s} a(M\alpha - \beta) v(\beta)$, where $\alpha \in \mathbb{Z}^s$ and v lies in $\ell_0(\mathbb{Z}^s)$, the linear space of all finitely supported sequences on $\mathbb{Z}^s$. In this paper we investigate the spectral properties of the transition operator T_a and apply these properties to the study of the approximation and smoothness properties of the normalized solution of the refinement equation $\phi = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\phi(M \cdot - \alpha)$. © 1999 Elsevier Science Inc. All rights reserved.

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1. Introduction

The purpose of this paper is to investigate the spectral properties of the transition operator associated to a multivariate refinement equation and their applications to the study of the approximation and smoothness properties of the corresponding refinable function.

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A refinement equation is a functional equation of the form

$$\phi = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\phi(M \cdot - \alpha), \tag{1.1}$$

where a is a finitely supported sequence on $\mathbb{Z}^s$, and M is an $s \times s$ integer matrix such that $\lim_{n\to\infty} M^{-n} = 0$. The matrix M is called a *dilation matrix*, and the sequence a is called the *refinement mask*. Any function satisfying a refinement equation is called a *refinable function*.

If a satisfies

$$\sum_{\alpha \in \mathbb{Z}^s} a(\alpha) = m := |\det M|, \tag{1.2}$$

then it is well known that there exists a unique compactly supported distribution ϕ satisfying the refinement equation (1.1) subject to the condition $\hat{\phi}(0) = 1$. This distribution is said to be *the normalized solution* to the refinement equation with mask a. Throughout this paper we assume that the condition (1.2) is satisfied.

In this paper, the Fourier transform of an integrable function f on $\mathbb{R}^s$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^s} f(x)\mathrm{e}^{-\mathrm{i}x\cdot\xi}\,\mathrm{d}x, \quad \xi \in \mathbb{R}^s,$$

where $x \cdot \xi$ denotes the inner product of two vectors x and ξ in $\mathbb{R}^s$. The domain of the Fourier transform can be naturally extended to include compactly supported distributions.

A multi-index is an s-tuple $\mu = (\mu_1, \ldots, \mu_s)$ with its components being nonnegative integers. Define

$$x^\mu := x_1^{\mu_1} \cdots x_s^{\mu_s} \quad \text{for } x = (x_1, \ldots, x_s) \in \mathbb{R}^s.$$

We may regard $x_1^{\mu_1} \cdots x_s^{\mu_s}$ as a monomial of total degree $|\mu| := \mu_1 + \cdots + \mu_s$. For a nonnegative integer k, let Π_k denote the set of all polynomials of (total) degree at most k. A sequence u on $\mathbb{Z}^s$ is called a polynomial sequence if there exists a polynomial p such that $u(\alpha) = p(\alpha)$ for all $\alpha \in \mathbb{Z}^s$. The degree of u is the same as the degree of p. For $j = 1, \ldots, s$, we use $D_j f$ to denote the partial derivative of f with respect to the jth coordinate. For a multi-index $\mu = (\mu_1, \ldots, \mu_s)$, D^μ stands for the differential operator $D_1^{\mu_1} \cdots D_s^{\mu_s}$. If $p = \sum_\mu c_\mu x^\mu$ is a polynomial, then we use $p(D)$ to denote the differential operator $\sum_\mu c_\mu D^\mu$.

We denote by $\ell(\mathbb{Z}^s)$ the linear space of all sequences on $\mathbb{Z}^s$, and by $\ell_0(\mathbb{Z}^s)$ the linear space of all finitely supported sequences on $\mathbb{Z}^s$. For $\gamma \in \mathbb{Z}^s$, we denote by δ_γ the element in $\ell_0(\mathbb{Z}^s)$ given by $\delta_\gamma(\gamma) = 1$ and $\delta_\gamma(\alpha) = 0$ for all $\alpha \in \mathbb{Z}^s \setminus \{\gamma\}$. In particular, we write δ for δ_0. For $j = 1, \ldots, s$, let e_j be the jth coordinate unit vector. The difference operator ∇_j on $\ell(\mathbb{Z}^s)$ is defined by $\nabla_j u := u - u(\cdot - e_j)$,

$u \in \ell(\mathbb{Z}^s)$. For a multi-index $\mu = (\mu_1, \ldots, \mu_s)$, ∇^μ is the difference operator $\nabla_1^{\mu_1} \cdots \nabla_s^{\mu_s}$.

Let a be an element in $\ell_0(\mathbb{Z}^s)$. The *transition operator* T_a is the linear operator on $\ell_0(\mathbb{Z}^s)$ defined by

$$T_a v(\alpha) := \sum_{\beta \in \mathbb{Z}^s} a(M\alpha - \beta) v(\beta), \qquad \alpha \in \mathbb{Z}^s, \tag{1.3}$$

where $v \in \ell_0(\mathbb{Z}^s)$. The *subdivision operator* S_a is the linear operator on $\ell(\mathbb{Z}^s)$ defined by

$$S_a u(\alpha) := \sum_{\beta \in \mathbb{Z}^s} a(\alpha - M\beta) u(\beta), \qquad \alpha \in \mathbb{Z}^s, \tag{1.4}$$

where $u \in \ell(\mathbb{Z}^s)$. We introduce a bilinear form on the pair of the linear spaces $\ell_0(\mathbb{Z}^s)$ and $\ell(\mathbb{Z}^s)$ as follows:

$$\langle u, v \rangle := \sum_{\alpha \in \mathbb{Z}^s} u(-\alpha) v(\alpha), \quad u \in \ell(\mathbb{Z}^s),\ v \in \ell_0(\mathbb{Z}^s). \tag{1.5}$$

Then $\ell(\mathbb{Z}^s)$ is the dual space of $\ell_0(\mathbb{Z}^s)$ with respect to this bilinear form. It is easily seen that

$$\langle S_a u, v \rangle = \langle u, T_a v \rangle \qquad \forall u \in \ell(\mathbb{Z}^s),\ v \in \ell_0(\mathbb{Z}^s).$$

Hence, S_a is the algebraic adjoint of T_a with respect to the bilinear form given in (1.5).

When the space dimension $s = 1$, Deslauriers and Dubuc [6] discussed the spectral properties of the transition operator and applied those properties to their study of interpolatory subdivision schemes. For the multivariate case ($s > 1$), the subdivision operator was introduced by Cavaretta et al. [3] in their investigation of stationary subdivision schemes. In [9], Goodman et al. established spectral radius formulas for subdivision operators.

In [10], Han and Jia showed that the transition operator T_a has only finitely many nonzero eigenvalues. The following is an outline of the proof. For a bounded subset H of $\mathbb{R}^s$, the set $\sum_{n=1}^\infty M^{-n} H$ is defined as

$$\left\{ \sum_{n=1}^\infty M^{-n} h_n : \ h_n \in H \quad \text{for} \quad n = 1, 2, \ldots \right\}.$$

If H is a compact set, then $\sum_{n=1}^\infty M^{-n} H$ is also compact. By $\operatorname{supp} a$ we denote the set $\{\alpha \in \mathbb{Z}^s : \ a(\alpha) \neq 0\}$. Let

$$\Omega := \left(\sum_{n=1}^\infty M^{-n} (\operatorname{supp} a) \right) \cap \mathbb{Z}^s. \tag{1.6}$$

We use $\ell(\Omega)$ to denote the linear space of all sequences supported in Ω. It is easily seen that $\ell(\Omega)$ is invariant under T_a. Moreover, if v is an eigenvector of T_a corresponding to a nonzero eigenvalue of T_a, then v must lie in $\ell(\Omega)$. Consequently, any nonzero eigenvalue of T_a must be an eigenvalue of the matrix

$$(a(M\alpha - \beta))_{\alpha,\beta\in\Omega}.$$

In particular, T_a has only finitely many nonzero eigenvalues. The spectral radius of T_a, denoted by $\rho(T_a)$, is defined as the spectral radius of the matrix $(a(M\alpha - \beta))_{\alpha,\beta\in\Omega}$.

In [16], using the subdivision and transition operators, Jia investigated the approximation properties of a refinable function in terms of its refinement mask. Let us review some basic results about approximation with shift-invariant spaces.

For a compactly supported distribution ϕ on $\mathbb{R}^s$ and a sequence $c \in \ell(\mathbb{Z}^s)$, the *semi-convolution* of ϕ with c is defined by

$$\phi *' c := \sum_{\alpha\in\mathbb{Z}^s} \phi(\cdot - \alpha)c(\alpha).$$

Let $\mathbb{S}(\phi)$ denote the linear space $\{\phi *' c : c \in \ell(\mathbb{Z}^s)\}$. We call $\mathbb{S}(\phi)$ the *shift-invariant space* generated by ϕ.

A compactly supported distribution ϕ on $\mathbb{R}^s$ is said to have *accuracy* k if $\mathbb{S}(\phi)$ contains Π_{k-1} (see [11]). If ϕ has accuracy k and $\hat{\phi}(0) \neq 0$, then for any polynomial sequence u of degree at most $k-1$, the semi-convolution $\phi *' u$ is a polynomial of the same degree. Conversely, for any $p \in \Pi_{k-1}$, there exists a *unique* polynomial sequence u such that $p = \phi *' u$. See [1, Proposition 1.1] and [15, Lemma 8.2] for these results. Suppose $1 \leqslant p \leqslant \infty$ and ϕ is a compactly supported function in $L_p(\mathbb{R}^s)$ such that $\hat{\phi}(0) \neq 0$. It was proved in [14] that $\mathbb{S}(\phi)$ provides approximation order k if and only if ϕ has accuracy k.

Let ϕ be the normalized solution of the refinement equation (1.1) with mask a and dilation matrix M. We say that a satisfies the *sum rules* of order k if

$$\sum_{\beta\in\mathbb{Z}^s} a(\gamma + M\beta)p(\gamma + M\beta) = \sum_{\beta\in\mathbb{Z}^s} a(M\beta)p(M\beta) \quad \forall p \in \Pi_{k-1} \text{ and } \gamma \in \mathbb{Z}^s.$$

It was proved in [16] that ϕ has accuracy k provided that a satisfies the sum rules of order k. Let

$$V_k := \left\{ v \in \ell_0(\mathbb{Z}^s) : \sum_{\alpha\in\mathbb{Z}^s} p(\alpha)v(\alpha) = 0 \;\forall p \in \Pi_k \right\}.$$

Then a satisfies the sum rules of order k if and only if V_{k-1} is invariant under the transition operator T_a.

In [17], Jia analyzed the smoothness of refinable functions in terms of their masks. For $v > 0$, we denote by $W_2^v(\mathbb{R}^s)$ the Sobolev space of all functions $f \in L_2(\mathbb{R}^s)$ for which

$$\int_{\mathbb{R}^s} |\hat{f}(\xi)|^2 (1 + |\xi|^v)^2 \, \mathrm{d}\xi < \infty.$$

The *smoothness order* $v(f)$ of a function $f \in L_2(\mathbb{R}^s)$ is defined by

$$v(f) := \sup\{v : \ f \in W_2^v(\mathbb{R}^s)\}.$$

Let ϕ be the normalized solution of the refinement equation (1.1) with mask a and dilation matrix M. We assume that M is isotropic, i.e., M is similar to a diagonal matrix $\operatorname{diag}\{\sigma_1, \ldots, \sigma_s\}$ with $|\sigma_1| = \cdots = |\sigma_s|$. Let b be the sequence given by

$$b(\alpha) := \sum_{\beta \in \mathbb{Z}^s} a(\alpha + \beta)\overline{a(\beta)}/m, \qquad \alpha \in \mathbb{Z}^s, \tag{1.7}$$

where $m = |\det M|$ and $\overline{a}$ denotes the complex conjugate of a. Suppose a satisfies the sum rules of order k. Then b satisfies the sum rules of order $2k$. Hence V_{2k-1} is invariant under T_b, the transition operator associated to b. Let ρ_k denote the spectral radius of $T_b|_{V_{2k-1}}$. Suppose ϕ lies in $L_2(\mathbb{R}^s)$. It was proved in [17] that

$$v(\phi) \geqslant -(\log_m \ \rho_k)s/2. \tag{1.8}$$

If, in addition, $k > -(\log_m \rho_k)s/2$ and the shifts of ϕ are stable, then equality holds in (1.8). Note that the shifts of ϕ are stable if and only if, for any $\xi \in \mathbb{R}^s$, there exists an element $\beta \in \mathbb{Z}^s$ such that $\hat{\phi}(\xi + 2\beta\pi) \neq 0$. Moreover, if the shifts of ϕ are linearly independent, that is,

$$\sum_{\alpha \in \mathbb{Z}^s} c(\alpha)\phi(\cdot - \alpha) = 0 \ \Rightarrow \ c(\alpha) = 0 \ \forall \alpha \in \mathbb{Z}^s,$$

then the shifts of ϕ are stable. See [18] for these facts.

The following is an outline of the paper.

Section 2 is devoted to a study of the spectrum of the transition operator. Suppose the dilation matrix M has eigenvalues $\sigma_1, \ldots, \sigma_s$. Write σ for the s-tuple $(\sigma_1, \ldots, \sigma_s)$. By convention, for a multi-index $\mu = (\mu_1, \ldots, \mu_s)$ we have

$$\sigma^\mu := \sigma_1^{\mu_1} \cdots \sigma_s^{\mu_s} \quad \text{and} \quad \sigma^{-\mu} := \sigma_1^{-\mu_1} \cdots \sigma_s^{-\mu_s}.$$

We shall show that the spectrum of the transition operator T_a contains $\{\sigma^{-\mu} : \ |\mu| < k\}$, provided ϕ has accuracy k. This gives an upper bound for the accuracy of ϕ in terms of the refinement mask a.

In Section 3 we shall investigate invariant subspaces of the subdivision and transition operators. We give a necessary and sufficient condition for a subspace of polynomial sequences to be invariant under the subdivision operator. Furthermore, we clarify the relationship among the spectra of the transition operator restricted to different invariant subspaces. In particular, we establish the following formula

$$\operatorname{spec}\Big(T_a|_{\ell(\Omega)}\Big) = \operatorname{spec}\Big(T_a|_{\ell(\Omega)\cap V_{k-1}}\Big) \cup \{\sigma^{-\mu} :\ |\mu| < k\},$$

where Ω is given by (1.6). Thus, the spectral radius of $T_a|_{V_{k-1}}$ can be found from the spectrum of $T_a|_{\ell(\Omega)}$. This result is significant for calculating the smoothness order of a refinable function in terms of its mask.

Section 4 is devoted to a study of the smoothness order of refinable functions which are convolutions of box splines with refinable distributions. Box splines are natural extensions of cardinal B-splines to multidimensional spaces. In the univariate case, a factorization technique can be used to compute the smoothness order of a refinable function by finding the dominant eigenvalue of a certain matrix. In the mutltivariate case, if a refinable function is the convolution of a box spline with a refinable distribution, we will give a method to compute its smoothness order by finding the *dominant* eigenvalues of certain transition matrices.

2. The spectrum of the transition operator

The spectrum of a square matrix A is denoted by $\operatorname{spec}(A)$, and it is understood to be the *multiset* of its eigenvalues. In other words, multiplicities of eigenvalues are counted in the spectrum of a square matrix. The transpose of a matrix A is denoted by A^{T}.

Suppose T is a linear mapping on a finite dimensional vector space V over $\mathbb{C}$. Let $\{v_1, \dots, v_n\}$ be an ordered basis of V. If

$$T(v_i) = b_{1i}v_1 + \cdots + b_{ni}v_n, \quad i = 1, \dots, n,$$

then $(b_{ij})_{1\leqslant i,j\leqslant n}$ is called the matrix representation of T with respect to $\{v_1, \dots, v_n\}$. The spectrum of T is the same as the spectrum of the matrix $(b_{ij})_{1\leqslant i,j\leqslant n}$.

Suppose ϕ is the normalized solution of the refinement equation (1.1) with mask a and dilation matrix M. Let $\operatorname{supp}\phi$ denote the support of ϕ. From (1.1) we observe that $\phi(x) \neq 0$ implies $\phi(Mx - \alpha) \neq 0$ for some $\alpha \in \operatorname{supp} a$. It follows that

$$x \in M^{-1}(\operatorname{supp} a) + M^{-1}(\operatorname{supp}\phi).$$

Hence we have

$$\operatorname{supp}\phi \subseteq M^{-1}(\operatorname{supp} a) + M^{-1}(\operatorname{supp}\phi).$$

A repeated use of the above relation yields

$$\operatorname{supp}\phi \subseteq \sum_{j=1}^{n} M^{-j}(\operatorname{supp} a) + M^{-n}(\operatorname{supp}\phi), \quad n = 1, 2, \ldots.$$

Consequently, we obtain

$$\operatorname{supp}\phi \subseteq \sum_{n=1}^{\infty} M^{-n}(\operatorname{supp} a). \tag{2.1}$$

It follows that $\mathbb{Z}^s \cap \operatorname{supp}\phi \subseteq \Omega$, where Ω is the set given in (1.6).

Let T_a and S_a be the transition operator and the subdivision operator given in (1.3) and (1.4), respectively. It was pointed out that S_a is the algebraic adjoint of T_a with respect to the bilinear form given in (1.5). Let Ω be a nonempty finite subset of $\mathbb{Z}^s$. Suppose $\ell(\Omega)$ is invariant under T_a. By $-\Omega$ we denote the set $\{-\alpha : \alpha \in \Omega\}$. Clearly, $\ell(-\Omega)$ is the dual space of $\ell(\Omega)$ with respect to the bilinear form

$$\langle u, v\rangle_{\Omega} := \sum_{\alpha\in\Omega} u(-\alpha)v(\alpha), \quad u \in \ell(-\Omega),\ v \in \ell(\Omega). \tag{2.2}$$

Let $Q := Q_{\Omega}$ be the linear mapping from $\ell(\mathbb{Z}^s)$ to $\ell(-\Omega)$ given by

$$Q_{\Omega}u(\alpha) = \begin{cases} u(-\alpha) & \text{for } \alpha \in -\Omega, \\ 0 & \text{for } \alpha \notin -\Omega. \end{cases} \tag{2.3}$$

Then QS_a maps $\ell(-\Omega)$ to $\ell(-\Omega)$. We claim that $(QS_a)|_{\ell(-\Omega)}$ is the algebraic adjoint of $T_a|_{\ell(\Omega)}$. Indeed, for $u \in \ell(-\Omega)$ and $v \in \ell(\Omega)$ we have

$$\langle QS_a u, v\rangle_{\Omega} = \langle QS_a u, v\rangle = \langle S_a u, v\rangle = \langle u, T_a v\rangle = \langle u, T_a v\rangle_{\Omega}.$$

This justifies our claim. Consequently, the spectra of $(QS_a)|_{\ell(-\Omega)}$ and $T_a|_{\ell(\Omega)}$ are the same. Moreover, for $u \in \ell(\mathbb{Z}^s)$ and $v \in \ell(\Omega)$ we have

$$\langle QS_a(Qu - u), v\rangle = \langle Qu - u, T_a v\rangle = 0,$$

since $T_a v \in \ell(\Omega)$ and $Qu - u$ vanishes on $\ell(-\Omega)$. Thus, $QS_a(Qu - u)(\alpha) = 0$ for all $\alpha \in -\Omega$. But, by the definition of Q, we have $QS_a(Qu - u)(\alpha) = 0$ for all $\alpha \in \mathbb{Z}^s \setminus (-\Omega)$. This shows $QS_a(Qu - u) = 0$. In other words,

$$QS_a Q = QS_a. \tag{2.4}$$

For $u \in \ell(\mathbb{Z}^s)$, we have

$$\sum_{\alpha\in\mathbb{Z}^s} u(\alpha)\phi(\cdot - \alpha) = \sum_{\alpha\in\mathbb{Z}^s} S_a u(\alpha)\phi(M\,\cdot - \alpha). \tag{2.5}$$

Indeed, since ϕ satisfies the refinement equation (1.1), we obtain

$$\sum_{\alpha\in\mathbb{Z}^s} u(\alpha)\phi(\cdot-\alpha) = \sum_{\alpha\in\mathbb{Z}^s} u(\alpha)\sum_{\beta\in\mathbb{Z}^s} a(\beta)\phi(M\,\cdot - M\alpha-\beta)$$
$$= \sum_{\gamma\in\mathbb{Z}^s} w(\gamma)\phi(M\,\cdot-\gamma),$$

where

$$w(\gamma) = \sum_{\alpha\in\mathbb{Z}^s} a(\gamma - M\alpha)u(\alpha), \qquad \gamma\in\mathbb{Z}^s.$$

Hence $w = S_a u$. This verifies (2.5).

By $K(\phi)$ we denote the linear space given by

$$K(\phi) := \{u\in\ell(\mathbb{Z}^s) : \phi *' u = 0\}.$$

It follows from (2.5) that $K(\phi)$ is invariant under the subdivision operator S_a.

Lemma 2.1. *Let $Q := Q_\Omega$ be the linear mapping from $\ell(\mathbb{Z}^s)$ to $\ell(-\Omega)$ given by* (2.3), *where $\Omega = \mathbb{Z}^s \cap \sum_{n=1}^{\infty} M^{-n}H$ for some compact set $H \supseteq \operatorname{supp} a$. If u is a sequence on $\mathbb{Z}^s$ such that $p := \phi *' u$ is a nonzero polynomial, then $Qu \notin Q(K(\phi))$.*

Proof. Set

$$G_r := \{(x_1,\ldots,x_s)\in\mathbb{R}^s : \ |x_1|+\cdots+|x_s| < r\}, \qquad r>0.$$

By (2.1), the compact set $\operatorname{supp}\phi$ is disjoint from the closed set $\mathbb{Z}^s\setminus\Omega$; hence there exists some $r>0$ such that

$$(\operatorname{supp}\phi + G_r)\cap(\mathbb{Z}^s\setminus\Omega) = \emptyset.$$

Suppose $x\in G_r$ and $\alpha\in\mathbb{Z}^s$. Then $\phi(x+\alpha)\neq 0$ implies $x+\alpha\in\operatorname{supp}\phi$. It follows that $\alpha\in\operatorname{supp}\phi + G_r$. Consequently,

$$\alpha\in\mathbb{Z}^s\cap(\operatorname{supp}\phi+G_r)\subseteq\Omega.$$

In other words, $x\in G_r$ and $\alpha\notin\Omega$ imply $\phi(x+\alpha)=0$. Therefore,

$$p(x) = \sum_{\alpha\in\mathbb{Z}^s} u(\alpha)\phi(x-\alpha)$$
$$= \sum_{\alpha\in\mathbb{Z}^s} u(-\alpha)\phi(x+\alpha)$$
$$= \sum_{\alpha\in\Omega} u(-\alpha)\phi(x+\alpha), \qquad x\in G_r.$$

If $Qu\in Q(K(\phi))$, then there would exist some $w\in K(\phi)$ such that $Qu = Qw$. It follows that $u(-\alpha) = w(-\alpha)$ for all $\alpha\in\Omega$. Thus, for all $x\in G_r$,

$$
\begin{aligned}
p(x) &= \sum_{\alpha\in\Omega} u(-\alpha)\phi(x+\alpha) = \sum_{\alpha\in\Omega} w(-\alpha)\phi(x+\alpha) \\
&= \sum_{\alpha\in\mathbb{Z}^s} w(-\alpha)\phi(x+\alpha) = 0,
\end{aligned}
$$

which is impossible, because p is a nonzero polynomial. This verifies $Qu \notin Q(K(\phi))$. $\square$

We are in a position to establish the main result of this section.

Theorem 2.2. *Let ϕ be the normalized solution of the refinement equation with mask a and dilation matrix M. If ϕ has accuracy k, then the spectrum of the transition operator T_a contains $\{\sigma^{-\mu}: \ |\mu| < k\}$, where $\sigma = (\sigma_1, \dots, \sigma_s)$ is the s-tuple of the eigenvalues of M and $\sigma^{-\mu} := \sigma_1^{-\mu_1}\cdots\sigma_s^{-\mu_s}$ for a multi-index $\mu = (\mu_1, \dots, \mu_s)$.*

Proof. Let Ω be the set given in (1.6), and let $Q := Q_\Omega$ be the linear mapping from $\ell(\mathbb{Z}^s)$ to $\ell(-\Omega)$ as defined in (2.3). Since the spectra of $(QS_a)|_{\ell(-\Omega)}$ and $T_a|_{\ell(\Omega)}$ are the same, it suffices to show that the spectrum of $(QS_a)|_{\ell(-\Omega)}$ contains $\{\sigma^{-\mu}: \ |\mu| < k\}$. For this purpose, we introduce the set

$$W := \{u \in \ell(\mathbb{Z}^s) : \phi *' u \in \Pi_{k-1}\}.$$

By (2.5), W is invariant under S_a. Clearly, $Q(W)$ is a subspace of $\ell(-\Omega)$. The theorem will be proved by finding the matrix representation of QS_a with respect to a suitable basis of $Q(W)$.

There exists an invertible matrix $H = (h_{ij})_{1\leqslant i,j\leqslant s}$ such that HMH^{-1} is a triangular matrix:

$$HMH^{-1} = \begin{bmatrix} \sigma_{11} & & \\ \vdots & \ddots & \\ \sigma_{s1} & \cdots & \sigma_{ss} \end{bmatrix}.$$

For $i = 1, \dots, s$ and $x = (x_1, \dots, x_s) \in \mathbb{R}^s$, let $l_i(x) := h_{i1}x_1 + \cdots + h_{is}x_s$. Then Hx can be represented as $[l_1(x), \dots, l_s(x)]^{\mathrm{T}}$. It follows that

$$\begin{bmatrix} l_1(Mx) \\ \vdots \\ l_s(Mx) \end{bmatrix} = HMx = \begin{bmatrix} \sigma_{11} & & \\ \vdots & \ddots & \\ \sigma_{s1} & \cdots & \sigma_{ss} \end{bmatrix} Hx = \begin{bmatrix} \sigma_{11} & & \\ \vdots & \ddots & \\ \sigma_{s1} & \cdots & \sigma_{ss} \end{bmatrix} \begin{bmatrix} l_1(x) \\ \vdots \\ l_s(x) \end{bmatrix}. \tag{2.6}$$

For simplicity, we write σ_j for σ_{jj}, $j = 1, \dots, s$. Thus, $\sigma_1, \dots, \sigma_s$ are the eigenvalues of the matrix M. For two multi-indices $\mu = (\mu_1, \dots, \mu_s)$ and

$\nu = (\nu_1, \dots, \nu_s)$, we write $\mu \prec \nu$ if there exists some j, $1 \leqslant j \leqslant s$, such that $\mu_j < \nu_j$, and $\mu_{j+1} = \nu_{j+1}, \dots, \mu_s = \nu_s$.

For a multi-index $\mu = (\mu_1, \dots, \mu_s)$, let p_μ be the polynomial given by

$$p_\mu := l_1^{\mu_1} \cdots l_s^{\mu_s}.$$

Clearly, p_μ ($|\mu| < k$) are linearly independent. With the help of (2.6) we obtain

$$p_\mu(Mx) = [l_1(Mx)]^{\mu_1} \cdots [l_s(Mx)]^{\mu_s} = \sigma^\mu p_\mu(x) + q_\mu(x), \qquad x \in \mathbb{R}^s,$$

where q_μ is a linear combination of p_ν with $|\nu| = |\mu|$ and $\nu \prec \mu$. It follows that

$$p_\mu(x) = \sigma^{-\mu} p_\mu(Mx) - \sigma^{-\mu} q_\mu(x).$$

A repeated use of the above relation yields

$$p_\mu(x) = \sigma^{-\mu} p_\mu(Mx) + r_\mu(Mx), \tag{2.7}$$

where r_μ is a linear combination of p_ν with $|\nu| = |\mu|$ and $\nu \prec \mu$.

By the assumption, ϕ has accuracy k. Thus, for each μ with $|\mu| < k$, the polynomial p_μ lies in $\mathbb{S}(\phi)$. Since $\hat{\phi}(0) \neq 0$, there exists a unique polynomial sequence $u_\mu \in \ell(\mathbb{Z}^s)$ such that

$$p_\mu = \sum_{\alpha \in \mathbb{Z}^s} u_\mu(\alpha) \phi(\cdot - \alpha). \tag{2.8}$$

It follows from (2.7) and (2.8) that

$$p_\mu(x) = \sigma^{-\mu} p_\mu(Mx) + r_\mu(Mx) = \sum_{\alpha \in \mathbb{Z}^s} \left[\sigma^{-\mu} u_\mu(\alpha) + v_\mu(\alpha)\right] \phi(Mx - \alpha), \tag{2.9}$$

where v_μ is a linear combination of u_ν with $|\nu| = |\mu|$ and $\nu \prec \mu$. On the other hand, we deduce from (2.5) and (2.8) that

$$p_\mu(x) = \sum_{\alpha \in \mathbb{Z}^s} S_a u_\mu(\alpha) \phi(Mx - \alpha).$$

Comparing this equation with (2.9), we obtain

$$S_a u_\mu = \sigma^{-\mu} u_\mu + v_\mu + w_\mu,$$

where $w_\mu \in K(\phi)$. By (2.4) it follows that

$$QS_a(Qu_\mu) = \sigma^{-\mu}(Qu_\mu) + Qv_\mu + Qw_\mu. \tag{2.10}$$

Let $U := U_0 + \cdots + U_{k-1}$, where each U_j ($j = 0, 1, \dots, k-1$) is the linear span of u_μ, $|\mu| = j$. Then $W = U + K(\phi)$. By Lemma 2.1, $Q(U) \cap Q(K(\phi)) = \{0\}$. Hence $Q(W)$ is the direct sum of $Q(U)$ and $Q(K(\phi))$. Moreover, $Q(U)$ is the direct sum of $Q(U_0), \dots, Q(U_{k-1})$. Choose a basis Y for $Q(K(\phi))$. For each j, the set $Y_j := \{Qu_\mu : |\mu| = j\}$ is a basis for $Q(U_j)$. The order of this basis is arranged in such a way that Qu_ν precedes Qu_μ whenever $\nu \prec \mu$. Consequently, $Y \cup Y_0 \cup \cdots \cup Y_{k-1}$ is a basis for $Q(W)$. With respect to this basis, (2.10) tells us that QS_a has the following matrix representation:

$$\begin{bmatrix} E & F_0 & F_1 & \cdots & F_{k-1} \\ & E_0 & 0 & \cdots & 0 \\ & & E_1 & \cdots & 0 \\ & & & \ddots & \vdots \\ & & & & E_{k-1} \end{bmatrix},$$

where each E_j $(j = 0, \ldots, k-1)$ is a triangular matrix with $\sigma^{-\mu}$ $(|\mu| = j)$ being the entries in its main diagonal. We conclude that the spectrum of $(QS_a)|_{Q(W)}$ contains $\{\sigma^{-\mu} : |\mu| < k\}$, as desired. $\square$

We emphasize that the conclusion of Theorem 2.2 is valid without any assumption of stability of ϕ.

Example 2.3. Let M be the matrix

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

and let a be the sequence on $\mathbb{Z}^2$ such that $a(\alpha) = 0$ for $\alpha \in \mathbb{Z}^2 \setminus [-2,2]^2$ and

$$(a(\alpha_1, \alpha_2))_{-2 \leqslant \alpha_1, \alpha_2 \leqslant 2} = \frac{1}{32} \begin{bmatrix} 0 & -1 & 0 & -1 & 0 \\ -1 & 0 & 10 & 0 & -1 \\ 0 & 10 & 32 & 10 & 0 \\ -1 & 0 & 10 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 \end{bmatrix}.$$

Let ϕ be the normalized solution of the refinement equation (1.1) with mask a and dilation matrix M given as above. Then ϕ has accuracy 4 but does not have accuracy 5.

It can be easily checked that a satisfies the sum rules of order 4. Hence ϕ has accuracy 4. Let us show that ϕ does not have accuracy 5. The matrix M has two eigenvalues $\sigma_1 = 1 + \mathrm{i}$ and $\sigma_2 = 1 - \mathrm{i}$, where i denotes the imaginary unit. We have $\operatorname{supp} a \subseteq [-2,2]^2$ and

$$\sum_{n=1}^{\infty} M^{-n}([-2,2]^2) = \big\{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \leqslant 6,\ |x_2| \leqslant 6,\ |x_1 - x_2| \leqslant 8,$$
$$|x_1 + x_2| \leqslant 8\big\}.$$

The set $\Omega := \mathbb{Z}^2 \cap (\sum_{n=1}^{\infty} M^{-n}([-2,2]^2))$ has exactly 129 points. Among the 129 eigenvalues of the matrix $A := (a(M\alpha - \beta))_{\alpha,\beta\in\Omega}$ the following are of the form $\sigma_1^{-\mu_1}\sigma_2^{-\mu_2}$ for some double-index (μ_1, μ_2) with $\mu_1 + \mu_2 \leqslant 4$:

$$1, \quad 0.5 - 0.5\mathrm{i}, \quad 0.5 + 0.5\mathrm{i}, \quad -0.5\mathrm{i}, \quad 0.5, \quad 0.5\mathrm{i},$$

$$-0.25 - 0.25\mathrm{i}, \quad 0.25 - 0.25\mathrm{i}, \quad 0.25 + 0.25\mathrm{i}, \quad -0.25 + 0.25\mathrm{i},$$

$$-0.25, \quad 0.25\mathrm{i}, \quad -0.25\mathrm{i}.$$

Since ϕ has accuracy 4, we expect that A has eigenvalues $\sigma_1^{-\mu_1}\sigma_2^{-\mu_2}$ for all double-indices (μ_1, μ_2) with $\mu_1 + \mu_2 \leqslant 3$. The above computation confirms our expectation. But A has only three eigenvalues of modulus 0.25. Therefore, by Theorem 2.2, ϕ does not have accuracy 5. $\square$

3. Invariant subspaces of the transition operator

In this section we investigate invariant subspaces of the subdivision and transition operators. We are particularly interested in invariant subspaces of the subdivision operator which consist of polynomial sequences. The results are then applied to smoothness analysis of refinable functions in terms of their masks.

Let Π denote the linear space of all polynomials of s variables. For a compactly supported distribution ϕ on $\mathbb{R}^s$, the intersection $\mathbb{S}(\phi) \cap \Pi$ is not of the form Π_k in general. But $\mathbb{S}(\phi) \cap \Pi$ is always shift-invariant, i.e., $p \in \mathbb{S}(\phi) \cap \Pi$ implies $p(\cdot - \alpha) \in \mathbb{S}(\phi) \cap \Pi$ for all $\alpha \in \mathbb{Z}^s$. It is easily seen that a shift-invariant subspace P of Π is D-invariant, that is, $p \in P$ implies all its partial derivatives belong to P.

Suppose ϕ is a compactly supported distribution on $\mathbb{R}^s$ such that $\hat{\phi}(0) \neq 0$. Let P be a finite dimensional D-invariant subspace of Π. Then $P \subset \mathbb{S}(\phi)$ if and only if

$$p(-iD)\hat{\phi}(2\pi\beta) = 0 \qquad \forall p \in P \text{ and } \beta \in \mathbb{Z}^s \setminus \{0\}.$$

Suppose $P \subset \mathbb{S}(\phi)$. Then $u \in P|_{\mathbb{Z}^s}$ implies $p := \phi *' u$ lies in P. Conversely, for each $p \in P$, there exists a unique polynomial sequence $u \in P|_{\mathbb{Z}^s}$ such that $p = \phi *' u$. See [1] for these results.

Now let ϕ be the normalized solution of the refinement equation with mask a and dilation matrix M, where $\sum_{\alpha\in\mathbb{Z}^s} a(\alpha) = m = |\det M|$. Let Γ be a complete set of representatives of the distinct cosets of $\mathbb{Z}^s/M\mathbb{Z}^s$, and let Θ be a complete set of representatives of the distinct cosets of $\mathbb{Z}^s/M^T\mathbb{Z}^s$. Recall that a satisfies the sum rules of order k implies ϕ has accuracy k. The converse of this statement is valid under the additional condition that

$$N(\phi) \cap (2\pi(M^T)^{-1}\Theta) = \emptyset, \tag{3.1}$$

where

$$N(\phi) := \{\xi \in \mathbb{R}^s : \ \hat{\phi}(\xi + 2\pi\beta) = 0 \ \forall \beta \in \mathbb{Z}^s\}.$$

These results can be extended to shift-invariant subspaces of Π. Let P be a finite dimensional shift-invariant subspace of Π. If

$$\sum_{\beta\in\mathbb{Z}^s} a(\gamma + M\beta)p(-\gamma - M\beta) = \sum_{\beta\in\mathbb{Z}^s} a(M\beta)p(-M\beta) \qquad \forall p \in P \ \text{ and } \ \gamma \in \Gamma, \tag{3.2}$$

then $P \subset \mathbb{S}(\phi)$. Conversely, if $P \subset \mathbb{S}(\phi)$ and (3.1) is valid, then a satisfies the conditions in (3.2). The proof is similar to the one given in [16].

It was proved in [16] that a satisfies the sum rules of order k if and only if $\Pi_{k-1}|_{\mathbb{Z}^s}$ is invariant under the subdivision operator S_a. In order to extend this result to shift-invariant subspaces of Π, additional work is needed.

Theorem 3.1. *Let M be a dilation matrix, and let a be an element in $\ell_0(\mathbb{Z}^s)$ such that $\sum_{\alpha\in\mathbb{Z}^s} a(\alpha) = m = |\det M|$. Suppose P is a finite dimensional shift-invariant subspace of Π. Then $P|_{\mathbb{Z}^s}$ is invariant under S_a if and only if a satisfies the conditions in* (3.2) *and $p \in P$ implies $p(M^{-1}\cdot) \in P$.*

Proof. Suppose $U := P|_{\mathbb{Z}^s}$ is invariant under S_a. Let us first show that $S_a|_U$ is one-to-one. For this purpose, choose an element u in U such that $S_a u = 0$. Then

$$\sum_{\beta\in\mathbb{Z}^s} a(\alpha - M\beta)u(\beta) = 0 \quad \forall \alpha \in \mathbb{Z}^s. \tag{3.3}$$

Suppose $u \neq 0$. Since u is a polynomial sequence, there exists a multi-index μ and a complex number $c \neq 0$ such that $\nabla^\mu u(\beta) = c$ for all $\beta \in \mathbb{Z}^s$. It follows from (3.3) that

$$\sum_{\beta\in\mathbb{Z}^s} a(\alpha - M\beta)\nabla^\mu u(\beta) = 0 \quad \forall \alpha \in \mathbb{Z}^s.$$

Hence $\sum_{\beta\in\mathbb{Z}^s} a(\alpha - M\beta) = 0$ for all $\alpha \in \mathbb{Z}^s$. This contradicts the assumption that $\sum_{\alpha\in\mathbb{Z}^s} a(\alpha) = m \neq 0$. Therefore, $S_a|_U$ is one-to-one. But U is finite dimensional. Hence $S_a|_U$ is one-to-one and onto.

Next, we show that $p \in P$ implies $p(M\cdot) \in P$. Let $p \in P$. Since $S_a|_U$ is onto, there exists $f \in P$ such that $p|_{\mathbb{Z}^s} = S_a(f|_{\mathbb{Z}^s})$, that is, $p(\alpha) = \sum_{\beta\in\mathbb{Z}^s} a(\alpha - M\beta)f(\beta)$ for all $\alpha \in \mathbb{Z}^s$. It follows that

$$p(M\alpha) = \sum_{\beta\in\mathbb{Z}^s} a(M\alpha - M\beta)f(\beta) = \sum_{\beta\in\mathbb{Z}^s} a(M\beta)f(\alpha - \beta) \qquad \forall \alpha \in \mathbb{Z}^s.$$

Let $q(x) := \sum_{\beta\in\mathbb{Z}^s} a(M\beta)f(x-\beta)$, $x \in \mathbb{R}^s$. Since P is shift-invariant, q belongs to P. Thus, q and $p(M\cdot)$ agree on the lattice $\mathbb{Z}^s$. Therefore, we have $p(M\cdot) = q \in P$.

For $p \in P$ let

$$u(\gamma) := \sum_{\beta\in\mathbb{Z}^s} a(M\beta+\gamma)p(-M\beta-\gamma), \qquad \gamma \in \mathbb{Z}^s.$$

We claim that u is a polynomial sequence. Indeed, by using Taylor's formula, we obtain

$$p(-M\beta-\gamma) = \sum_{\mu} t_\mu(-M\beta)(-\gamma)^\mu,$$

where $t_\mu := D^\mu p/\mu!$. Since P is D-invariant, $t_\mu \in P$ for every multi-index μ. For $x \in \mathbb{R}^s$, set $q_\mu(x) := t_\mu(Mx)$. By what has been proved, we have $q_\mu \in P$. Let $u_\mu := q_\mu|_{\mathbb{Z}^s}$. Then for $\gamma \in \mathbb{Z}^s$,

$$\begin{aligned} u(\gamma) &= \sum_{\beta\in\mathbb{Z}^s} a(M\beta+\gamma)p(-M\beta-\gamma) \\ &= \sum_{\beta\in\mathbb{Z}^s}\sum_{\mu} a(\gamma+M\beta)\,u_\mu(-\beta)(-\gamma)^\mu = \sum_{\mu} S_a u_\mu(\gamma)\,(-\gamma)^\mu. \end{aligned}$$

Since U is invariant under S_a, $S_a u_\mu \in U$. Hence u is a polynomial sequence. By the definition of u, we have $u(\gamma+M\eta) = u(\gamma)$ for all $\eta \in \mathbb{Z}^s$ and $\gamma \in \mathbb{Z}^s$. In other words, u is a constant sequence on the lattice $\gamma + M\mathbb{Z}^s$ for each $\gamma \in \mathbb{Z}^s$. Therefore, u itself must be a constant sequence. This shows that a satisfies the conditions in (3.2). Consequently, $P \subset \mathbb{S}(\phi)$, where ϕ is the normalized solution of the refinement equation with mask a and dilation matrix M.

It remains to prove that $p \in P$ implies $p(M^{-1}\cdot) \in P$. Let $p \in P$. Then there exists a unique $u \in U$ such that $p = \phi *' u$. From (2.5) we deduce that

$$p(M^{-1}x) = \sum_{\alpha\in\mathbb{Z}^s} u(\alpha)\phi(M^{-1}x-\alpha) = \sum_{\alpha\in\mathbb{Z}^s} S_a u(\alpha)\phi(x-\alpha).$$

Since U is invariant under S_a, we have $S_a u \in U$. This shows $p(M^{-1}\cdot) \in P$.

Now suppose a satisfies the conditions in (3.2) and $p \in P$ implies $p(M^{-1}\cdot) \in P$. We wish to show that $U = P|_{\mathbb{Z}^s}$ is invariant under S_a. Let $p \in P$ and $u = p|_{\mathbb{Z}^s}$. We first show that $S_a u$ is a polynomial sequence. Set $q(x) := p(M^{-1}x)$, $x \in \mathbb{R}^s$. By our assumption, $q \in P$. An application of Taylor's formula gives

$$q(M\beta) = q(-\alpha+M\beta+\alpha) = \sum_{\mu} q_\mu(-\alpha+M\beta)\alpha^\mu,$$

where $q_\mu = D^\mu p/\mu!$. Since P is D-invariant, we have $q_\mu \in P$ for all multi-indices μ. It follows that

$$S_a u(\alpha) = \sum_{\beta\in\mathbb{Z}^s} a(\alpha - M\beta)p(\beta) = \sum_{\beta\in\mathbb{Z}^s} a(\alpha - M\beta)q(M\beta)$$

$$= \sum_{\mu} \left[\sum_{\beta\in\mathbb{Z}^s} a(\alpha - M\beta)q_\mu(-\alpha + M\beta) \right] \alpha^\mu.$$

Since a satisfies the conditions in (3.2), $c_\mu := \sum_{\beta\in\mathbb{Z}^s} a(\alpha - M\beta)q_\mu(\alpha - M\beta)$ is independent of α. Therefore, $S_a u(\alpha) = \sum_\mu c_\mu \alpha^\mu$ for all $\alpha \in \mathbb{Z}^s$. This shows that $S_a u$ is a polynomial sequence.

To finish the proof, we observe that

$$S_a u(M\gamma) = \sum_{\beta\in\mathbb{Z}^s} a(M(\gamma - \beta))u(\beta) = \sum_{\beta\in\mathbb{Z}^s} a(M\beta)p(\gamma - \beta), \qquad \gamma \in \mathbb{Z}^s.$$

Since P is shift-invariant, there exists $f \in P$ such that

$$\sum_{\beta\in\mathbb{Z}^s} a(M\beta)p(\gamma - \beta) = f(\gamma) \qquad \forall\gamma \in \mathbb{Z}^s.$$

Let $g(x) := f(M^{-1}x)$, $x \in \mathbb{R}^s$. Then $g \in P$ and

$$S_a u(M\gamma) = f(\gamma) = g(M\gamma) \qquad \forall\gamma \in \mathbb{Z}^s.$$

This shows that $S_a u$ and g agree on the lattice $M\mathbb{Z}^s$. But both $S_a u$ and $g|_{\mathbb{Z}^s}$ are polynomial sequences. Therefore, $S_a u = g|_{\mathbb{Z}^s} \in U$. We conclude that U is invariant under S_a. $\square$

The following theorem clarifies the relationship among the spectra of the transition operator restricted to different invariant subspaces.

Theorem 3.2. *Let U be a finite dimensional subspace of $\ell(\mathbb{Z}^s)$, and let*

$$V := \left\{ v \in \ell_0(\mathbb{Z}^s) : \sum_{\alpha\in\mathbb{Z}^s} u(-\alpha)v(\alpha) = 0 \; \forall u \in U \right\}. \tag{3.4}$$

Then U is invariant under the subdivision operator S_a if and only if V is invariant under the transition operator T_a. Let Ω be a finite subset of $\mathbb{Z}^s$ such that $\ell(\Omega)$ is invariant under T_a, and let $Q := Q_\Omega$ be the linear mapping from $\ell(\mathbb{Z}^s)$ to $\ell(-\Omega)$ as defined in (2.3). *If U is invariant under S_a, and if $Q|_U$ is one-to-one, then*

$$\operatorname{spec}\Big(T_a|_{\ell(\Omega)}\Big) = \operatorname{spec}\Big(T_a|_{\ell(\Omega)\cap V}\Big) \cup \operatorname{spec}\big(S_a|_U\big). \tag{3.5}$$

In particular, the above relation is valid when $\Omega = \mathbb{Z}^s \cap \sum_{n=1}^\infty M^{-n}H$ for some compact set $H \supset \operatorname{supp} a$ and $U = P|_{\mathbb{Z}^s}$ for some finite dimensional shift-invariant subspace P of Π .

Proof. Let $\langle u, v\rangle$ be the bilinear form defined in (1.5). Then $v \in V$ if and only if $\langle u, v\rangle = 0$ for all $u \in U$. Suppose U is invariant under S_a. Then for $v \in V$ we have

$$\langle u, T_a v\rangle = \langle S_a u, v\rangle = 0 \qquad \forall u \in U.$$

Hence $v \in V$ implies $T_a v \in V$. This shows that V is invariant under T_a.

Choose a basis $\{u_1, \ldots, u_n\}$ for U. Then there exist $v_1, \ldots, v_n \in \ell_0(\mathbb{Z}^s)$ such that $\langle u_j, v_k\rangle = \delta_{jk}$ for $j, k = 1, \ldots, n$, where δ_{jk} stands for the Kronecker sign. It is easily seen that $\ell_0(\mathbb{Z}^s)$ is the direct sum of V and the linear span of $v_1, \ldots, v_n$. Suppose V is invariant under T_a. We wish to show that U is invariant under S_a. Let $u \in U$ and $w = S_a u$. Then

$$\langle w, v\rangle = \langle S_a u, v\rangle = \langle u, T_a v\rangle = 0 \qquad \forall v \in V.$$

Moreover, with $c_j := \langle w, v_j\rangle$, $j = 1, \ldots, n$, we have

$$\langle w - (c_1 u_1 + \cdots + c_n u_n), v_j\rangle = 0 \qquad \forall j = 1, \ldots, n.$$

It follows that $\langle w - (c_1 u_1 + \cdots + c_n u_n), y\rangle = 0$ for all $y \in \ell_0(\mathbb{Z}^s)$. This shows that $w = c_1 u_1 + \cdots + c_n u_n \in U$. In other words, U is invariant under S_a. This proves the first statement of the theorem.

Now suppose U is invariant under S_a. Choose a basis $\{u_1, \ldots, u_r\}$ for U. Since $Q|_U$ is one-to-one, $\{Qu_1, \ldots, Qu_r\}$ is a basis for $Q(U)$. We supplement elements $u_{r+1}, \ldots, u_n$ in $\ell(-\Omega)$ such that $\{Qu_1, \ldots, Qu_r, u_{r+1}, \ldots, u_n\}$ forms a basis for $\ell(-\Omega)$. Clearly, $Qu_j = u_j$ for $j = r+1, \ldots, n$. Suppose

$$QS_a(Qu_j) = \sum_{k=1}^{n} b_{jk}(Qu_k), \qquad j = 1, \ldots, n. \tag{3.6}$$

Let $B := (b_{jk})_{1 \leqslant j,k \leqslant n}$. Then B^{T}, the transpose of B, is the matrix of the linear mapping $(QS_a)|_{\ell(-\Omega)}$ with respect to the basis $\{Qu_1, \ldots, Qu_n\}$. Since U is invariant under S_a, $Q(U)$ is invariant under QS_a in light of (2.4). Therefore, $b_{jk} = 0$ for $j = 1, \ldots, r$ and $k = r+1, \ldots, n$. In other words, B is a block triangular matrix:

$$B = \begin{bmatrix} E & 0 \\ G & F \end{bmatrix},$$

where $E = (b_{jk})_{1 \leqslant j,k \leqslant r}$ and $F = (b_{jk})_{r+1 \leqslant j,k \leqslant n}$. Since $\ell(\Omega)$ is invariant under T_a, by (2.4) we have $QS_a Q = QS_a$. By our assumption, $Q|_U$ is one-to-one. Thus, it follows from (3.6) that

$$S_a u_j = \sum_{k=1}^{r} b_{jk} u_k, \qquad j = 1, \ldots, r.$$

Therefore, E^{T} is the matrix of $S_a|_U$ with respect to the basis $\{u_1, \ldots, u_r\}$.

Note that $\ell(-\Omega)$ is the dual space of $\ell(\Omega)$ with respect to the bilinear form $\langle u, v\rangle_\Omega$ defined in (2.2). Let $\{v_1, \ldots, v_n\}$ be the basis of $\ell(\Omega)$ dual to $\{Qu_1, \ldots, Qu_n\}$, that is,

$$\langle Qu_j, v_k\rangle = \delta_{jk} \qquad \text{for } j, k = 1, \ldots, n.$$

Clearly, $\{v_{r+1}, \ldots, v_n\}$ is a basis for $\ell(\Omega) \cap V$. It was proved in Section 2 that $(QS_a)|_{\ell(-\Omega)}$ is the adjoint of $T_a|_{\ell(\Omega)}$ with respect to the bilinear form $\langle u, v\rangle_\Omega$. Consequently, by (3.6) we have

$$\langle Qu_j, T_a v_k\rangle_\Omega = \langle QS_a Qu_j, v_k\rangle_\Omega = b_{jk}, \qquad j, k = 1, \ldots, n.$$

This shows that $B = (b_{jk})_{1 \leqslant j,k \leqslant n}$ is the matrix of $T_a|_{\ell(\Omega)}$ with respect to the basis $\{v_1, \ldots, v_n\}$. But $\{v_{r+1}, \ldots, v_n\}$ is a basis for $\ell(\Omega) \cap V$. Hence $F = (b_{jk})_{r+1 \leqslant j,k \leqslant n}$ is the matrix of $T_a|_{\ell(\Omega)\cap V}$ with respect to this basis. To summarize, we obtain

$$\begin{aligned}\operatorname{spec}(T_a|_{\ell(\Omega)}) &= \operatorname{spec}(B) = \operatorname{spec}(E) \cup \operatorname{spec}(F)\\ &= \operatorname{spec}\big(S_a|_U\big) \cup \operatorname{spec}\Big(T_a|_{\ell(\Omega)\cap V}\Big).\end{aligned}$$

This verifies (3.5).

Finally, suppose $U = P|_{\mathbb{Z}^s}$ for some shift-invariant subspace P of Π and U is invariant under S_a. Theorem 3.2 tells us that $P \subset \mathbb{S}(\phi)$, where ϕ is the normalized solution of the refinement equation (1.1) with mask a and dilation matrix M. Suppose that $\Omega = \mathbb{Z}^s \cap \sum_{n=1}^\infty M^{-n}H$ for some compact set $H \supset \operatorname{supp} a$. Let $u \in U$ and $p := \phi *' u$. If $u \neq 0$, then $p \neq 0$; hence $Qu \neq 0$ by Lemma 2.1. This shows that $Q|_U$ is one-to-one. Therefore, (3.5) is valid for this case. $\square$

The case $U = \Pi_{k-1}|_{\mathbb{Z}^s}$ is of particular interest. Suppose a satisfies the sum rules of order k. Then U is invariant under S_a, by Theorem 3.2. By Theorem 2.2 we have $\operatorname{spec}(S_a|_U) = \{\sigma^{-\mu} : |\mu| < k\}$. Thus, (3.5) reads as follows:

$$\operatorname{spec}\Big(T_a|_{\ell(\Omega)}\Big) = \operatorname{spec}\Big(T_a|_{\ell(\Omega)\cap V_{k-1}}\Big) \cup \{\sigma^{-\mu} : \ |\mu| < k\}. \tag{3.7}$$

For the univariate case ($s = 1$), this formula was established by Deslauriers and Dubuc in [6, Theorem 8.2.]

Theorem 3.2 has useful applications to smoothness analysis of refinable functions in terms of their masks.

Let ϕ be the normalized solution of the refinement equation (1.1) with a mask a and an isotropic dilation matrix M. Let b be the sequence given by (1.7). Suppose a satisfies the sum rules of order k. Then b satisfies the sum rules of order $2k$. Hence V_{2k-1} is invariant under the transition operator T_b. Let ρ_k denote the spectral radius of $T_b|_{V_{2k-1}}$. It follows from (3.7) that

$$\mathrm{spec}\Big(T_b|_{\ell(\Omega)\cap V_1}\Big) = \mathrm{spec}\Big(T_b|_{\ell(\Omega)\cap V_{2k-1}}\Big) \cup \{\sigma^{-\mu} : 2 \leqslant |\mu| < 2k\},$$

where $\Omega := \mathbb{Z}^s \cap \sum_{n=1}^{\infty} M^{-n}(\mathrm{supp}\, b)$. If $\rho_k < 1$, then the above relation tells us that $\rho_1 < 1$, which implies that the subdivision scheme associated to mask a and dilation matrix M converges in the L_2-norm (see [10]). In particular, $\rho_k < 1$ implies $\phi \in L_2(\mathbb{R}^s)$.

We can find ρ_k from $\mathrm{spec}(T_b|_{\ell(\Omega)})$ by using the following formula:

$$\mathrm{spec}\Big(T_b|_{\ell(\Omega)}\Big) = \mathrm{spec}\Big(T_b|_{\ell(\Omega)\cap V_{2k-1}}\Big) \cup \{\sigma^{-\mu} : |\mu| < 2k\}.$$

The following example illustrates this technique.

Example 3.3. Let M be the matrix

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

and let a be the mask given in Example 2.3. Let us determine the smoothness order of the normalized solution ϕ of the refinement equation with mask a and dilation matrix M.

Let b be the mask computed from a by using (1.7). Then $\mathrm{supp}\, b \subseteq [-4,4]^2$ and the set $\sum_{n=1}^{\infty} M^{-n}([-4,4]^2)$ is

$$\{(x_1,x_2) \in \mathbb{R}^2 : |x_1| \leqslant 12,\ |x_2| \leqslant 12,\ |x_1 - x_2| \leqslant 16,\ |x_1 + x_2| \leqslant 16\}.$$

The set $\Omega := \mathbb{Z}^2 \cap \Big(\sum_{n=1}^{\infty} M^{-n}([-4,4]^2)\Big)$ has exactly 481 points. We use MATLAB to compute the eigenvalues of the matrix $(b(M\alpha - \beta))_{\alpha,\beta\in\Omega}$. These eigenvalues are arranged in the order of descending absolute values. The following is a list of the first 22 eigenvalues.

$$1,\quad 0.5+0.5\mathrm{i},\quad 0.5-0.5\mathrm{i},\quad 0.5,\quad 0.5\mathrm{i},\quad -0.5\mathrm{i},$$

$$-0.25+0.25\mathrm{i},\quad -0.25-0.25\mathrm{i},\quad 0.25+0.25\mathrm{i},\quad 0.25-0.25\mathrm{i},$$

$$-0.25,\quad -0.25,\quad 0.25\mathrm{i},\quad -0.25\mathrm{i},\quad 0.25,$$

$$0.1832744177,\quad 0.125+0.125\mathrm{i},\quad 0.125-0.125\mathrm{i},$$

$$-0.125+0.125\mathrm{i},\quad -0.125-0.125\mathrm{i},\quad -0.125+0.125\mathrm{i},$$
$$-0.125-0.125\mathrm{i}.$$

Note that the matrix M has two eigenvalues $\sigma_1 = 1+i$ and $\sigma_2 = 1-i$. In the above list, 21 eigenvalues are of the form $\sigma_1^{-\mu_1}\sigma_2^{-\mu_2}$ for double indices (μ_1,μ_2) with $\mu_1+\mu_2 \leqslant 5$. Therefore, $\rho_4 \approx 0.1832744177$. By (1.8) we obtain

$$\nu(\phi) \geqslant -\log_2 0.1832744178 \approx 2.44792267.$$

From the results in [10] we know that the subdivision scheme associated to mask a and dilation matrix M converges uniformly. Moreover, the mask a is interpolatory, i.e., $a(0) = 1$ and $a(M\alpha) = 0$ for $\alpha \in \mathbb{Z}^s \setminus \{0\}$. Consequently, ϕ is continuous, $\phi(0) = 1$, and $\phi(\alpha) = 0$ for $\alpha \in \mathbb{Z}^s \setminus \{0\}$. Hence, the shifts of ϕ are linearly independent. We conclude that $\nu(\phi) \approx 2.44792267$. □

4. Refinable functions induced by box splines

In the univariate case, a factorization technique can be used to compute the smoothness order (regularity) of a refinable function by finding the dominant eigenvalue of a certain matrix. In this regard, the reader is referred to the work of Daubechies and Lagarias [5], Eirola [8], and Villemoes [20].

In the multivariate case, if a refinable function is the convolution of a box spline with a refinable distribution, then it is still possible to compute its smoothness order by finding the dominant eigenvalues of certain transition matrices.

For an element $a \in \ell_0(\mathbb{Z}^s)$ we use $\tilde{a}(z)$ to denote its symbol:

$$\tilde{a}(z) := \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)z^\alpha, \qquad z \in (\mathbb{C} \setminus \{0\})^s.$$

The convolution of two sequences a and b in $\ell_0(\mathbb{Z}^s)$ is defined by

$$a * b(\alpha) := \sum_{\beta\in\mathbb{Z}^s} a(\alpha-\beta)b(\beta), \qquad \alpha \in \mathbb{Z}^s.$$

If $c = a * b$, then

$$\tilde{c}(z) = \tilde{a}(z)\tilde{b}(z), \qquad z \in (\mathbb{C} \setminus \{0\})^s.$$

For $r = 1, 2, \ldots$, let a_r be the element in $\ell_0(\mathbb{Z})$ defined by its symbol:

$$\tilde{a}_r(z) = (1 + z)^r / 2^{r-1}.$$

The cardinal B-spline B_r of order r can be viewed as the normalized solution of the refinement equation $\phi = \sum_{\alpha\in\mathbb{Z}} a_r(\alpha)\phi(2 \cdot -\alpha)$.

Box splines are natural extensions of cardinal B-splines. The reader is referred to the monograph [2] by de Boor et al. for a comprehensive study of box splines.

In this section we are particularly interested in box splines on the three-direction mesh on $\mathbb{R}^2$. For $r, s, t \geqslant 1$, let $a_{r,s,t}$ be the element in $\ell_0(\mathbb{Z}^2)$ defined by its symbol:

$$\tilde{a}_{r,s,t}(z_1, z_2) := (1 + z_1)^r (1 + z_2)^s (1 + z_1 z_2)^t / 2^{r+s+t-2}, \qquad (z_1, z_2) \in \mathbb{C}^2.$$

The box spline $B_{r,s,t}$ is defined as the normalized solution of the refinement equation

$$\phi = \sum_{\alpha\in\mathbb{Z}^2} a_{r,s,t}(\alpha)\phi(2\cdot -\alpha).$$

In [7], Dyn et al. analyzed convergence of the so-called butterfly scheme which is induced by the box spline $B_{1,1,1}$. More generally, using convolutions of box splines with distributions, Riemenschneider and Shen [19] constructed a family of bivariate interpolatory subdivision schemes with symmetry.

The following theorem provides a method to simplify the computation of the smoothness order of refinable functions which are convolutions of box splines $B_{r,r,r}$ with refinable distributions. In what follows we use $\mathbb{T}^2$ to denote the torus

$$\{(z_1,z_2)\in\mathbb{C}^2 : |z_1| = 1, |z_2| = 1\}.$$

Theorem 4.1. *Let c be an element in $\ell_0(\mathbb{Z}^2)$ such that $\sum_{\alpha\in\mathbb{Z}^2} c(\alpha) = 4$, and let a be given by its symbol*

$$\tilde{a}(z) = \left(\frac{1+z_1}{2}\right)^r\left(\frac{1+z_2}{2}\right)^r\left(\frac{1+z_1z_2}{2}\right)^r\tilde{c}(z), \qquad z=(z_1,z_2)\in\mathbb{T}^2,$$

where r is a positive integer. Let ϕ be the normalized solution of the refinement equation $\phi = \sum_{\alpha\in\mathbb{Z}^2} a(\alpha)\phi(2\cdot -\alpha)$. Write z_3 for z_1z_2. Let a_j $(j = 1,2,3)$ *be given by*

$$\tilde{a}_j(z) = \left(\frac{1+z_j}{2}\right)^r\tilde{c}(z), \qquad z\in\mathbb{T}^2,$$

and let b_j $(j = 1,2,3)$ be given by $\tilde{b}_j(z) = |a_j(z)|^2/4$, $z\in\mathbb{T}^2$. Let $\rho := \max_{1\leqslant j\leqslant 3}\{\rho(T_{b_j})\}$. If $\rho > 1$ and if the shifts of ϕ are stable, then

$$v(\phi) = 2r - \log_4\rho. \tag{4.1}$$

Proof. Let $b\in\ell_0(\mathbb{Z}^2)$ be given by

$$\tilde{b}(z) = |\tilde{a}(z)|^2/4, \qquad z\in\mathbb{T}^2,$$

and let f be the normalized solution to the refinement equation with mask b. Then $\hat{f}(\xi) = |\hat{\phi}(\xi)|^2$ for all $\xi\in\mathbb{R}^s$. Thus, if the shifts of ϕ are stable, then so are the shifts of f.

Let $\mathbb{P}_{r,s,t} := \Pi\cap\mathbb{S}(B_{r,s,t})$. It is known (see, e.g., [2]) that

$$\mathbb{P}_{r,s,t} = \left\{p\in\Pi : D_1^rD_2^sp = 0,\, D_1^r(D_1+D_2)^tp = 0,\, D_2^s(D_1+D_2)^tp = 0\right\}.$$

In particular, $\mathbb{P}_{r,s,t}\subseteq\Pi_{r+s+t-2}$.

For $j = 1, 2, 3$, we use Δ_j to denote the difference operator on $\ell_0(\mathbb{Z}^2)$ given by

$$\Delta_j v := -v(\cdot - e_j) + 2v - v(\cdot + e_j), \qquad v \in \ell_0(\mathbb{Z}^2),$$

where $e_1 = (1,0)$, $e_2 = (0,1)$, and $e_3 = (1,1)$. Let V be the linear span of $\Delta_1^r \Delta_2^r \delta_\beta$, $\Delta_2^r \Delta_3^r \delta_\beta$, and $\Delta_1^r \Delta_3^r \delta_\beta$, $\beta \in \mathbb{Z}^2$, and let

$$U := \{ u \in \ell(\mathbb{Z}^2) : \langle u, v \rangle = 0 \quad \forall v \in V \},$$

where $\langle u, v \rangle$ is the bilinear form given in (1.5). Then u belongs to U if and only if u satisfies the following system of partial difference equations:

$$\Delta_1^r \Delta_2^r u = 0, \quad \Delta_1^r \Delta_3^r u = 0, \quad \Delta_2^r \Delta_3^r u = 0.$$

By [4, Proposition 2.1] we have $U = \mathbb{P}_{2r,2r,2r}|_{\mathbb{Z}^2}$. Also see [12, §5] for properties of partial difference equations associated to box splines. Note that $\ell(\mathbb{Z}^2)$ is the dual space of $\ell_0(\mathbb{Z}^2)$ with respect to the bilinear form $\langle u, v \rangle$. Suppose $w \in \ell_0(\mathbb{Z}^2) \setminus V$. Then there exists an element $u \in \ell(\mathbb{Z}^2)$ such that $\langle u, w \rangle = 1$ and $\langle u, v \rangle = 0$ for all $v \in V$. This shows

$$V = \{ v \in \ell_0(\mathbb{Z}^2) : \langle u, v \rangle = 0 \ \forall u \in U \}.$$

Since U is invariant under the subdivision operator S_b, V is invariant under the transition operator T_b, by Theorem 3.2. Let $U_k := \Pi_k|_{\mathbb{Z}^2}$. Then we have

$$V_k = \{ v \in \ell_0(\mathbb{Z}^2) : \langle u, v \rangle = 0 \ \forall u \in U_k \}.$$

Consequently, $U_{4r-1} \subseteq U$ and $V \subseteq V_{4r-1}$.

We observe that $\sum_{n=1}^{\infty} 2^{-n} \operatorname{supp} b$ is contained in the convex hull of $\operatorname{supp} b$. Let Ω be the intersection of $\mathbb{Z}^2$ with the convex hull of $\operatorname{supp} b$. By Theorem 3.2 we have

$$\operatorname{spec}\Big(T_b|_{\ell(\Omega)} \Big) = \operatorname{spec}\Big(T_b|_{\ell(\Omega) \cap V} \Big) \cup \operatorname{spec}\big(S_b|_U \big) \tag{4.2}$$

and

$$\operatorname{spec}\Big(T_b|_{\ell(\Omega)} \Big) = \operatorname{spec}\Big(T_b|_{\ell(\Omega) \cap V_{4r-1}} \Big) \cup \operatorname{spec}\Big(S_b|_{U_{4r-1}} \Big). \tag{4.3}$$

Let us find the difference between $\operatorname{spec}(S_b|_U)$ and $\operatorname{spec}(S_b|_{U_{4r-1}})$.

For $j = 0, 1, \ldots$, by $\mathbb{H}_j$ we denote the linear space of *homogeneous* polynomials of degree j. Let

$$E_j := \{ u \in U : f *' u \in \mathbb{H}_j \}.$$

Then U is the direct sum of E_j, $j = 0, 1, \ldots, 6r-2$. Let $u \in E_j$ and $p := f *' u \in \mathbb{H}_j$. Since $f = \sum_{\alpha \in \mathbb{Z}^2} b(\alpha) f(2 \cdot - \alpha)$, by (2.5) we obtain

$$p(x) = \sum_{\alpha\in\mathbb{Z}^2} u(\alpha)f(x-\alpha) = \sum_{\alpha\in\mathbb{Z}^2} S_b u(\alpha)f(2x-\alpha), \qquad x \in \mathbb{R}^2.$$

On the other hand, since p is a homogeneous polynomial of degree j, we have

$$p(x) = 2^{-j}p(2x) = \sum_{\alpha\in\mathbb{Z}^2} 2^{-j}u(\alpha)f(2x-\alpha), \qquad x \in \mathbb{R}^2.$$

But the shifts of f are stable. Thus, the above two equations yield $S_b u = 2^{-j}u$ for all $u \in E_j$. In particular, E_j is invariant under S_b. Therefore,

$$\begin{aligned} \operatorname{spec}\left(S_b|_U\right) &= \cup_{j=0}^{6r-2}\operatorname{spec}\left(S_b|_{E_j}\right) \\ &= \operatorname{spec}\left(S_b|_{U_{4r-1}}\right) \cup \left(\cup_{j=4r}^{6r-2}\operatorname{spec}\left(S_b|_{E_j}\right)\right). \end{aligned} \tag{4.4}$$

Note that each element in $\operatorname{spec}(S_b|_{E_j})$ is equal to 2^{-j}. Write ρ_V for $\rho(T_b|_{\ell(\Omega)\cap V})$. Combining (4.2)–(4.4) together, we obtain

$$\rho_{2r} = \rho\left(T_b|_{\ell(\Omega)\cap V_{4r-1}}\right) = \max\left\{\rho_V, 2^{-4r}\right\}. \tag{4.5}$$

For convenience, we set $\Delta_{j+3} := \Delta_j$, $j = 1, 2, 3$. In order to find ρ_V, let W_j be the minimal invariant subspace of T_b generated by the sequences $\Delta_{j+1}^r\Delta_{j+2}^r\delta_\beta$, $\beta \in \mathbb{Z}^2$. Then $V = W_1 + W_2 + W_3$, so

$$\rho_V = \max_{1\leqslant j\leqslant 3}\left\{\rho(T_b|_{W_j})\right\}.$$

Let S_a denote the subdivision operator associated to a as defined in (1.4). It follows from [10, Theorem 4.1] and [17, Theorem 3.2] that

$$\lim_{n\to\infty}\left\|\nabla_1^r\nabla_2^r S_a^n\delta\right\|_2^{1/n} = \sqrt{\rho(T_b|_{W_3})}.$$

Since $\tilde{a}(z) = 2^{-2r}(1+z_1)^r(1+z_2)^r\tilde{a}_3(z)$, by [13, Theorem 3.2] we have

$$\lim_{n\to\infty}\left\|\nabla_1^r\nabla_2^r S_a^n\delta\right\|_2^{1/n} = 2^{-2r}\lim_{n\to\infty}\left\|S_{a_3}^n\delta\right\|_2^{1/n}.$$

But $\tilde{b}_3(z) = |\tilde{a}_3(z)|^2/4$, $z \in \mathbb{T}^2$. Hence $\lim_{n\to\infty}\left\|S_{a_3}^n\delta\right\|_2^{1/n} = \sqrt{\rho(T_{b_3})}$. The preceding discussion tells us that $\rho(T_b|_{W_j}) = 2^{-4r}\rho(T_{b_j})$ is true for $j = 3$. Clearly, this relation is also valid for $j = 1$ or $j = 2$. It follows that

$$\rho_V = \max_{1\leqslant j\leqslant 3}\left\{\rho(T_b|_{W_j})\right\} = 2^{-4r}\max_{1\leqslant j\leqslant 3}\left\{\rho(T_{b_j})\right\} = 2^{-4r}\rho.$$

By our assumption, $\rho > 1$. Hence, (4.5) tells us that $\rho_{2r} = \max\{\rho_V, 2^{-4r}\} = \rho_V > 2^{-4r}$. It follows that $2r > -\log_4\rho_{2r}$. If, in addition, the shifts of ϕ are stable, then

Table 1

r	$v(\varphi_r)$	$v(f_r)$
9	5.89529419	6.33524331
10	6.42640635	6.81143594
11	6.17848062	7.28259907
12	6.68092993	7.74953085
13	6.41506309	8.21284369
14	6.89718935	8.67302201
15	6.61823707	9.13045707
16	7.08520104	9.58546997

$$v(\phi) = -\log_4 \rho_{2r} = -\log_4 \rho_V = 2r - \log_4 \rho.$$

This verifies (4.1). $\square$

Example 4.2. For $r = 1, 2, \ldots$, let h_r be the mask on $\mathbb{Z}^2$ given by its symbol

$$\tilde{h}_r(z_1, z_2) = z_1^{-r} z_2^{-r} (1 + z_1)^r (1 + z_2)^r (1 + z_1 z_2)^r / 2^{3r-2}.$$

There exists a unique sequence c_r supported in $[1 - r, r - 1]^2$ such that $q_r := h_r * c_r$ is an interpolatory mask. Let φ_r be the normalized solution of the refinement equation associated with mask q_r. The smoothness order $v(\varphi_r)$ was computed in [19] for $r = 2, \ldots, 8$. Theorem 4.1 enables us to simplify the computation significantly so that we obtain $v(\varphi_r)$ for $r = 9, \ldots, 16$ as shown in Table 1.

In [6] Deslauriers and Dubuc showed that, for each $r = 1, 2, \ldots$, there exists a unique interpolatory mask b_r supported on $[1 - 2r, 2r - 1]$ such that b_r is symmetric about the origin and its symbol $\tilde{b}_r(z)$ is divisible by $(1 + z)^{2r}$. Let f_r be the normalized solution of the refinement equation $\phi = \sum_{\alpha \in \mathbb{Z}} b_r(\alpha) \phi(2 \cdot -\alpha)$. The smoothness order $v(f_r)$ was computed in [8] for $r = 1, 2, \ldots, 20$. For the purpose of comparison, we have listed the values of $v(f_r)$ $(r = 9, \ldots, 16)$ in Table 1. $\square$

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