Using the Refinement Equation for the Construction of Pre-wavelets V: Extensibility of Trigonometric Polynomials

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§1 Introduction

By a trigonometric polynomial on $\mathbb{R}^s$ we mean a function of the form

$$f(w) = \sum_{\alpha\in\mathbb{Z}^s} a_\alpha e^{i\alpha\cdot w}, \quad w \in \mathbb{R}^s$$

where the coefficients a_α ($\alpha \in \mathbb{Z}^s$) of complex numbers are assumed to be zero except on some finite subset of $\mathbb{Z}^s$. The purpose of this paper is to prove the following result:

Theorem 1.1. *Let $F(w) := (f_1(w), \ldots, f_n(w))$, $w \in \mathbb{R}^s$ be a vector whose coordinates are trigonometric polynomials having no common zeros on $\mathbb{R}^s$. If*

$$s < 2n - 1,$$

then there exists an $n \times n$ matrix $M(w) = (M_{jk}(w))$, $j, k = 1, \ldots, n$ such that every entry of $M(w)$ is a trigonometric polynomial, the first row of $M(w)$ is given by

$$M_{1k}(w) = f_k(w), \quad 1 \le k \le n,$$

and

$$\det\ M(w) \ne 0, \quad w \in \mathbb{R}^s.$$

In the language of [4] this means that the trigonometric polynomials $f_1(w),\dots,f_n(w)$ are *extensible* over $\mathbb{R}^s$ when $s<2n-1$ if and only if they have no common zeros on $\mathbb{R}^s$. By extensibility we mean the existence of a nonsingular matrix $M(w)$ of trigonometric polynomials whose first row is $f_1(w),\dots,f_n(w)$.

It has been observed recently in [4] and [5] that decomposition of multivariate wavelet spaces hinges upon extensibility of maps defined on either $\mathbb{R}^s$ or $(\mathbb{C}\backslash\{0\})^s$, complex s-space minus its coordinate axes. Thus the above result has direct application to wavelet decomposition which we will discuss at the end of the paper.

§2 Extensibility of Hölder Maps

In this section we prove a result which subsumes Theorem 1.1. Let Ω be a subset of $\mathbb{R}^s$. We say that a map $F:\Omega\to\mathbb{C}^n$ is Hölder continuous provided there is some $\rho\in(0,1]$ and constant $\kappa>0$ such that

$$|F(x)-F(y)|\le\kappa|x-y|^\rho,\quad x,y\in\Omega. \tag{2.1}$$

The absolute value sign above represents any norm defined on either $\mathbb{R}^s$ or $\mathbb{C}^n$ whichever is appropriate.

Theorem 2.1. *Let Ω be any compact subset of $\mathbb{R}^s$ and F a Hölder continuous map from Ω into $\mathbb{C}^n\backslash\{0\}$ with $s<2n-1$. Then there exists an $n\times n$ nonsingular matrix $M(w)$ of complex-valued continuous functions on Ω such that the first row of M is F.*

At the end of the proof of this result we will explain how it yields Theorem 1.1.

The proof of Theorem 2.1 is divided in two parts. The first part concerns some familiar facts about Householder transformations, cf. [6]. For its statement we first establish some notational conventions.

For the complex inner product of vectors $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$ in $\mathbb{C}^n$ we use

$$(x,y):=\sum_{j=1}^n\bar{x}_jy_j,$$

and for the euclidean norm $|x|:=\sqrt{(x,x)}$. We use $S_{\mathbb{C}}^{n-1}$ for the complex $n-1$ sphere, that is, the set of all vectors $x\in\mathbb{C}^n$ such that $|x|=1$. The "north pole" of $S_{\mathbb{C}}^{n-1}$ is chosen to be the vector $e = (1, 0, \ldots, 0)$. Also, we use tensor product notation for rank one matrix

$$(x \otimes x)_{jk} := x_j \bar{x}_k, \quad j, k = 1, \ldots, n.$$

The matrix $x \otimes x$ is clearly complex hermitian. Generally, the hermitian transpose of an $n \times n$ complex matrix A is denoted by A^H.

Proposition 2.1. *For every $x \in S_{\mathbb{C}}^{n-1} \backslash \{e\}$ we define*

$$y = \frac{e - x}{\sqrt{2(1 - Re\ x_1)}}, \quad x = (x_1, \ldots, x_n),$$

and

$$\zeta = \frac{2(1 - Re\ x_1)}{1 - x_1}. \tag{2.2}$$

Then the matrix

$$Q_x = I - \zeta y \otimes y$$

has the following properties:

(i) $Q_x x = e$*;*

(ii) $Q_x v = v$ *for any* $v \in \mathbb{C}^n$ *with* $(v, e) = (v, x) = 0$*;*

(iii) $Q_x Q_x^H = Q_x^H Q_x = I =$ *the identity matrix;*

(iv) $|\det Q_x| = 1$.

Remark 2.1. Note that both y and ζ are continuous functions on $S_{\mathbb{C}}^{n-1} \backslash \{e\}$. Moreover, whenever x is real then both y and ζ are real. In fact it is obvious from the definition of ζ that it must be equal to two in this case.

Proof: For the proof of (i) we note that

$$\begin{aligned} Q_x x &= x - \zeta y(y, x) \\ &= x - \frac{2(1 - Re\, x_1)}{1 - x_1} \frac{(x_1 - 1)(e - x)}{2(1 - Re\, x_1)} \\ &= x + e - x = e. \end{aligned}$$

The second claim is equally simple to see since y is chosen to be in the span of e and x. For the third claim we note first that ζ satisfies the equation

$$\zeta + \bar{\zeta} = |\zeta|^2 \tag{2.3}$$

which follows from the definition (2.2).

Since

$$Q_x^H = I - \bar{\zeta} y \otimes y$$

we have

$$\begin{aligned} Q_x Q_x^H &= I - \bar{\zeta} y \otimes y - \zeta y \otimes y + |\zeta^2||y|^2 y \otimes y \\ &= I + (|\zeta|^2|y|^2 - \zeta - \bar{\zeta}) y \otimes y. \end{aligned} \tag{2.4}$$

However,

$$\begin{aligned} |y|^2 &= \frac{|1-x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}{2(1 - Re\, x_1)} \\ &= \frac{1 + |x_1|^2 - 2Re\, x_1 + 1 - |x_1|^2}{2(1 - Re\, x_1)} = 1 \end{aligned}$$

and so by (2.3) and (2.4) we obtain (iii).

The last claim follows immediately from (iii). Thus the proposition is established.

The next fact we need is the following "dimension" result.

Proposition 2.2. *Let f be a Hölder map from some subset Ω of $\mathbb{R}^m$ into $\mathbb{R}^n$. If $n > m$, then $f(\Omega)$ has Lebesgue measure zero in $\mathbb{R}^n$.*

Proof: For the proof of this fact we make use of Hausdorff measure of a set $A \subseteq \mathbb{R}^m$. We recall this concept from [1, Chap.10]. For every $\alpha > 0$ and $\varepsilon > 0$ let

$$H_\alpha^{(\varepsilon)}(A) = \inf\big\{ \sum_k \delta(A_k)^\alpha : \ \delta(A_k) \le \varepsilon \big\}.$$

Here $\delta(A_k)$ denotes the diameter of A_k, and the infimum is taken over all countable covers of the set A. Then $H_\alpha^{(\varepsilon)}(A)$ is a nondecreasing function of ε whose limit as $\varepsilon \to 0$ is called the Hausdorff measure of A, which we shall denote by $H_\alpha(A)$. It is known that there is a constant $\gamma_n > 0$ such that $\gamma_n H_n$ is Lebesgue measure on $\mathbb{R}^n$ (see [1, p. 325]).

Using the inequality (2.1) we see that

$$H_\alpha^{(\varepsilon')}(f(\Omega)) \le H_\alpha^{(\varepsilon)}(\Omega)$$

where $\varepsilon' := \kappa \varepsilon^\rho$ and so

$$H_\alpha(f(\Omega)) \le H_\alpha(\Omega). \tag{2.5}$$

It is easily seen that $H_n(\Omega) = 0$ for any subset Ω of $\mathbb{R}^m$, $m < n$. Hence by (2.5), $H_n(f(\Omega)) = 0$. This shows that $f(\Omega)$ has Lebesgue measure zero in $\mathbb{R}^n$, and proves the result.

Remark 2.2. Proposition 2.2 is a variation of Sard's Theorem. See, for example, [2, pp. 204–205], in which the corresponding result was proved for smooth maps. However, the above proposition is not true for continuous maps. It is well-known, as a special case of the Hahn-Mazurkiewicz theorem, cf. [3, p. 129] that there is a continuous map of the unit interval $\Omega = [0,1] \subset \mathbb{R}$ onto the n-cube $[0,1]^n \subset \mathbb{R}^n$ for any n.

Propositions 2.1 and 2.2 are the main facts we need to prove Theorem 2.1.

Proof of Theorem 2.1: We first scale the map F so that its image is contained in $S^{n-1}_{\mathbb{C}}$ by setting

$$G = F/|F|.$$

We embed $S^{n-1}_{\mathbb{C}}$ into $S^{2n-1}_{\mathbb{R}}$ and express G in the form

$$G = (g_1, \ldots, g_{2n})$$

where $g_1, \ldots, g_{2n}$ are real-valued continuous maps on Ω. Since Ω is compact and $|F| > 0$ on Ω it follows that G is Hölder continuous. Specifically, we have

$$|G(x) - G(y)| \leq 2|F(x) - F(y)|/|F(y)|,$$

for any $x, y \in \Omega$. Next, we trim the map G to its first $2n-1$ coordinates and define

$$G_0 = (g_1, \ldots, g_{2n-1}).$$

Since $s < 2n-1$, Proposition 2.2 applied to G_0 tells us that $G_0(\Omega)$ has measure zero in $\mathbb{R}^{2n-1}$. In particular, $G_0(\Omega)$ is *not* the unit ball in $\mathbb{R}^{2n-1}$. Hence we conclude that there is a point $x^0 \in S^{n-1}_{\mathbb{C}}$ such that $x^0 \notin G(\Omega)$. We wish to arrange x^0 to be the north pole of $S^{n-1}_{\mathbb{C}}$ so that we can apply Proposition 2.1. If indeed we are fortunate to have $x^0 = e$, there is nothing to do. Otherwise, we consider the map

$$G_Q := Q_{x^0} G \tag{2.6}$$

and note that the north pole is omitted from the range of G_Q on Ω. We now consider the $n \times n$ matrix $Q_{G_Q(w)}$ $(w \in \Omega)$. According to Proposition 2.1 the entries of Q_{G_Q} are continuous functions on Ω and Q_{G_Q} is everywhere nonsingular on Ω. Moreover

$$Q_{G_Q} G_Q = e,$$

and so $G_Q = Q^H_{G_Q} e$; that is, G_Q is the first column of $Q^H_{G_Q}$. To get back to the map F we note that $F = |F| Q^H_{x^0} G_Q$ by (2.6). Hence the matrix $M := |F| Q^H_{x^0} Q^H_{G_Q}$ has the vector F as its first column. Thus M^t, the transpose of M, satisfies all the requirements of Theorem 2.1, thereby completing the proof.

The following version of Theorem 1.1 will be the one used later for wavelet decomposition.

Corollary 2.1. *Let $F(w) := (f_1(w), \ldots, f_n(w))$, $w \in \mathbb{R}^s$ be a vector whose coordinates are trigonometric series*

$$f_k(w) = \sum_{\alpha \in \mathbb{Z}^s} a^k_\alpha e^{i\alpha \cdot w}, \quad w \in \mathbb{R}^s$$

such that $F(w) \in \mathbb{C}^n \backslash \{0\}$ for all $w \in \mathbb{R}^s$. If

$$s < 2n - 1$$

and for some $\rho > 0$,

$$\sum_{\alpha \in \mathbb{Z}^s} |a^k_\alpha| |\alpha|^\rho < \infty, \quad k = 1, \ldots, n,$$

then there exists an $n \times n$ matrix $M(w) = (M_{jk}(w))$, $j, k = 1, \ldots, n$ such that each of the entries $M_{jk}(w)$, $2 \leq j \leq n$, $1 \leq k \leq n$ is a trigonometric polynomial, the first row of $M(w)$ is given by

$$M_{1k}(w) = f_k(w), \quad 1 \leq k \leq n,$$

and

$$\det M(w) \neq 0, \quad w \in \mathbb{R}^s.$$

Proof: Our condition on the coefficients of the trigonometric series ensure that the map F is Hölder continuous, see (2.1). Let $C_{2\pi}$ denote the Banach space of all continuous

2π-periodic functions equipped with the maximum norm. By Theorem 2.1 there exists an $n \times n$ nonsingular matrix $M(w)$ of complex-valued continuous functions on $\mathbb{R}^s$ such that the first row of M is F. Since every component of F is in $C_{2\pi}$, from the proof of Theorem 2.1 we see that M can be made in such a way that all its entries lie in $C_{2\pi}$. As is well-known trigonometric polynomials are dense in $C_{2\pi}$ and we may approximate all the entries in $M(w)$, except in the first row, arbitrarily closely by trigonometric polynomials. In this fashion we create a matrix near $M(w)$, all of whose entries except in the first row are trigonometric polynomials. This proves Corollary 2.1, from which Theorem 1.1 is easily derived.

Next we note without proof the following versions of Theorem 2.1 and Theorem 1.1 for maps whose range is $\mathbb{R}^s$.

Theorem 2.2. *Let Ω be any compact subset of $\mathbb{R}^s$ and F a Hölder continuous map from Ω into $\mathbb{R}^n \backslash \{0\}$ with $s < n-1$. Then there exists an $n \times n$ nonsingular matrix $M(w)$ of real-valued continuous functions on Ω such that the first row of M is F.*

Theorem 2.3. *Let $F(w) := (f_1(w), \ldots, f_n(w))$, $w \in \mathbb{R}^s$ be a vector whose coordinates are real-valued trigonometric polynomials having no common zeros on $\mathbb{R}^s$. If*

$$s < n-1,$$

then there exists an $n \times n$ matrix $M(w) = (M_{jk}(w))$, $j, k = 1, \ldots, n$ such that every entry of $M(w)$ is a real-valued trigonometric polynomial, the first row of $M(w)$ is given by

$$M_{1k}(w) = f_k(w), \quad 1 \le k \le n,$$

and

$$\det\ M(w) \neq 0, \quad w \in \mathbb{R}^s.$$

The proofs of these results are essentially the same as those given for Theorems 2.1 and 1.1 and therefore we omit the details.

§3 Multivariate Wavelet Decomposition

In this section we will review and extend some of the main results contained in [3] for the purpose of applying Theorem 1.1 and Theorem 2.3 to wavelet decomposition.

We recall the multiresolution point of view. Let T be an $s \times s$ invertible matrix with integer entries such that the spectral radius of T^{-1} is less than one. Then T has the property that for any nonempty bounded set $B \subset \mathbb{R}^s$,

$$\lim_{j\to\infty} T^{-j}(B) = \{0\}. \tag{3.1}$$

We call T a *scaling matrix*. A scaling matrix determines a scaling mapping

$$(sc\, f)(x) := f(Tx), \quad x \in \mathbb{R}^s.$$

We also need the shift mapping

$$(sh^y f)(x) := f(x-y), \quad x, y \in \mathbb{R}^s.$$

We say $\phi \in L^2(\mathbb{R}^s)$ *admits multiresolution* provided the following conditions hold:

(i) There exist constants m and M, $0 < m < M$, such that for all $c \in \ell^2(\mathbb{Z}^s)$

$$m\|c\|_2 \le \|[c,\phi]\|_2 \le M\|c\|_2, \tag{3.2}$$

where

$$[c,\phi] := \sum_{\alpha\in\mathbb{Z}^s} c_\alpha sh^\alpha \phi$$

and, of course, the ℓ^p and L^p norms are denoted by $\|\cdot\|_p$.

(ii) For $V^k := sc^k V$, $k \in \mathbb{Z}$, where

$$V := V(\phi) := \left\{ [c,\phi] : \ c \in \ell^2(\mathbb{Z}^s) \right\},$$

we have

$$V^k \subset V^{k+1}, \quad k \in \mathbb{Z}.$$

(iii)

$$\cap_{k\in\mathbb{Z}} V^k = \{0\}, \qquad \overline{\cup_{k\in\mathbb{Z}} V^k} = L^2(\mathbb{R}^s). \tag{3.3}$$

Following [4], we set for any function ϕ on $\mathbb{R}^s$

$$\phi^\circ(x) := \sum_{\alpha\in\mathbb{Z}^s} |\phi(x-\alpha)|, \quad x \in \mathbb{R}^s$$

and define $\mathcal{L}^2 = \mathcal{L}^2(\mathbb{R}^s)$ to be the Banach space of all functions ϕ for which

$$|\phi|_2 := \|\phi^\circ\|_{L^2([0,1)^s)}$$

is finite.

Theorem 3.1. *Let T be a scaling matrix and ϕ a function in $\mathcal{L}^2$ which satisfies (3.2) for some constants m and M. Suppose there is a sequence $a = (a_\alpha : \alpha \in \mathbb{Z}^s) \in \ell^1(\mathbb{Z}^s)$ such that*

$$\phi = sc[a, \phi]. \tag{3.4}$$

Then ϕ admits multiresolution.

Proof: According to Theorem 2.1 of [4]

$$\|[c, \phi]\|_2 \le \|c\|_2 |\phi|_2,$$

so we can actually choose $M = |\phi|_2$ in the upper inequality in (3.2). According to (3.4) we have

$$[c, \phi] = [c, sc[a, \phi]] = sc[c, [a, \phi]] = sc[c{*}a, \phi],$$

where $c{*}a$ is the convolution of c and a. Generally, we have

$$sc^k[c, \phi] = sc^{k+1}[c{*}a, \phi].$$

Since $\|c{*}a\|_2 \le \|a\|_1 \|c\|_2$, (ii) follows.

To prove the first part of (iii) we suppose f has the property that $sc^j f \in V^0$ for all $j \in \mathbb{Z}$. Thus for each j there is a $d \in \ell^2(\mathbb{Z}^s)$ such that $sc^j f = [d, \phi]$. According to (i)

$$\|d\|_2 \le m^{-1} \|sc^j f\|_2 = m^{-1} n^{-j/2} \|f\|_2 \tag{3.5}$$

where $n = |\det T|$. Since for each $x \in \mathbb{R}^s$

$$|[d, \phi](x)| \le \|d\|_\infty \phi^\circ(x) \le \|d\|_2 \phi^\circ(x),$$

we have $|sc^j f(x)| \le \|d\|_2 \phi^\circ(x)$ and so for any ball B in $\mathbb{R}^s$

$$\|f\|_{L^2(B)} = n^{j/2}\|sc^j f\|_{L^2(T^{-j}B)} \le n^{j/2}\|d\|_2\|\phi^\circ\|_{L^2(T^{-j}B)}.$$

Combining this inequality with (3.5) gives

$$\|f\|_{L^2(B)} \le m^{-1}\|\phi^\circ\|_{L^2(T^{-j}B)}\|f\|_2. \tag{3.6}$$

Letting $j \to \infty$ in (3.6) and using (3.1) proves that $f = 0$, thereby establishing the first part of (iii).

As for the second part of (iii) we introduce the operators T_k defined by

$$T_k f := sc^k[ssc^{-k}f, \phi], \quad k = 1, 2, \ldots$$

where we interpret $ssc^{-k}f$ as the sequence given by $(ssc^{-k}f)(\alpha) := (sc^{-k}f)(\alpha)$, $\alpha \in \mathbb{Z}^s$. For $r > 0$ we denote by B_r the cube $\{x \in \mathbb{R}^s : |x|_\infty \le r\}$, where

$$|x|_\infty := \max\{|x_j| : 1 \le j \le s\}, \quad x = (x_1, \ldots, x_s) \in \mathbb{R}^s.$$

Suppose that $f \in C(\mathbb{R}^s)$ is supported on a cube B_N for some positive integer N. Then according to (3.2)

$$\|T_k f\|_2 \le n^{-k/2}M\|ssc^{-k}f\|_2 \le (n^{-k}\#(J_N))^{1/2}M\|f\|_\infty, \tag{3.7}$$

where $\#(J_N)$ denotes the cardinality of the set

$$J_N := \{\, \alpha \in \mathbb{Z}^s : |T^{-k}\alpha|_\infty \le N \,\}.$$

To estimate $\#(J_N)$, we observe that the sets $T^{-k}(\alpha + [0,1)^s)$, $\alpha \in \mathbb{Z}^s$, are pairwise disjoint. Moreover, according to (3.1) for sufficiently large k

$$\bigcup_{\alpha \in J_N} T^{-k}(\alpha + [0,1)^s) \subseteq B_N + T^{-k}([0,1)^s) \subseteq B_{N+1}.$$

For each $\alpha \in J_N$, the volume of $T^{-k}(\alpha + [0,1)^s)$ is $|\det(T^{-k})| = n^{-k}$; hence it follows from the above inclusion relation that

$$\#(J_N)n^{-k} \le (2N+2)^s. \tag{3.8}$$

This together with (3.7) shows that for sufficiently large k

$$\|T_k f\|_2 \leq (2N+2)^{s/2} M \|f\|_\infty. \tag{3.9}$$

Next, we demonstrate that $T_k f$ converges to $\hat{\phi}(0)f$ weakly in $L^2(\mathbb{R}^s)$ as $k \to \infty$, where

$$\hat{\phi}(w) := \int_{\mathbb{R}^s} \phi(x) e^{-iw\cdot x}\, dx$$

is the Fourier transform of ϕ. It suffices by (3.9) above to prove that

$$\lim_{k\to\infty} \int_{\mathbb{R}^s} (T_k f)(x)\bar{g}(x)\, dx = \hat{\phi}(0) \int_{\mathbb{R}^s} f(x)\bar{g}(x)\, dx$$

for any function g in the Schwartz class $\mathcal{S}(\mathbb{R}^s)$. According to the Plancherel formula

$$\int_{\mathbb{R}^s} (T_k f)(x)\bar{g}(x)\, dx = \frac{1}{(2\pi)^s} \int_{\mathbb{R}^s} \big(n^{-k} \sum_{\alpha\in\mathbb{Z}^s} f(T^{-k}\alpha) e^{-iw\cdot T^{-k}\alpha}\big) \hat{\phi}((T^{-k})^t w) \bar{\hat{g}}(w)\, dw.$$

Since $\phi \in \mathcal{L}^2$, it follows that $\phi \in \mathcal{L}^1 = L^1$ and so $\|\hat{\phi}\|_\infty \leq \|\phi\|_1$. This together with the estimate (3.8) tells us that the integrand is bounded by

$$(2N+2)^s \|\phi\|_1 \|f\|_\infty |\hat{g}(w)|.$$

However, since $f \in C_c(\mathbb{R}^s)$, the integrand converges pointwise to $\hat{f}(w)\bar{\hat{g}}(w)\hat{\phi}(0)$ as $k \to \infty$, which proves the desired result.

The theorem will follow once we show $\hat{\phi}(0) \neq 0$. In this case the weak closure of the subspace

$$V^\infty := \cup_{k\in\mathbb{Z}} V^k$$

would contain $C_c(\mathbb{R}^s)$. It then follows that V^∞ is dense in $L^2(\mathbb{R}^s)$. The proof of the fact that $\hat{\phi}(0) \neq 0$ follows along the lines of Theorem 2.4 and Theorem 3.5 of [4]. Theorem 3.5 directly applies to our situation. In particular it guarantees that whenever $\phi \in \mathcal{L}^2$ satisfies the stability inequality (i) then

$$\sup_{\alpha\in\mathbb{Z}^s} |\hat{\phi}(2\pi\alpha)| > 0.$$

We now follow the proof of Theorem 2.4 of [4] and show that actually $\hat{\phi}(2\pi\alpha) = 0$, $\alpha \in \mathbb{Z}^s\backslash\{0\}$. This will then imply $\hat{\phi}(0) \neq 0$ by the above remark.

Taking the Fourier transform of both sides of the refinement equation (3.4) gives

$$\hat{\phi}(\xi) = n^{-1} a(e^{-i(T^{-1})^t \xi}) \hat{\phi}((T^{-1})^t \xi), \quad \xi \in \mathbb{R}^s, \tag{3.10}$$

where a is the function given by

$$a(e^{iw}) := \sum_{\alpha \in \mathbb{Z}^s} a_\alpha e^{i\alpha \cdot w}, \quad w \in \mathbb{R}^s.$$

It follows that for any positive integer k,

$$\hat{\phi}(\xi) = \prod_{j=1}^{k} \left(n^{-1} a(e^{-i(T^{-j})^t \xi}) \right) \hat{\phi}((T^{-k})^t \xi), \quad \xi \in \mathbb{R}^s. \tag{3.11}$$

If $n^{-1}|a(1)| < 1$, then choosing $\xi = 0$ in (3.10) gives $\hat{\phi}(0) = 0$. Also (3.11) implies $\hat{\phi}(\xi) = 0$ for any $\xi \in \mathbb{R}^s$, because when ξ is fixed, $n^{-1}|a(e^{-i(T^{-j})^t \xi})| < 1$ for sufficiently large j. Hence, in view of (3.2), we must have $|a(1)| \geq n$. Now, we choose $\xi = (T^k)^t (2\pi\beta)$ for some $\beta \in \mathbb{Z}^s \backslash \{0\}$ in (3.11) to obtain

$$\hat{\phi}\big((T^k)^t (2\pi\beta)\big) = (n^{-1} a(1))^k \hat{\phi}(2\pi\beta).$$

Thus $|\hat{\phi}(2\pi\beta)| \leq |\hat{\phi}\big((T^k)^t (2\pi\beta)\big)|$ and by the Riemann-Lebesgue lemma we conclude that $\hat{\phi}(2\pi\beta) = 0$ for $\beta \in \mathbb{Z}^s \backslash \{0\}$, because $(T^k)^t (2\pi\beta) \to \infty$ as $k \to \infty$. This proves the theorem.

We are now ready to provide the application of the ideas of Section 2 to wavelet decomposition. We restrict ourselves to a function $\phi \in L^2(\mathbb{R}^s)$ of compact support which has integer translates that are stable in the sense of (i) and satisfies the refinement equation (3.4) for some $a = (a_\alpha : \ \alpha \in \mathbb{Z}^s)$. We observe that a necessarily decays exponentially fast, that is, there are constants $A > 0$, $\zeta \in (0, 1)$ such that

$$|a_\alpha| \leq A \zeta^{|\alpha|} \quad \text{for all } \alpha \in \mathbb{Z}^s.$$

To see this we recall, as noted in [4] and [5], that by the Poisson summation formula

$$d(w) := \sum_{\alpha \in \mathbb{Z}^s} |\hat{\phi}(w + 2\pi\alpha)|^2 = \sum_{\alpha \in \mathbb{Z}^s} d_\alpha e^{i\alpha \cdot w}, \quad w \in \mathbb{R}^s$$

where

$$d_\alpha = \int_{\mathbb{R}^s} \phi(x)\bar{\phi}(x+\alpha)\,dx.$$

Thus $d(w)$ is a strictly positive trigonometric polynomial. Hence, its reciprocal $c(w) := 1/d(w)$ has an expansion

$$c(w) = \sum_{\alpha\in\mathbb{Z}^s} c_\alpha e^{i\alpha\cdot w}$$

whose coefficients decay exponentially fast. The function $g = [c, \phi]$ has the property that

$$\int_{\mathbb{R}^s} g(x)\bar{\phi}(x-\alpha)dx = \delta_{0\alpha}, \quad \alpha \in \mathbb{Z}^s, \tag{3.12}$$

see the proof of Theorem 3.3 of [4]. Hence by (3.4) and (3.12) we get

$$a_\alpha = n\int_{\mathbb{R}^s} \bar{g}(Tx-\alpha)\phi(x)\,dx, \quad \alpha \in \mathbb{Z}^s,$$

from which it follows that a decays exponentially fast.

Next, we consider the orthogonal complement of V^0 in V^1 which we call W^0. Our main result is

Theorem 3.2. *Suppose $\phi \in L^2(\mathbb{R}^s)$ has compact support, stable integer translates and satisfies the refinement equation (3.4) with a scaling matrix T with $|\det T| = n > (s+1)/2$. Then W^0 has an unconditional basis consisting of the integer translates of $n-1$ functions $\psi_2, \ldots, \psi_n$ of compact support such that $V(\psi_j)$ is orthogonal to $V(\psi_k)$, $j \neq k$. Moreover, the scaled spaces*

$$W^k := sc^k W^0, \quad k \in \mathbb{Z}$$

are mutually orthogonal and their sum is dense in $L^2(\mathbb{R}^s)$.

Proof: The proof follows closely the proof of Theorem 6.1 of [4]. For every $s\times s$ integer matrix T with $n := |\det T|$ there exist $\delta^1, \ldots, \delta^n \in \mathbb{Z}^s$ such that $\delta^1 = 0$ and the lattices $\delta^j + T\mathbb{Z}^s$, $j = 1, \ldots, n$ partition $\mathbb{Z}^s$. We consider the trigonometric series

$$a_{\delta^j}(w) := \sum_{\alpha\in\mathbb{Z}^s} a_{\delta^j+T\alpha} e^{i\alpha\cdot w}, \quad 1 \le j \le n.$$

In view of the fact that $a = (a_\alpha : \alpha \in \mathbb{Z}^s)$ decays exponentially fast, these trigonometric series are in $C^\infty(\mathbb{R}^s)$. Moreover, as observed in [4] and [5], they have no common zeros in $\mathbb{R}^s$. For the argument we recall the identity from [5]

$$d(w) = \sum_{j=1}^{n} e^{i\delta^j \cdot w} a_{\delta^j}(w) \mu_{-\delta^j}(w)$$

where

$$\mu_\delta(w) := \sum_{\alpha \in \mathbb{Z}^s} \mu_{\delta + T\alpha} e^{i\alpha \cdot w}, \quad \delta \in \mathbb{Z}^s$$

and

$$\mu_\alpha := \int_{\mathbb{R}^s} \phi(Tx + \alpha) \bar{\phi}(x)\, dx, \quad \alpha \in \mathbb{Z}^s.$$

As observed above, $d(w) > 0$ for all $w \in \mathbb{R}^s$, and so $a_{\delta^j}(w)$, $j = 1, \ldots, n$ have no common zeros on $\mathbb{R}^s$. Thus according to Corollary 2.1, there is an $n \times n$ matrix $A(w) = (a_{jk}(w))$, $j, k = 1, \ldots, n$ such that $\det A(w) \neq 0$ for all $w \in \mathbb{R}^s$, $a_{1k}(w) = a_{\delta^k}(w)$, $k = 1, \ldots, n$, and each of the entries $a_{jk}(w)$, $2 \le j \le n$, $1 \le k \le n$ is a trigonometric polynomial. We introduce the function $\phi_j(x) := \phi(Tx - \delta^j)$, $j = 1, \ldots, n$, and so it follows that

$$V^1 = \{ \sum_{j=1}^{n} [c^j, \phi_j] : \ c^1, \ldots, c^n \in \ell^2(\mathbb{Z}^s) \}.$$

We set

$$\rho_j := \sum_{k=1}^{n} [a^{jk}, \phi_k], \quad 1 \le j \le n,$$

where $(a_\alpha^{jk} : \alpha \in \mathbb{Z}^s)$ are the sequences of the coefficients in the expansion of $a_{jk}(w)$:

$$a_{jk}(w) = \sum_{\alpha \in \mathbb{Z}^s} a_\alpha^{jk} e^{i\alpha \cdot w}, \quad 1 \le j, k \le n.$$

According to the refinement equation (3.4) it follows that $\rho_1 = \phi$ and since $a_{jk}(w)$ are trigonometric polynomials we conclude that $\rho_2, \ldots, \rho_n$ are of compact support. Thus Theorem 4.3 of [4] implies that each ρ_j, $1 \le j \le n$, satisfies an estimate of the type (3.2) stability. Furthermore, Theorem 4.4 of [4] guarantees that there are functions $\psi_1, \ldots, \psi_n \in V^1$ of compact support such that (i) $\psi_1 = \phi$, (ii) the spaces $V(\psi_j)$, $j = 1, \ldots, n$ are mutually orthogonal, and (iii) V^1 is the sum of $V(\psi_1), \ldots, V(\psi_n)$. Hence W^0 is the direct sum of

$V(\psi_2), \ldots, V(\psi_n)$. This proves our first claim. Also, the scaled spaces $W^k = sc^k W^0$ ($k \in \mathbb{Z}$) are mutually orthogonal. The fact that the sum of the spaces W^k, $k \in \mathbb{Z}$, is dense follows from (3.3), using general principles, see [5]. This completes the proof.

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