Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT

Appl. Comput. Harmon. Anal. 15 (2003) 224–241

Applied and Computational Harmonic Analysis

www.elsevier.com/locate/acha

Compactly supported wavelet bases for Sobolev spaces

Rong-Qing Jia,[a,1] Jianzhong Wang,[b,2] and Ding-Xuan Zhou[c,3]

[a] *Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Canada T6G 2G1*
[b] *Department of Mathematics and Statistics, Sam Houston State University, Huntsville, TX 77341, USA*
[c] *Department of Mathematics, City University of Hong Kong, Hong Kong, China*

Received 17 March 2003; revised 1 August 2003; accepted 26 August 2003

Communicated by W.R. Madych

Abstract

In this paper we investigate compactly supported wavelet bases for Sobolev spaces. Starting with a pair of compactly supported refinable functions ϕ and $\tilde{\phi}$ in $L_2(\mathbb{R})$ satisfying a very mild condition, we provide a general principle for constructing a wavelet ψ such that the wavelets $\psi_{jk} := 2^{j/2}\psi(2^j \cdot - k)$ $(j, k \in \mathbb{Z})$ form a Riesz basis for $L_2(\mathbb{R})$. If, in addition, ϕ lies in the Sobolev space $H^m(\mathbb{R})$, then the derivatives $2^{j/2}\psi^{(m)}(2^j \cdot - k)$ $(j, k \in \mathbb{Z})$ also form a Riesz basis for $L_2(\mathbb{R})$. Consequently, $\{\psi_{jk}\colon\ j, k \in \mathbb{Z}\}$ is a stable wavelet basis for the Sobolev space $H^m(\mathbb{R})$. The pair of ϕ and $\tilde{\phi}$ are not required to be biorthogonal or semi-orthogonal. In particular, ϕ and $\tilde{\phi}$ can be a pair of B-splines. The added flexibility on ϕ and $\tilde{\phi}$ allows us to construct wavelets with relatively small supports.

Keywords: Wavelets; Spline wavelets; Multiresolution analysis; Stable wavelet bases; Riesz bases; Sobolev spaces

1. Introduction

The purpose of this paper is to investigate compactly supported wavelet bases for Sobolev spaces.

E-mail addresses: rjia@ualberta.ca (R.-Q. Jia), mth_jxw@shsu.edu (J. Wang), mazhou@math.cityu.edu.hk (D.-X. Zhou).
1 Research was supported in part by NSERC Canada under Grant OGP 121336.
2 Research was supported by EGR Grant-2003 at Sam Houston State University.
3 Research was supported in part by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China, Project No. CityU 1144/01P.

1063-5203/$ – see front matter
doi:10.1016/j.acha.2003.08.003

As usual, by $L_2(\mathbb{R})$ we denote the linear space of all (complex-valued) square integrable functions on $\mathbb{R}$. The space $L_2(\mathbb{R})$ is a Hilbert space with the inner product given by

$$\langle f, g\rangle := \int_{\mathbb{R}} f(x)\,\overline{g(x)}\,\mathrm{d}x, \quad f, g \in L_2(\mathbb{R}).$$

The norm of a function f in $L_2(\mathbb{R})$ is given by $\|f\|_2 := \sqrt{\langle f, f\rangle}$.

The Fourier transform of an integrable function f is defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}} f(x)\,\mathrm{e}^{-ix\xi}\,\mathrm{d}x, \quad \xi \in \mathbb{R}.$$

The Fourier transform can be naturally extended to functions in $L_2(\mathbb{R})$. For $\mu > 0$, we denote by $H^\mu(\mathbb{R})$ the Sobolev space of all functions $f \in L_2(\mathbb{R})$ such that

$$\int_{\mathbb{R}} \bigl|\hat{f}(\xi)\bigr|^2 \bigl(1 + |\xi|^{2\mu}\bigr)\,\mathrm{d}\xi < \infty.$$

The space $H^\mu(\mathbb{R})$ is a Hilbert space with the inner product given by

$$\langle f, g\rangle_{H^\mu(\mathbb{R})} := \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)\,\overline{\hat{g}(\xi)}\bigl(1 + |\xi|^{2\mu}\bigr)\,\mathrm{d}\xi, \quad f, g \in H^\mu(\mathbb{R}).$$

We are interested in wavelet bases for the Sobolev space $H^m(\mathbb{R})$, where m is a positive integer. In this case, we have

$$\langle f, g\rangle_{H^m(\mathbb{R})} = \langle f, g\rangle + \bigl\langle f^{(m)}, g^{(m)}\bigr\rangle,$$

where $f^{(m)}$ denotes the mth derivative of f.

Let H be a Hilbert space equipped with the norm $\|\cdot\|$. A sequence $(f_n)_{n=1,2,\ldots}$, in H is called a *Riesz sequence* if there exist two positive constants A and B such that

$$A\left(\sum_{n=1}^{\infty}|c_n|^2\right)^{1/2} \leqslant \left\|\sum_{n=1}^{\infty} c_n f_n\right\| \leqslant B\left(\sum_{n=1}^{\infty}|c_n|^2\right)^{1/2}$$

for every sequence $(c_n)_{n=1,2,\ldots}$, with only finitely many nonzero terms. A Riesz sequence $(f_n)_{n=1,2,\ldots}$, is called a *Riesz basis* if additionally the linear span of $\{f_n\colon n = 1, 2, \ldots\}$ is dense in H.

Smooth orthogonal wavelets with compact support were constructed by Daubechies (see [8]). Let ψ be an orthogonal wavelet in $L_2(\mathbb{R})$, and let

$$\psi_{jk}(x) := 2^{j/2}\psi\bigl(2^j x - k\bigr), \quad x \in \mathbb{R}. \tag{1.1}$$

Then $\{\psi_{jk}\colon j, k \in \mathbb{Z}\}$ is a Riesz basis for $L_2(\mathbb{R})$.

Suppose ψ is a function in the Sobolev space $H^m(\mathbb{R})$, where m is a positive integer. For $j, k \in \mathbb{Z}$, let ψ_{jk} be given as in (1.1). Then we have

$$\|\psi_{jk}\|_{H^m(\mathbb{R})} \geqslant \bigl\|\psi_{jk}^{(m)}\bigr\|_{L_2(\mathbb{R})} = 2^{jm}\bigl\|\psi^{(m)}\bigr\|_{L_2(\mathbb{R})}.$$

Since $\lim_{j\to\infty} 2^{jm} = \infty$, $\{\psi_{jk}\colon j, k \in \mathbb{Z}\}$ will *not* be a Riesz basis for $H^m(\mathbb{R})$. It is possible to construct Riesz bases for $H^m(\mathbb{R})$ only if ψ_{jk} are given by $2^{j/2}\psi_j(2^j\cdot -k)$ for $j, k \in \mathbb{Z}$, where the wavelets ψ_j are different for different j. See the work of Bastin and Boigelot [1], and Micchelli [17] for research in this direction.

The preceding discussion tells us that there is *no* function ψ in $H^m(\mathbb{R})$ such that $\{\psi_{jk}\colon\ j,k\in\mathbb{Z}\}$ forms a Riesz basis for $H^m(\mathbb{R})$. However, the Sobolev space $H^m(\mathbb{R})$ can be characterized by means of orthogonal wavelets in $L_2(\mathbb{R})$. Let ψ be an orthogonal wavelet in $L_2(\mathbb{R})$. It was shown in [8,15] that a function f lies in $H^m(\mathbb{R})$ if and only if

$$\sum_{j,k\in\mathbb{Z}} \bigl|\langle f,\psi_{jk}\rangle\bigr|^2\bigl(1+2^{2mj}\bigr)<\infty.$$

Moreover, there exist two positive constants A and B such that the inequalities

$$A\sum_{j,k\in\mathbb{Z}}\bigl(1+2^{2mj}\bigr)|b_{jk}|^2\leqslant\Bigl\|\sum_{j,k\in\mathbb{Z}} b_{jk}\psi_{jk}\Bigr\|^2_{H^m(\mathbb{R})}\leqslant B\sum_{j,k\in\mathbb{Z}}\bigl(1+2^{2mj}\bigr)|b_{jk}|^2 \tag{1.2}$$

hold for every sequence $(b_{jk})_{j,k\in\mathbb{Z}}$ with $\sum_{j,k\in\mathbb{Z}}(1+2^{2mj})|b_{jk}|^2<\infty$.

If ψ is a function in $H^m(\mathbb{R})$ satisfying (1.2), then $\{\psi_{jk}\colon\ j,k\in\mathbb{Z}\}$ is said to be a *stable wavelet basis* for $H^m(\mathbb{R})$. Although stable wavelet bases are not Riesz bases for $H^m(\mathbb{R})$, they still play a vital role in the study of function spaces as well as in many applications. In light of the preceding discussion we see that orthogonality is no longer a significant issue for wavelet bases of Sobolev spaces. Instead, the size of the support of a wavelet turns out to be an important criterion for its performance. From a numerical point of view, a wavelet with smaller support usually generates more efficient algorithms for wavelet transforms than that with a larger support.

In this paper we aim to provide a general principle for the construction of stable wavelet bases for Sobolev spaces and, by applying the principle to concrete problems, to construct stable wavelet bases with relatively small supports.

As usual, we begin our construction with refinable functions. Let ϕ be a compactly supported function in $L_2(\mathbb{R})$ such that $\hat{\phi}(0)=1$. Suppose that ϕ is refinable

$$\phi(x)=\sum_{j\in\mathbb{Z}} a(j)\phi(2x-j),\quad x\in\mathbb{R}, \tag{1.3}$$

where the mask a is finitely supported and $\sum_{j\in\mathbb{Z}} a(j)=2$. Similarly, let $\tilde{\phi}$ be a compactly supported function in $L_2(\mathbb{R})$ such that $\hat{\tilde{\phi}}(0)=1$. Suppose that $\tilde{\phi}$ is refinable

$$\tilde{\phi}(x)=\sum_{j\in\mathbb{Z}} \tilde{a}(j)\tilde{\phi}(2x-j),\quad x\in\mathbb{R}, \tag{1.4}$$

where the mask $\tilde{a}$ is finitely supported and $\sum_{j\in\mathbb{Z}}\tilde{a}(j)=2$.

Let us recall the concept of bracket products from [2,14]. The *bracket product* of two compactly supported functions f and g in $L_2(\mathbb{R})$ is given by

$$[f,g](\xi):=\sum_{j\in\mathbb{Z}}\bigl\langle f,g(\cdot-j)\bigr\rangle \mathrm{e}^{-ij\xi}=\sum_{k\in\mathbb{Z}}\hat{f}(\xi+2k\pi)\overline{\hat{g}(\xi+2k\pi)},\quad \xi\in\mathbb{R}.$$

Clearly, $[f,g]$ is a 2π-periodic function on $\mathbb{R}$. Our main assumption about ϕ and $\tilde{\phi}$ is

$$[\phi,\tilde{\phi}](\xi)\neq 0\quad \forall\xi\in\mathbb{R}. \tag{1.5}$$

Section 2 will be devoted to multiresolution analysis related to the refinable functions ϕ and $\tilde{\phi}$ satisfying the above condition.

In Sections 3 and 4, using the refinable functions ϕ and $\tilde{\phi}$, we will give an explicit construction of wavelet bases for $L_2(\mathbb{R})$. That is, we will prove the following. Let

$$\psi := \sum_{j\in\mathbb{Z}} (-1)^j \overline{\mu(1-j)}\,\phi(2\cdot -j) \quad \text{with } \mu(j) := \langle \tilde{\phi}, \phi(2\cdot -j)\rangle,\ j \in \mathbb{Z}.$$

Then $\{2^{j/2}\psi(2^j\cdot -k)\colon\ j,k \in \mathbb{Z}\}$ is a Riesz basis for $L_2(\mathbb{R})$.

In Section 5 we will show that, for $\phi \in H^m(\mathbb{R})$, $\{2^{j/2}\psi^{(m)}(2^j\cdot -k)\colon\ j,k \in \mathbb{Z}\}$ is a Riesz basis for $L_2(\mathbb{R})$. Consequently, $\{\psi_{jk}\colon\ j,k \in \mathbb{Z}\}$ forms a stable wavelet bases for $H^m(\mathbb{R})$.

Two important classes of wavelets satisfy the condition in (1.5). In the study of biorthogonal wavelets by Cohen, Daubechies, and Feauveau (see [6,7]), the shifts of ϕ and $\tilde{\phi}$ are biorthogonal. Hence, $[\phi,\tilde{\phi}](\xi) = 1$ for all $\xi \in \mathbb{R}$. In the investigation of semi-orthogonal wavelets by Chui and Wang (see [4,5]), $\tilde{\phi} = \phi$ and $[\phi,\phi](\xi) > 0$ for all $\xi \in \mathbb{R}$. By relaxing conditions on ϕ and $\tilde{\phi}$ we gain flexibility to construct desirable stable wavelet bases for Sobolev spaces.

Finally, in Section 6, we shall apply the above theory to B-splines. For a positive integer m, let M_m denote the B-spline of order m, which is the convolution of m copies of $\chi_{[0,1]}$, the characteristic function of the interval $[0,1]$.

Suppose N is an odd number. Choosing $\phi = M_N$ and $\tilde{\phi} = M_1$, we obtain

$$\psi_N = \sum_{j=0}^{N} \frac{(-1)^j}{2}\bigl[M_{N+1}(j) + M_{N+1}(j+1)\bigr] M_N(2\cdot -j).$$

The wavelet ψ_N is supported on $[0,N]$ and is antisymmetric about $N/2$. Moreover, for $r = 0,1,\ldots,N-1$, the set $\{2^{j/2}\psi_N^{(r)}(2^j\cdot -k)\colon\ j,k \in \mathbb{Z}\}$ is a Riesz basis for $L_2(\mathbb{R})$.

For an even integer N, with $\phi = M_N$ and $\tilde{\phi} = M_2(\cdot + 1)$ we get

$$\psi_N = \sum_{j=0}^{N+2} \frac{(-1)^j}{4}\bigl[M_{N+2}(j-1) + 2M_{N+2}(j) + M_{N+2}(j+1)\bigr] M_N(2\cdot -j).$$

The wavelet ψ_N is supported on $[0,N+1]$ and is symmetric about $(N+1)/2$. Moreover, for $r = 0,1,\ldots,N-1$, the set $\{2^{j/2}\psi_N^{(r)}(2^j\cdot -k)\colon\ j,k \in \mathbb{Z}\}$ is a Riesz basis for $L_2(\mathbb{R})$.

By comparison, the support of the semi-orthogonal C^N spline wavelet constructed in [4] has length at least $2N$, and the C^N biorthogonal wavelet constructed in [7] has length greater than $4N$.

2. Multiresolution analysis

In this section we provide a self-contained treatment for the multiresolution analysis induced by a pair of compactly supported refinable functions in $L_2(\mathbb{R})$.

Let H be a Hilbert space. The inner product of two elements f and g in H is denoted by $\langle f,g\rangle$. The norm of an element f in H is given by $\|f\| := \sqrt{\langle f,f\rangle}$. For an element $f \in H$ and a subset E of H, the distance from f to E is defined as

$$\operatorname{dist}(f,E) := \inf\bigl\{\|f-g\|\colon\ g \in E\bigr\}.$$

A linear mapping $P:H\to H$ is called a *projection*, if $P^2=P$, i.e., $P(Ph)=Ph$ for all $h\in H$. Suppose H is the direct sum of two closed linear subspaces V and U. Then there exists a unique continuous projection P such that the range of P is V and the null space of P is U. Let I denote the identity operator on H. Then each element $h\in H$ can be decomposed as $h=Ph+(I-P)h$, where $Ph\in V$ and $(I-P)h\in U$.

Let $(V_n)_{n\in\mathbb{Z}}$ be a family of closed linear subspaces of a Hilbert space H. We say that $(V_n)_{n\in\mathbb{Z}}$ forms a *multiresolution* of H if it satisfies the following conditions:

(a) $(V_n)_{n\in\mathbb{Z}}$ is nested, i.e., $V_n\subseteq V_{n+1}$ for every $n\in\mathbb{Z}$;
(b) $\bigcup_{n\in\mathbb{Z}} V_n$ is dense in H;
(c) $\bigcap_{n\in\mathbb{Z}} V_n=\{0\}$.

See [15] for the concept of multiresolution analysis. The above definition of multiresolution is adopted from [3].

Suppose $(V_n)_{n\in\mathbb{Z}}$ and $(\widetilde{V}_n)_{n\in\mathbb{Z}}$ are two families of closed linear subspaces of H. Each of them forms a multiresolution of H. Furthermore, we assume that H is the direct sum of V_n and $\widetilde{V}_n^{\perp}$ for each $n\in\mathbb{Z}$:

$$H=V_n+\widetilde{V}_n^{\perp}\quad\text{and}\quad V_n\cap\widetilde{V}_n^{\perp}=\{0\}. \tag{2.1}$$

Let $W_n:=V_{n+1}\cap\widetilde{V}_n^{\perp}$. Then V_{n+1} is the direct sum of V_n and W_n.

Lemma 2.1. *Let P_n be the continuous projection with range V_n and null space $\widetilde{V}_n^{\perp}$. If there exists a positive constant A such that $\|P_n\|\leqslant A$ for all $n\in\mathbb{Z}$, then for each $f\in H$,*

$$\lim_{n\to\infty}\|P_nf-f\|=0\quad\textit{and}\quad\lim_{n\to-\infty}\|P_nf\|=0. \tag{2.2}$$

Moreover, $\sum_{n\in\mathbb{Z}} W_n$ is dense in H.

Proof. Let $f\in H$. Since $\bigcup_{n\in\mathbb{Z}} V_n$ is dense in H, for given $\varepsilon>0$ there exists an integer N such that $\operatorname{dist}(f,V_N)<\varepsilon$. But $V_N\subseteq V_n$ for all $n\geqslant N$. Hence, $\operatorname{dist}(f,V_n)<\varepsilon$ for $n\geqslant N$. Thus, for $n\geqslant N$, there exists some $v_n\in V_n$ such that $\|f-v_n\|<\varepsilon$. It follows from $P_nv_n=v_n$ that:

$$\|P_nf-f\|=\big\|P_n(f-v_n)-(f-v_n)\big\|\leqslant\big\|P_n(f-v_n)\big\|+\|f-v_n\|.$$

By our assumption, $\|P_n\|\leqslant A$ for all $n\in\mathbb{Z}$. Therefore,

$$\|P_nf-f\|\leqslant(A+1)\|f-v_n\|<(A+1)\varepsilon\quad\text{for } n\geqslant N.$$

In other words, $\lim_{n\to\infty}\|P_nf-f\|=0$.

In order to prove $\lim_{n\to-\infty}\|P_nf\|=0$, we argue as follows. Let $U_n:=\widetilde{V}_n^{\perp}$ and $\widetilde{U}_n:=V_n^{\perp}$ for $n\in\mathbb{Z}$. Then $U_{n+1}\subseteq U_n$ for every $n\in\mathbb{Z}$. Also, $\bigcup_{n\in\mathbb{Z}} U_n$ is dense in H, since $\bigcap_{n\in\mathbb{Z}}\widetilde{V}_n=\{0\}$. Moreover, $\bigcap_{n\in\mathbb{Z}} U_n=\{0\}$, because $\bigcup_{n\in\mathbb{Z}}\widetilde{V}_n$ is dense in H. Thus, $(U_{-n})_{n\in\mathbb{Z}}$ forms a multiresolution of H. Similarly, $(\widetilde{U}_{-n})_{n\in\mathbb{Z}}$ forms a multiresolution of H. It follows from (2.1) that

$$H=U_n+\widetilde{U}_n^{\perp}\quad\text{and}\quad U_n\cap\widetilde{U}_n^{\perp}=\{0\}.$$

Let $Q_n:=I-P_n$, $n\in\mathbb{Z}$. Then Q_n is the continuous projection with range U_n and null space $\widetilde{U}_n^{\perp}$. Clearly, $\|Q_n\|\leqslant\|P_n\|+1\leqslant A+1$ for all $n\in\mathbb{Z}$. By what has been proved before we have

$$\lim_{n\to-\infty}\|Q_nf-f\|=\lim_{n\to\infty}\|Q_{-n}f-f\|=0.$$

This shows that $\lim_{n\to-\infty}\|P_n f\| = 0$.

It follows from (2.2) that:

$$\lim_{n\to\infty}\big\|(P_n f - P_{-n} f) - f\big\| = 0$$

for each $f \in H$. We have $P_n f - P_{-n} f = \sum_{k=-n}^{n-1}(P_{k+1} f - P_k f)$. But

$$P_{k+1} f - P_k f = (I - P_k)(P_{k+1} f) \in V_{k+1} \cap \widetilde{V}_k^{\perp} = W_k.$$

Hence, $P_n f - P_{-n} f \in \sum_{k=-n}^{n-1} W_k$. This shows that $\sum_{n\in\mathbb{Z}} W_n$ is dense in H. □

Now let us turn to multiresolution analysis for the Hilbert space $H = L_2(\mathbb{R})$. Let ϕ and $\tilde{\phi}$ be compactly supported functions in $L_2(\mathbb{R})$ satisfying the refinement equations (1.3) and (1.4), respectively. In addition, we assume that $\hat{\phi}(0) = \hat{\tilde{\phi}}(0) = 1$. For $n \in \mathbb{Z}$, let V_n be the closure of the linear span of $\{\phi(2^n \cdot -j)\colon j \in \mathbb{Z}\}$ in $L_2(\mathbb{R})$. Then $(V_n)_{n\in\mathbb{Z}}$ forms a multiresolution of $L_2(\mathbb{R})$ (see [14, Theorem 2.2]). Similarly, let $\widetilde{V}_n$ be the closure of the linear span of $\{\tilde{\phi}(2^n \cdot -j)\colon j \in \mathbb{Z}\}$ in $L_2(\mathbb{R})$. Then $(\widetilde{V}_n)_{n\in\mathbb{Z}}$ also forms a multiresolution of $L_2(\mathbb{R})$. The following lemma asserts that $(V_n)_{n\in\mathbb{Z}}$ and $(\widetilde{V}_n)_{n\in\mathbb{Z}}$ satisfy the conditions of Lemma 2.1, provided $[\phi, \tilde{\phi}](\xi) \neq 0$ for all $\xi \in [0, 2\pi]$.

Lemma 2.2. *If $[\phi, \tilde{\phi}](\xi) \neq 0$ for all $\xi \in [0, 2\pi]$, then $L_2(\mathbb{R})$ is the direct sum of V_n and $\widetilde{V}_n^{\perp}$ for every $n \in \mathbb{Z}$*

$$L_2(\mathbb{R}) = V_n + \widetilde{V}_n^{\perp} \quad \textit{and} \quad V_n \cap \widetilde{V}_n^{\perp} = \{0\}.$$

Let P_n be the continuous projection with range V_n and null space $\widetilde{V}_n^{\perp}$. Then there exists a positive number B such that $\|P_n\| \leqslant B$ for all $n \in \mathbb{Z}$.

In order to prove this lemma we need to review some basic properties of shift-invariant spaces. For $\phi \in L_2(\mathbb{R})$, we denote by $\mathbb{S}(\phi)$ the closure of the linear span of $\{\phi(\cdot - j)\colon j \in \mathbb{Z}\}$ in $L_2(\mathbb{R})$. Then $\mathbb{S}(\phi)$ is shift-invariant, that is,

$$f \in \mathbb{S}(\phi) \quad \Rightarrow \quad f(\cdot - k) \in \mathbb{S}(\phi) \quad \forall k \in \mathbb{Z}.$$

We call $\mathbb{S}(\phi)$ the shift-invariant space generated by ϕ. The shifts of ϕ are said to be *stable* if $\{\phi(\cdot - j)\colon j \in \mathbb{Z}\}$ is a Riesz sequence in $L_2(\mathbb{R})$. For a compactly supported function ϕ in $L_2(\mathbb{R})$, the shifts of ϕ are stable if and only if, for each $\xi \in \mathbb{R}$, there exists some $k \in \mathbb{Z}$ such that $\hat{\phi}(\xi + 2k\pi) \neq 0$ (see, e.g., [14]).

We denote by $\ell(\mathbb{Z})$ the linear space of complex-valued sequences on $\mathbb{Z}$, and by $\ell_0(\mathbb{Z})$ the linear space of all finitely supported sequences on $\mathbb{Z}$. Given $b \in \ell(\mathbb{Z})$, we define

$$\|b\|_p := \left(\sum_{j\in\mathbb{Z}} \big|b(j)\big|^p\right)^{1/p}$$

for $1 \leqslant p < \infty$, and define $\|b\|_\infty$ to be the supremum of $\{|b(j)|\colon j \in \mathbb{Z}\}$. For $1 \leqslant p \leqslant \infty$ we denote by $\ell_p(\mathbb{Z})$ the Banach space of all sequences b on $\mathbb{Z}$ such that $\|b\|_p < \infty$. Given $b \in \ell_1(\mathbb{Z})$, we use $\hat{b}$ to denote the corresponding Fourier series:

$$\hat{b}(\xi) := \sum_{j\in\mathbb{Z}} b(j)\,\mathrm{e}^{-ij\xi}, \quad \xi \in \mathbb{R}.$$

For $a, b \in \ell(\mathbb{Z})$, we define the *convolution* of a and b by

$$a * b(j) := \sum_{k\in\mathbb{Z}} a(j-k)b(k), \quad j \in \mathbb{Z},$$

whenever the above series is absolutely convergent. For example, if we use δ to denote the sequence given by $\delta(0) = 1$ and $\delta(j) = 0$ for $j \in \mathbb{Z} \setminus \{0\}$, then $a * \delta = a$ for all $a \in \ell(\mathbb{Z})$. If $a \in \ell_p(\mathbb{Z})$, $1 \leqslant p \leqslant \infty$, and $b \in \ell_1(\mathbb{Z})$, then $a * b$ is well defined and $\|a * b\|_p \leqslant \|a\|_p \|b\|_1$. In particular, if both a and b lie in $\ell_1(\mathbb{Z})$, then $a * b \in \ell_1(\mathbb{Z})$ and $(a * b)\hat{}(\xi) = \hat{a}(\xi)\hat{b}(\xi)$, $\xi \in \mathbb{R}$.

The proof of Lemma 2.2 requires some knowledge about discrete convolution equations (see [11]). Let a be an element in $\ell_0(\mathbb{Z})$ such that $\hat{a}(\xi) \neq 0$ for all $\xi \in \mathbb{R}$. For given $v \in \ell_p(\mathbb{Z})$, the discrete convolution equation

$$a * u = v \tag{2.3}$$

has a unique solution for $u \in \ell_p(\mathbb{Z})$. To see this, let

$$c(j) := \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\hat{a}(\xi)} \mathrm{e}^{ij\xi} \, \mathrm{d}\xi, \quad j \in \mathbb{Z}.$$

Then the sequence c decays exponentially fast, and $\hat{c}(\xi)\hat{a}(\xi) = 1$ for all $\xi \in \mathbb{Z}$. Hence $c * a = \delta$. If $a * u = v$, then it follows that:

$$u = \delta * u = (c * a) * u = c * (a * u) = c * v.$$

This shows that $u = c * v$ is the unique solution of the discrete convolution equation (2.3). Since the sequence c decays exponentially fast, c lies in $\ell_1(\mathbb{Z})$. Consequently, u lies in $\ell_p(\mathbb{Z})$ and $\|u\|_p \leqslant \|c\|_1 \|v\|_p$. Note that $\|c\|_1$ is independent of v.

Proof of Lemma 2.2. For $j \in \mathbb{Z}$, let $\lambda(j) := \langle \phi, \tilde{\phi}(\cdot - j) \rangle$. Then we have

$$\hat{\lambda}(\xi) = [\phi, \tilde{\phi}](\xi) = \sum_{k\in\mathbb{Z}} \hat{\phi}(\xi + 2k\pi)\overline{\hat{\tilde{\phi}}(\xi + 2k\pi)} \neq 0 \quad \forall \xi \in \mathbb{R}.$$

Consequently, for each $\xi \in [0, 2\pi]$, there exists some $k \in \mathbb{Z}$ such that $\hat{\phi}(\xi + 2k\pi) \neq 0$. This shows that the shifts of ϕ are stable. In other words, $\{\phi(\cdot - j)\colon\ j \in \mathbb{Z}\}$ is a Riesz sequence in $L_2(\mathbb{R})$.

Given $f \in L_2(\mathbb{R})$, we wish to find $v_n \in V_n$ such that $f - v_n$ is orthogonal to $\widetilde{V}_n$. Since $\{\phi(\cdot - j)\colon\ j \in \mathbb{Z}\}$ is a Riesz sequence, an element v_n in V_n can be uniquely represented as

$$v_n = \sum_{j\in\mathbb{Z}} b_n(j) 2^{n/2} \phi(2^n \cdot -j), \tag{2.4}$$

where $b_n \in \ell_2(\mathbb{Z})$. Moreover, there exist positive constants A_1 and B_1 independent of n such that $A_1\|b_n\|_2 \leqslant \|v_n\|_2 \leqslant B_1\|b_n\|_2$. Note that $(f - v_n) \perp \widetilde{V}_n$ if and only if

$$\langle v_n, 2^{n/2}\tilde{\phi}(2^n \cdot -k) \rangle = \langle f, 2^{n/2}\tilde{\phi}(2^n \cdot -k) \rangle \quad \forall k \in \mathbb{Z}. \tag{2.5}$$

It follows from (2.4) that:

$$\begin{aligned}\big\langle v_n, 2^{n/2}\tilde{\phi}\big(2^n\cdot -k\big)\big\rangle &= \sum_{j\in\mathbb{Z}} b_n(j)2^n\big\langle \phi\big(2^n\cdot -j\big), \tilde{\phi}\big(2^n\cdot -k\big)\big\rangle\\ &= \sum_{j\in\mathbb{Z}} b_n(j)\big\langle \phi(\cdot - j), \tilde{\phi}(\cdot - k)\big\rangle.\end{aligned}$$

Hence, (2.5) is equivalent to the discrete convolution equation $b_n * \lambda = c_n$, where

$$c_n(k) := \big\langle f, 2^{n/2}\tilde{\phi}\big(2^n\cdot -k\big)\big\rangle = \big\langle 2^{-n/2} f\big(2^{-n}\cdot\big), \tilde{\phi}(\cdot - k)\big\rangle, \quad k\in\mathbb{Z}.$$

Applying [14, Theorem 3.1] to the sequence c_n, we obtain

$$\|c_n\|_2 \leqslant B_2\big\|2^{-n/2} f\big(2^{-n}\cdot\big)\big\|_2 = B_2\|f\|_2,$$

where

$$B_2 := \Big\|\sum_{k\in\mathbb{Z}}\big|\tilde{\phi}(\cdot - k)\big|\Big\|_{L_2([0,1))} < \infty.$$

By our assumption, $\hat{\lambda}(\xi)\neq 0$ for all $\xi\in\mathbb{R}$. Therefore, the discrete convolution equation $b_n * \lambda = c_n$ has a unique solution for b_n and $\|b_n\|_2 \leqslant B_3\|c_n\|_2$ for some constant B_3 independent of n. Consequently, there exists a unique element v_n in V_n such that $(f - v_n)\perp \widetilde{V}_n$. Let $u_n := f - v_n$. Then f can be uniquely decomposed as the sum of v_n and u_n with $v_n\in V_n$ and $u_n\in \widetilde{V}_n^{\perp}$. This shows that $L_2(\mathbb{R})$ is the direct sum of V_n and $\widetilde{V}_n^{\perp}$.

Let P_n be the continuous projection with range V_n and null space $\widetilde{V}_n^{\perp}$. Then $P_n f = v_n$ for $f\in L_2(\mathbb{R})$, where v_n is the element in V_n such that $f - v_n$ is orthogonal to $\widetilde{V}_n$. Hence, $\|P_n f\|_2 = \|v_n\|_2 \leqslant B_1\|b_n\|_2$. Combining the above estimates together, we obtain $\|P_n f\|_2 \leqslant B_1B_2B_3\|f\|_2$. Therefore, with $B := B_1B_2B_3$, $\|P_n\|\leqslant B$ for all $n\in\mathbb{Z}$. □

3. Riesz sequences

In this section we investigate Riesz sequences generated by a pair of compactly supported functions in $L_2(\mathbb{R})$.

A sequence $(g_n)_{n=1,2,\dots}$, in a Hilbert space H is called a **Bessel sequence** if there exists a positive constant B such that

$$\sum_{n=1}^{\infty}\big|\langle f, g_n\rangle\big|^2 \leqslant B\|f\|^2 \quad \forall f\in H. \tag{3.1}$$

Let ψ and $\tilde{\psi}$ be two compactly supported functions in $H^{\mu}(\mathbb{R})$ for some $\mu > 0$. Suppose

$$\int_{\mathbb{R}}\psi(x)\,\mathrm{d}x = 0 \quad\text{and}\quad \int_{\mathbb{R}}\tilde{\psi}(x)\,\mathrm{d}x = 0.$$

For $j,k\in\mathbb{Z}$, let

$$\psi_{jk}(x) := 2^{j/2}\psi\big(2^j x - k\big) \quad\text{and}\quad \tilde{\psi}_{jk}(x) := 2^{j/2}\tilde{\psi}\big(2^j x - k\big), \quad x\in\mathbb{R}.$$

Then $(\psi_{jk})_{j,k\in\mathbb{Z}}$ is a Bessel sequence in $L_2(\mathbb{R})$, and so is $(\tilde{\psi}_{jk})_{j,k\in\mathbb{Z}}$. This result was established by Villemoes [19]. Also, see [12, Section 2] for a discussion related to this problem.

Theorem 3.1. *Suppose that ψ and $\tilde{\psi}$ are compactly supported functions in $H^\mu(\mathbb{R})$ for some $\mu > 0$ and $\hat{\psi}(0) = \hat{\tilde{\psi}}(0) = 0$. Then the sequences $(\psi_{jk})_{j,k\in\mathbb{Z}}$ and $(\tilde{\psi}_{jk})_{j,k\in\mathbb{Z}}$ are Riesz sequences in $L_2(\mathbb{R})$, provided the following two conditions are satisfied*:

(a) $\langle \psi_{jk}, \tilde{\psi}_{j'k'} \rangle = 0$ *for* $j \neq j'$;
(b) $[\psi, \tilde{\psi}](\xi) \neq 0$ *for all* $\xi \in [0, 2\pi]$.

Proof. Since $(\psi_{jk})_{j,k\in\mathbb{Z}}$ and $(\tilde{\psi}_{jk})_{j,k\in\mathbb{Z}}$ are Bessel sequences in $L_2(\mathbb{R})$, there exist two positive constants A and B such that

$$\sum_{j,k\in\mathbb{Z}} \bigl|\langle f, \psi_{jk}\rangle\bigr|^2 \leqslant A\|f\|_2^2 \quad \text{and} \quad \sum_{j,k\in\mathbb{Z}} \bigl|\langle f, \tilde{\psi}_{jk}\rangle\bigr|^2 \leqslant B\|f\|_2^2 \quad \forall f \in L_2(\mathbb{R}). \tag{3.2}$$

Let $(c_{jk})_{j,k\in\mathbb{Z}}$ be a sequence such that $\sum_{j,k\in\mathbb{Z}} |c_{jk}|^2 < \infty$. Then $f := \sum_{j,k\in\mathbb{Z}} c_{jk}\psi_{jk}$ is an element in $L_2(\mathbb{R})$ and

$$\|f\|_2^2 \leqslant A \sum_{j,k\in\mathbb{Z}} |c_{jk}|^2. \tag{3.3}$$

By condition (a), $\langle \psi_{jk}, \tilde{\psi}_{j'k'} \rangle = 0$ for $j \neq j'$. Consequently,

$$\langle f, \tilde{\psi}_{jk'}\rangle = \sum_{k\in\mathbb{Z}} c_{jk}\langle \psi_{jk}, \tilde{\psi}_{jk'}\rangle = \sum_{k\in\mathbb{Z}} \tau(k'-k)c_{jk}, \tag{3.4}$$

where $\tau(k) := \langle \psi, \tilde{\psi}(\cdot - k)\rangle$, $k \in \mathbb{Z}$. Indeed, we have

$$\langle \psi_{jk}, \tilde{\psi}_{jk'}\rangle = \bigl\langle 2^{j/2}\psi\bigl(2^j\cdot - k\bigr), 2^{j/2}\tilde{\psi}\bigl(2^j\cdot - k'\bigr)\bigr\rangle = \bigl\langle \psi(\cdot - k), \tilde{\psi}(\cdot - k')\bigr\rangle = \tau(k'-k).$$

By condition (b),

$$\hat{\tau}(\xi) = \sum_{k\in\mathbb{Z}} \tau(k)\,\mathrm{e}^{-ik\xi} = [\psi, \tilde{\psi}](\xi) \neq 0 \quad \forall \xi \in [0, 2\pi].$$

Fix $j \in \mathbb{Z}$ for the time being. We may view (3.4) as a discrete convolution equation for $(c_{jk})_{k\in\mathbb{Z}}$. By the discussion on discrete convolution equations in Section 2 we see that there exists a positive constant K independent of j such that

$$\sum_{k\in\mathbb{Z}} |c_{jk}|^2 \leqslant K \sum_{k'\in\mathbb{Z}} \bigl|\langle f, \tilde{\psi}_{jk'}\rangle\bigr|^2.$$

Therefore, with the help of (3.2), we obtain

$$\sum_{j,k\in\mathbb{Z}} |c_{jk}|^2 \leqslant K \sum_{j,k'\in\mathbb{Z}} \bigl|\langle f, \tilde{\psi}_{jk'}\rangle\bigr|^2 \leqslant KB\|f\|_2^2.$$

This together with (3.3) shows that $(\psi_{jk})_{j,k\in\mathbb{Z}}$ is a Riesz sequence in $L_2(\mathbb{R})$. The same argument shows that $(\tilde{\psi}_{jk})_{j,k\in\mathbb{Z}}$ is also a Riesz sequence in $L_2(\mathbb{R})$. □

4. Wavelet bases

In this section we give an explicit construction of wavelet bases for $L_2(\mathbb{R})$. Our construction is motivated by the work of Micchelli in [16].

Let ϕ and $\tilde{\phi}$ be compactly supported functions in $L_2(\mathbb{R})$ satisfying the refinement equations (1.3) and (1.4), respectively. Taking Fourier transform of both sides of the refinement equations (1.3) and (1.4), we obtain

$$\hat{\phi}(\xi)=\frac{1}{2}\hat{a}(\xi/2)\hat{\phi}(\xi/2)\quad\text{and}\quad\hat{\tilde{\phi}}(\xi)=\frac{1}{2}\hat{\tilde{a}}(\xi/2)\hat{\tilde{\phi}}(\xi/2),\quad\xi\in\mathbb{R}.\tag{4.1}$$

Throughout this section we assume that

$$[\phi,\tilde{\phi}](\xi)\neq 0\quad\forall\xi\in\mathbb{R}.$$

Note that $[\phi,\tilde{\phi}](\xi)=\hat{\lambda}(\xi)$ for all $\xi\in\mathbb{R}$, where λ is the sequence given by

$$\lambda(j):=\langle\phi,\tilde{\phi}(\cdot-j)\rangle,\quad j\in\mathbb{Z}.$$

Lemma 4.1. *Let μ and ν be the sequences given by*

$$\mu(j):=\langle\tilde{\phi},\phi(2\cdot-j)\rangle\quad\textit{and}\quad\nu(j):=\langle\phi,\tilde{\phi}(2\cdot-j)\rangle,\quad j\in\mathbb{Z},$$

and let ψ and $\tilde{\psi}$ be the functions given by

$$\psi:=\sum_{j\in\mathbb{Z}}(-1)^j\overline{\mu(1-j)}\phi(2\cdot-j)\quad\textit{and}\quad\tilde{\psi}:=\sum_{j\in\mathbb{Z}}(-1)^j\overline{\nu(1-j)}\,\tilde{\phi}(2\cdot-j).\tag{4.2}$$

Then the following statements are true:

(a) $\langle\psi,\tilde{\phi}(\cdot-k)\rangle=0$ *for all $k\in\mathbb{Z}$, and $\hat{\mu}(\xi)=\hat{\tilde{a}}(\xi)\,\overline{\hat{\lambda}(\xi)}/2$ for all $\xi\in\mathbb{R}$;*
(b) $\langle\tilde{\psi},\phi(\cdot-k)\rangle=0$ *for all $k\in\mathbb{Z}$, and $\hat{\nu}(\xi)=\hat{a}(\xi)\hat{\lambda}(\xi)/2$ for all $\xi\in\mathbb{R}$.*

Proof. We have

$$\langle\tilde{\phi}(\cdot-k),\psi\rangle=\sum_{j\in\mathbb{Z}}(-1)^j\mu(1-j)\langle\tilde{\phi}(\cdot-k),\phi(2\cdot-j)\rangle=\sum_{j\in\mathbb{Z}}(-1)^j\mu(1-j)\mu(j-2k)=0.$$

Next, for $\xi\in\mathbb{R}$ we have

$$\begin{aligned}\hat{\mu}(\xi)&=\sum_{j\in\mathbb{Z}}\langle\tilde{\phi},\phi(2\cdot-j)\rangle\,\mathrm{e}^{-ij\xi}=\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\tilde{a}(k)\langle\tilde{\phi}(2\cdot-k),\phi(2\cdot-j)\rangle\,\mathrm{e}^{-ij\xi}\\&=\frac{1}{2}\sum_{k\in\mathbb{Z}}\tilde{a}(k)\,\mathrm{e}^{-ik\xi}\sum_{j\in\mathbb{Z}}\overline{\lambda(k-j)\,\mathrm{e}^{-i(k-j)\xi}}.\end{aligned}$$

The proves part (a). The proof for part (b) is similar. □

Lemma 4.2. *Let ψ and $\tilde{\psi}$ be the functions given in* (4.2). *Then*

$$[\psi,\tilde{\psi}](\xi)=\frac{1}{4}\hat{\lambda}(\xi/2)\hat{\lambda}(\xi/2+\pi)\hat{\lambda}(\xi)\quad\forall\xi\in\mathbb{R}.$$

Proof. By the definition of ψ, for $\xi\in\mathbb{R}$ we have

$$\hat{\psi}(\xi)=\frac{1}{2}\hat{\phi}(\xi/2)\sum_{j\in\mathbb{Z}}(-1)^j\overline{\mu(1-j)}\,\mathrm{e}^{-ij\xi/2}=-\frac{1}{2}\hat{\phi}(\xi/2)\,\mathrm{e}^{-i\xi/2}\sum_{j\in\mathbb{Z}}\overline{\mu(j)\,\mathrm{e}^{-ij(\xi/2-\pi)}}.$$

It follows that:

$$\hat{\psi}(\xi) = -\frac{1}{2}\hat{\phi}(\xi/2)\,\mathrm{e}^{-i\xi/2}\overline{\hat{\mu}(\xi/2-\pi)}, \quad \xi \in \mathbb{R}. \tag{4.3}$$

Similarly,

$$\hat{\tilde{\psi}}(\xi) = -\frac{1}{2}\hat{\tilde{\phi}}(\xi/2)\,\mathrm{e}^{-i\xi/2}\overline{\hat{\nu}(\xi/2-\pi)}, \quad \xi \in \mathbb{R}. \tag{4.4}$$

By using the above expressions of $\hat{\psi}$ and $\hat{\tilde{\psi}}$, for $\xi \in \mathbb{R}$ we have

$$\begin{aligned}
[\psi, \tilde{\psi}](\xi) &= \frac{1}{4}\sum_{k\in\mathbb{Z}} \overline{\hat{\mu}(\xi/2+k\pi-\pi)}\,\hat{\nu}(\xi/2+k\pi-\pi)\hat{\phi}(\xi/2+k\pi)\,\overline{\hat{\tilde{\phi}}(\xi/2+k\pi)} \\
&= \frac{1}{4}\sum_{k\in\mathbb{Z}} \overline{\hat{\mu}(\xi/2+2k\pi-\pi)}\,\hat{\nu}(\xi/2+2k\pi-\pi)\,\hat{\phi}(\xi/2+2k\pi)\,\overline{\hat{\tilde{\phi}}(\xi/2+2k\pi)} \\
&\quad + \frac{1}{4}\sum_{k\in\mathbb{Z}} \overline{\hat{\mu}(\xi/2+2k\pi)}\,\hat{\nu}(\xi/2+2k\pi)\,\hat{\phi}(\xi/2+2k\pi+\pi)\,\overline{\hat{\tilde{\phi}}(\xi/2+2k\pi+\pi)} \\
&= \frac{1}{4}\big[\overline{\hat{\mu}(\xi/2+\pi)}\,\hat{\nu}(\xi/2+\pi)\,\hat{\lambda}(\xi/2) + \overline{\hat{\mu}(\xi/2)}\,\hat{\nu}(\xi/2)\,\hat{\lambda}(\xi/2+\pi)\big].
\end{aligned}$$

By Lemma 4.1 we have $\hat{\mu}(\xi) = \hat{\tilde{a}}(\xi)\,\overline{\hat{\lambda}(\xi)}/2$ and $\hat{\nu}(\xi) = \hat{a}(\xi)\hat{\lambda}(\xi)/2$. Consequently,

$$[\psi, \tilde{\psi}](\xi) = \frac{1}{16}\hat{\lambda}(\xi/2)\,\hat{\lambda}(\xi/2+\pi)\big[\hat{a}(\xi/2+\pi)\,\overline{\hat{\tilde{a}}(\xi/2+\pi)}\hat{\lambda}(\xi/2+\pi) + \hat{a}(\xi/2)\,\overline{\hat{\tilde{a}}(\xi/2)}\hat{\lambda}(\xi/2)\big].$$

But for $\xi \in \mathbb{R}$ we have

$$\begin{aligned}
&\hat{a}(\xi/2)\,\overline{\hat{\tilde{a}}(\xi/2)}\,\hat{\lambda}(\xi/2) + \hat{a}(\xi/2+\pi)\,\overline{\hat{\tilde{a}}(\xi/2+\pi)}\,\hat{\lambda}(\xi/2+\pi) \\
&\quad + \sum_{k\in\mathbb{Z}} \hat{a}(\xi/2+\pi)\,\overline{\hat{\tilde{a}}(\xi/2+\pi)}\hat{\phi}(\xi/2+\pi+2k\pi)\,\overline{\hat{\tilde{\phi}}(\xi/2+\pi+2k\pi)} \\
&= 4\bigg(\sum_{k\in\mathbb{Z}}\hat{\phi}(\xi+4k\pi)\overline{\hat{\tilde{\phi}}(\xi+4k\pi)} + \sum_{k\in\mathbb{Z}}\hat{\phi}(\xi+4k\pi+2\pi)\,\overline{\hat{\tilde{\phi}}(\xi+4k\pi+2\pi)}\bigg) \\
&= 4\hat{\lambda}(\xi),
\end{aligned}$$

where (4.1) has been used to derive the second equality. This completes the proof of the lemma. □

Let ψ and $\tilde{\psi}$ be as given in (4.2). By Lemma 4.1 we have $\hat{\psi}(0) = \hat{\tilde{\psi}}(0) = 0$. For $j, k \in \mathbb{Z}$, let

$$\psi_{jk} := 2^{j/2}\psi\big(2^j\cdot - k\big) \quad \text{and} \quad \tilde{\psi}_{jk} := 2^{j/2}\tilde{\psi}\big(2^j\cdot - k\big).$$

Since ϕ and $\tilde{\phi}$ are compactly supported refinable functions in $L_2(\mathbb{R})$, they belong to $H^\mu(\mathbb{R})$ for some $\mu > 0$ (see [13,20]). Consequently, ψ and $\tilde{\psi}$ lie in $H^\mu(\mathbb{R})$. We are in a position to establish the following result.

Lemma 4.3. *$(\psi_{jk})_{j,k\in\mathbb{Z}}$ is a Riesz sequence in $L_2(\mathbb{R})$.*

Proof. This result will be established by showing that $(\psi_{jk})_{j,k\in\mathbb{Z}}$ and $(\tilde\psi_{jk})_{j,k\in\mathbb{Z}}$ satisfy conditions (a) and (b) in Theorem 3.1.

Suppose $j, j', k, k' \in \mathbb{Z}$. For $j > j'$, $\tilde\psi_{j'k'}$ is a linear combination of $\tilde\phi_{jr}$, $r \in \mathbb{Z}$. By Lemma 4.1, $\langle\psi_{jk}, \tilde\phi_{jr}\rangle = 0$ for all $r \in \mathbb{Z}$. Hence,

$$\langle\psi_{jk}, \tilde\psi_{j'k'}\rangle = 0. \tag{4.5}$$

For $j < j'$, ψ_{jk} is a linear combination of $\phi_{j's}$, $s \in \mathbb{Z}$. But $\langle\phi_{j's}, \tilde\psi_{j'k'}\rangle = 0$. Hence, (4.5) is true for $j \neq j'$.

Since $\hat\lambda(\xi) \neq 0$ for all $\xi \in \mathbb{R}$, by Lemma 4.2 we have $[\psi, \tilde\psi](\xi) \neq 0$ for all $\xi \in \mathbb{R}$. Therefore, $(\psi_{jk})_{j,k\in\mathbb{Z}}$ is a Riesz sequence in $L_2(\mathbb{R})$, by Theorem 3.1. □

Our goal is to demonstrate that $\{\psi_{jk}\colon j, k \in \mathbb{Z}\}$ is a Riesz basis for $L_2(\mathbb{R})$. For this purpose it remains to show that the linear span of $\{\psi_{jk}\colon j, k \in \mathbb{Z}\}$ is dense in $L_2(\mathbb{R})$. For each $j \in \mathbb{Z}$, let W_j denotes the closure of the linear span of $\{\psi_{jk}\colon k \in \mathbb{Z}\}$ in $L_2(\mathbb{R})$. The rest of this section is devoted to proving that $\sum_{j\in\mathbb{Z}} W_j$ is dense in $L_2(\mathbb{R})$.

Let $V := \mathbb{S}(\phi)$, $\widetilde V := \mathbb{S}(\tilde\phi)$, and $W := \mathbb{S}(\psi)$. For $n \in \mathbb{Z}$, by σ_n we denote the scaling operator given by $\sigma_n f(x) := f(2^n x)$ for $x \in \mathbb{R}$ and a function f on $\mathbb{R}$. Then $W_n = \sigma_n(W)$. Let $V_n := \sigma_n(V)$ and $\widetilde V_n := \sigma_n(\widetilde V)$. It was proved in Lemma 2.2 that $(V_n)_{n\in\mathbb{Z}}$ and $(\tilde V_n)_{n\in\mathbb{Z}}$ satisfy the conditions in Lemma 2.1. Therefore, if we can show $W_n = V_{n+1} \cap \widetilde V_n^{\perp}$, then Lemma 2.1 tells us that $\sum_{n\in\mathbb{Z}} W_n$ is dense in $L_2(\mathbb{R})$.

In order to prove $W_n = V_{n+1} \cap \widetilde V_n^{\perp}$, it suffices to show that this statement is true for $n = 0$, i.e., $W = V_1 \cap \widetilde V^{\perp}$. By Lemma 4.1, $\langle\psi, \tilde\phi(\cdot - k)\rangle = 0$ for all $k \in \mathbb{Z}$. Hence, $W \subseteq V_1 \cap \widetilde V^{\perp}$. By Lemma 2.2, $L_2(\mathbb{R})$ is the direct sum of V_n and $\widetilde V_n^{\perp}$. It follows that V_1 is the direct sum of V and $V_1 \cap \widetilde V^{\perp}$. Thus, in order to prove $W = V_1 \cap \widetilde V^{\perp}$, it suffices to show that $V_1 = V + W$. For this purpose we need a result on characterization of shift-invariant subspaces of $L_2(\mathbb{R})$.

For a finite subset Φ of $L_2(\mathbb{R})$, we use $\mathbb{S}(\Phi)$ to denote the closure of the linear span of $\{\phi(\cdot - j)\colon \phi \in \Phi,\ j \in \mathbb{Z}\}$ in $L_2(\mathbb{R})$. It was proved in [9] that a function $f \in L_2(\mathbb{R})$ lies in $\mathbb{S}(\Phi)$ if and only if there exist 2π-periodic functions τ_ϕ $(\phi \in \Phi)$ such that

$$\hat f(\xi) = \sum_{\phi\in\Phi} \tau_\phi(\xi)\hat\phi(\xi)$$

for almost every $\xi \in \mathbb{R}$. Also, see [2,11] for some elementary proofs of this result.

Let $f \in V_1$. Then $f(\cdot/2) \in V$. Hence, there exists a 2π-periodic function τ such that $\hat f(2\xi) = \tau(\xi)\hat\phi(\xi)$ for almost every $\xi \in \mathbb{R}$. In other words,

$$\hat f(\xi) = \tau(\xi/2)\hat\phi(\xi/2). \tag{4.6}$$

In order to show $f \in V + W$, it suffices to find 2π-periodic functions η_1 and η_2 such that

$$\hat f(\xi) = \eta_1(\xi)\hat\phi(\xi) + \eta_2(\xi)\hat\psi(\xi) \tag{4.7}$$

for almost every $\xi \in \mathbb{R}$. Recall from (4.1) and (4.3) that

$$\hat\phi(\xi) = \frac{1}{2}\hat a(\xi/2)\hat\phi(\xi/2) \quad \text{and} \quad \hat\psi(\xi) = -\frac{1}{2}\mathrm{e}^{-i\xi/2}\overline{\hat\mu(\xi/2+\pi)}\,\hat\phi(\xi/2), \quad \xi \in \mathbb{R}.$$

Comparing (4.7) with (4.6), we see that (4.7) is valid if

$$\begin{bmatrix} \frac{1}{2}\hat a(\xi/2) & -\frac{1}{2}\mathrm{e}^{-i\xi/2}\overline{\hat\mu(\xi/2+\pi)} \\ \frac{1}{2}\hat a(\xi/2+\pi) & \frac{1}{2}\mathrm{e}^{-i\xi/2}\overline{\hat\mu(\xi/2)} \end{bmatrix} \begin{bmatrix} \eta_1(\xi) \\ \eta_2(\xi) \end{bmatrix} = \begin{bmatrix} \tau(\xi/2) \\ \tau(\xi/2+\pi) \end{bmatrix}. \tag{4.8}$$

The determinant of the above 2×2 matrix is

$$\Delta(\xi) = \frac{1}{4} e^{-i\xi/2} \big[\hat{a}(\xi/2)\,\overline{\hat{\mu}(\xi/2)} + \hat{a}(\xi/2+\pi)\,\overline{\hat{\mu}(\xi/2+\pi)}\big].$$

Clearly, $\Delta(\xi + 2\pi) = -\Delta(\xi)$ for all $\xi \in \mathbb{R}$. Moreover, we have $\Delta(0) = 1/2 \neq 0$, since $\hat{a}(0) = 2$, $\hat{a}(\pi) = 0$, and $\hat{\mu}(0) = 1$. But there exists a Laurent polynomial p such that $\Delta(\xi) = p(e^{-i\xi/2})$. Hence, $\Delta(\xi) \neq 0$ for almost every $\xi \in \mathbb{R}$. Thus, (4.8) will be true for almost every $\xi \in \mathbb{R}$ if η_1 and η_2 are given by

$$\eta_1(\xi) = \frac{1}{\Delta(\xi)} \begin{vmatrix} \tau(\xi/2) & -\frac{1}{2} e^{-i\xi/2}\overline{\hat{\mu}(\xi/2+\pi)} \\ \tau(\xi/2+\pi) & \frac{1}{2} e^{-i\xi/2}\overline{\hat{\mu}(\xi/2)} \end{vmatrix}$$

and

$$\eta_2(\xi) = \frac{1}{\Delta(\xi)} \begin{vmatrix} \frac{1}{2}\hat{a}(\xi/2) & \tau(\xi/2) \\ \frac{1}{2}\hat{a}(\xi/2+\pi) & \tau(\xi/2+\pi) \end{vmatrix}.$$

Since $\Delta(\xi + 2\pi) = -\Delta(\xi)$ for all $\xi \in \mathbb{R}$, the functions η_1 and η_2 are 2π-periodic. Consequently, $f \in V + W$. This shows $V_1 = V + W$.

The above discussions are summarized in the following theorem.

Theorem 4.4. *The set* $\{\psi_{jk}\colon\ j, k \in \mathbb{Z}\}$ *is a Riesz basis for* $L_2(\mathbb{R})$.

5. Wavelet bases for Sobolev spaces

Let ϕ (respectively $\tilde{\phi}$) be the refinable function associated with the refinement mask a (respectively $\tilde{a}$), as given in (1.3) and (1.4). Let ψ and $\tilde{\psi}$ be the wavelets constructed in Section 4. Assuming $\phi \in H^m(\mathbb{R})$, we will show that $\{2^{j/2}\psi^{(m)}(2^j \cdot - k)\colon\ j, k \in \mathbb{Z}\}$ is a Riesz basis for $L_2(\mathbb{R})$. Consequently, $\{\psi_{jk}\colon\ j, k \in \mathbb{Z}\}$ forms a stable wavelet basis for the Sobolev space $H^m(\mathbb{R})$, that is, $\{\psi_{jk}\colon\ j, k \in \mathbb{Z}\}$ satisfies the inequalities in (1.2).

A lifting technique will be used in our arguments. Suppose f is a function in $L_2(\mathbb{R})$ and f is supported on the closed interval $[a, b]$ $(-\infty < a < b < \infty)$. If $\int_{\mathbb{R}} f(x)\,dx = 0$, then f can be lifted in the following sense. Let

$$g(x) := \int_{-\infty}^{x} f(t)\,dt, \quad x \in \mathbb{R}.$$

We have $g(x) = 0$ for $x < a$ and $x > b$. In other words, g is also supported on $[a, b]$. Moreover, $g' = f$. It follows that $\hat{f}(\xi) = (i\xi)\hat{g}(\xi)$ for all $\xi \in \mathbb{R}$. More generally, suppose $D^j \hat{f}(0) = 0$ for $j = 0, 1, \ldots, k-1$. By using an induction argument we see that there exists $h \in H^k(\mathbb{R})$ such that h is supported on $[a, b]$ and $D^k h = f$. Consequently, $\hat{f}(\xi) = (i\xi)^k \hat{h}(\xi)$ for all $\xi \in \mathbb{R}$.

Theorem 5.1. *Suppose* ϕ *lies in the Sobolev space* $H^m(\mathbb{R})$, *where* m *is a positive integer. If* $[\phi, \tilde{\phi}](\xi) \neq 0$ *for all* $\xi \in \mathbb{R}$, *then*

$$\big\{2^{j/2}\psi^{(m)}\big(2^j \cdot - k\big)\colon\ j, k \in \mathbb{Z}\big\}$$

is a Riesz basis for $L_2(\mathbb{R})$.

Proof. Since ϕ is a compactly supported refinable function in $H^m(\mathbb{R})$, there exists some $\mu > 0$ such that ϕ lies in $H^{m+\mu}(\mathbb{R})$ (see [13,20]). Consequently, ψ lies in $H^{m+\mu}(\mathbb{R})$.

By our assumption, $[\phi, \tilde{\phi}](\xi) \neq 0$ for all $\xi \in \mathbb{R}$; hence, the shifts of ϕ are stable. This together with the fact $\phi \in H^m(\mathbb{R})$ tells us (see [20])

$$D^r \hat{a}(\pi) = 0 \quad \text{for } r = 0, 1, \ldots, m.$$

Recall that $\lambda(j) = \langle \phi, \tilde{\phi}(\cdot - j)\rangle$ and $\nu(j) = \langle \phi, \tilde{\phi}(2\cdot - j)\rangle$, $j \in \mathbb{Z}$. By Lemma 4.1, we have $\hat{\nu}(\xi) = \hat{a}(\xi)\hat{\lambda}(\xi)/2$ for all $\xi \in \mathbb{R}$. This in connection with (4.4) gives

$$\hat{\tilde{\psi}}(\xi) = -\frac{1}{4}\mathrm{e}^{-i\xi/2}\hat{\tilde{\phi}}(\xi/2)\overline{\hat{a}(\xi/2+\pi)\hat{\lambda}(\xi/2+\pi)}, \quad \xi \in \mathbb{R}.$$

Hence, $D^r\hat{\tilde{\psi}}(0) = 0$ for $r = 0, 1, \ldots, m$. Therefore, there exists a compactly supported function $h \in H^m(\mathbb{R})$ such that

$$(-1)^m D^m h = \tilde{\psi}.$$

Consequently, $\hat{\tilde{\psi}}(\xi) = (-i\xi)^m \hat{h}(\xi)$, $\xi \in \mathbb{R}$. But $D^m\hat{\tilde{\psi}}(0) = 0$; hence, $\hat{h}(0) = 0$. For $j \neq j'$ we obtain

$$\begin{aligned}\left\langle \psi^{(m)}\left(2^j \cdot -k\right), h\left(2^{j'} \cdot -k'\right)\right\rangle &= \left\langle \psi\left(2^j \cdot -k\right), (-1)^m 2^{j'm} h^{(m)}\left(2^{j'} \cdot -k'\right)\right\rangle \\ &= 2^{j'm}\left\langle \psi\left(2^j \cdot -k\right), \tilde{\psi}\left(2^{j'} \cdot -k'\right)\right\rangle = 0.\end{aligned}$$

Moreover, we have

$$\left\langle \psi^{(m)}, h(\cdot - k)\right\rangle = \left\langle \psi, (-1)^m h^{(m)}(\cdot - k)\right\rangle = \left\langle \psi, \tilde{\psi}(\cdot - k)\right\rangle, \quad k \in \mathbb{Z}.$$

It follows that:

$$\left[\psi^{(m)}, h\right](\xi) = \sum_{k\in\mathbb{Z}}\left\langle \psi^{(m)}, h(\cdot - k)\right\rangle \mathrm{e}^{-ik\xi} = \sum_{k\in\mathbb{Z}}\left\langle \psi, \tilde{\psi}(\cdot - k)\right\rangle \mathrm{e}^{-ik\xi} = [\psi, \tilde{\psi}](\xi) \neq 0,$$

by Lemma 4.2. Since $\psi \in H^{m+\mu}(\mathbb{R})$, we have $\psi^{(m)} \in H^{\mu}(\mathbb{R})$. Furthermore, h lies in $H^m(\mathbb{R})$. Therefore, $\{2^{j/2}\psi^{(m)}(2^j \cdot -k)\colon j, k \in \mathbb{Z}\}$ is a Riesz sequence in $L_2(\mathbb{R})$, by Theorem 3.1. It remains to prove that the linear span of this set is dense in $L_2(\mathbb{R})$.

Since $D^r\hat{a}(\pi) = 0$ for $r = 0, 1, \ldots, m$, we can find a finitely supported mask b such that

$$\hat{a}(\xi) = \left(\frac{1+\mathrm{e}^{-i\xi}}{2}\right)^m \hat{b}(\xi), \quad \xi \in \mathbb{R}.$$

There exists a compactly supported function g in $L_2(\mathbb{R})$ such that $\hat{g}(0) = 1$ and

$$g(x) = \sum_{j\in\mathbb{Z}} b(j) g(2x - j), \quad x \in \mathbb{R}.$$

If we use ∇ to denote the difference operator given by $\nabla g := g - g(\cdot - 1)$, then $\nabla^m g = D^m\phi$ (see [10]). Hence,

$$\hat{\phi}(\xi) = \left(\frac{1-\mathrm{e}^{-i\xi}}{i\xi}\right)^m \hat{g}(\xi), \quad \xi \in \mathbb{R}.$$

Let $\tilde{b}$ be the mask given by

$$\hat{\tilde{b}}(\xi) = \left(\frac{1+\mathrm{e}^{-i\xi}}{2}\right)^m \hat{\tilde{a}}(\xi), \quad \xi \in \mathbb{R}.$$

There exists a compactly supported function $\tilde{g}$ in $L_2(\mathbb{R})$ such that $\hat{\tilde{g}}(0) = 1$ and

$$\tilde{g}(x) = \sum_{j\in\mathbb{Z}} \tilde{b}(j)\tilde{g}(2x - j), \quad x \in \mathbb{R}.$$

Again, $\tilde{g} \in H^m(\mathbb{R})$ and $D^m\tilde{g} = \nabla^m\tilde{\phi}$ (see [10]).

Let $U := \mathbb{S}(g)$ and $\widetilde{U} := \mathbb{S}(\tilde{g})$. For $n \in \mathbb{Z}$, let $U_n := \sigma_n(U)$ and $\widetilde{U}_n := \sigma_n(\widetilde{U})$, where σ_n is the scaling operator given in Section 4. We claim that $(U_n)_{n\in\mathbb{Z}}$ and $(\widetilde{U}_n)_{n\in\mathbb{Z}}$ satisfy the conditions of Lemma 2.1. To justify our claim, by Lemma 2.2, it suffices to show $[g, \tilde{g}](\xi) \neq 0$ for all $\xi \in \mathbb{R}$. Since $D^m\tilde{g} = \nabla^m\tilde{\phi}$, we have

$$\hat{\tilde{g}}(\xi) = \left(\frac{1 - \mathrm{e}^{-i\xi}}{i\xi}\right)^m \hat{\tilde{\phi}}(\xi), \quad \xi \in \mathbb{R}.$$

It follows that:

$$\hat{g}(\xi)\hat{\tilde{g}}(\xi) = \hat{g}(\xi)\left(\frac{1 - \mathrm{e}^{-i\xi}}{i\xi}\right)^m \hat{\tilde{\phi}}(\xi) = \hat{\phi}(\xi)\hat{\tilde{\phi}}(\xi), \quad \xi \in \mathbb{R}.$$

Therefore,

$$[g, \tilde{g}](\xi) = \sum_{k\in\mathbb{Z}} \hat{g}(\xi + 2k\pi)\overline{\hat{\tilde{g}}(\xi + 2k\pi)} = \sum_{k\in\mathbb{Z}} \hat{\phi}(\xi + 2k\pi)\overline{\hat{\tilde{\phi}}(\xi + 2k\pi)} \neq 0 \quad \forall \xi \in \mathbb{R}.$$

Let D^m denote the differential operator given by $D^m f := f^{(m)}$, $f \in H^m(\mathbb{R})$. Let $Y := \mathbb{S}(\psi^{(m)})$ and $Y_n := \sigma_n(Y)$, $n \in \mathbb{Z}$. Clearly, $U = D^m(V)$ and $Y = D^m(W)$, where $V = \mathbb{S}(\phi)$ and $W = \mathbb{S}(\psi)$. We observe that

$$\left\langle \psi^{(m)}, \tilde{g}(\cdot - k)\right\rangle = \left\langle \psi, (-1)^m \tilde{g}^{(m)}(\cdot - k)\right\rangle = \left\langle \psi, (-1)^m \nabla^m \tilde{\phi}(\cdot - k)\right\rangle = 0 \quad \forall k \in \mathbb{Z}.$$

Hence, $Y \subseteq \widetilde{U}^{\perp}$. Since $V_1 = V + W$, we have $U_1 = U + Y$. On the other hand, U_1 is the direct sum of U and $U_1 \cap \widetilde{U}^{\perp}$. This shows $Y = U_1 \cap \widetilde{U}^{\perp}$. Consequently, for every $n \in \mathbb{Z}$, $Y_n = U_{n+1} \cap \widetilde{U}_n^{\perp}$. By Lemma 2.1, $\sum_{n\in\mathbb{Z}} Y_n$ is dense in $L_2(\mathbb{R})$. We conclude that $\{2^{j/2}\psi^{(m)}(2^j\cdot - k)\colon\ j, k \in \mathbb{Z}\}$ is a Riesz basis for $L_2(\mathbb{R})$. □

6. Examples

In this section we apply the general theory to B-splines and construct spline wavelets with nice properties.

Recall that M_m, the B-spline of order m, is the convolution of m copies of $\chi_{[0,1]}$. The B-spline M_m is supported on $[0, m]$ and is symmetric about $m/2$; i.e., $M_m(m - x) = M_m(x)$ for all $x \in \mathbb{R}$. Moreover, M_m lies in $H^{m-1}(\mathbb{R})$. For positive integers m and n, we have $M_{m+n} = M_m * M_n$. It follows that $\widehat{M}_{m+n}(\xi) = \widehat{M}_m(\xi)\widehat{M}_n(\xi)$, $\xi \in \mathbb{R}$. In particular,

$$\widehat{M}_m(\xi) = \left(\widehat{M}_1(\xi)\right)^m = \left[\left(1 - \mathrm{e}^{-i\xi}\right)/(i\xi)\right]^m, \quad \xi \in \mathbb{R}.$$

Consequently, M_m satisfies the following refinement equation:

$$\widehat{M}_m(\xi) = \left(\frac{1 + \mathrm{e}^{-i\xi/2}}{2}\right)^m \widehat{M}_m(\xi/2), \quad \xi \in \mathbb{R}.$$

In what follows we need the following result of Schoenberg [18] on cardinal interpolation: if m is an even integer, then

$$\sum_{j\in\mathbb{Z}} M_m(j)\,\mathrm{e}^{-ij\xi} \neq 0 \quad \forall\xi\in\mathbb{R}.$$

Example 6.1. Suppose N is an odd integer. Let

$$\psi_N := \sum_{j=0}^{N} \frac{(-1)^j}{2}\bigl[M_{N+1}(j) + M_{N+1}(j+1)\bigr]M_N(2\cdot - j).$$

For $r = 0, 1, \ldots, N-1$, the set

$$\bigl\{2^{j/2}\psi_N^{(r)}\bigl(2^j\cdot - k\bigr):\ j,k\in\mathbb{Z}\bigr\}$$

is a Riesz basis for $L_2(\mathbb{R})$. Moreover, ψ_N is supported on $[0, N]$, and ψ_N is antisymmetric about $N/2$.

To verify our assertions, we choose $\phi = M_N$ and $\tilde{\phi} = M_1$. For $j\in\mathbb{Z}$ we have

$$\bigl\langle\phi, \tilde{\phi}(\cdot - j)\bigr\rangle = \int_{\mathbb{R}} M_N(x)M_1(x-j)\,\mathrm{d}x = \int_{\mathbb{R}} M_N(x)M_1(1+j-x)\,\mathrm{d}x = M_{N+1}(1+j).$$

Since $N+1$ is an even integer, we have

$$[\phi,\tilde{\phi}](\xi) = \sum_{j\in\mathbb{Z}} M_{N+1}(j+1)\,\mathrm{e}^{-ij\xi} \neq 0 \quad \forall\,\xi\in\mathbb{R}.$$

Recall that $\mu(j) = \langle\tilde{\phi}, \phi(2\cdot - j)\rangle$, $j\in\mathbb{Z}$. Hence,

$$\mu(j) = \int_{\mathbb{R}} M_1(x)M_N(2x-j)\,\mathrm{d}x = \frac{1}{2}\bigl[M_{N+1}(1-j) + M_{N+1}(2-j)\bigr].$$

It follows that:

$$\overline{\mu(1-j)} = \frac{1}{2}\bigl[M_{N+1}(j) + M_{N+1}(1+j)\bigr], \quad j\in\mathbb{Z}.$$

Therefore,

$$\psi_N = \sum_{j\in\mathbb{Z}} (-1)^j\,\overline{\mu(1-j)}\,\phi(2\cdot - j).$$

By Theorems 4.4 and 5.1, for $r = 0, 1, \ldots, N-1$, $\{2^{j/2}\psi_N^{(r)}(2^j\cdot - k):\ j,k\in\mathbb{Z}\}$ is a Riesz basis for $L_2(\mathbb{R})$. Clearly, ψ_N is supported on $[0, N]$. Furthermore, ψ_N is antisymmetric about $N/2$. Indeed, for $x\in\mathbb{R}$ we have

$$\begin{aligned}
\psi_N(N-x) &= \sum_{j=0}^{N} \frac{(-1)^j}{2}\bigl[M_{N+1}(j) + M_{N+1}(j+1)\bigr]M_N(2N-2x-j)\\
&= \sum_{j=0}^{N} \frac{(-1)^{N-j}}{2}\bigl[M_{N+1}(N-j) + M_{N+1}(N-j+1)\bigr]M_N(2x-j)\\
&= -\psi_N(x).
\end{aligned}$$

For the special case $N = 3$ we have

$$\psi_3(x) = \frac{1}{6} M_3(2x) - \frac{5}{6} M_3(2x-1) + \frac{5}{6} M_3(2x-2) - \frac{1}{6} M_3(2x-3), \quad x \in \mathbb{R}.$$

For $r = 0, 1, 2$, $\{2^{j/2}\psi_3^{(r)}(2^j \cdot - k):\ j, k \in \mathbb{Z}\}$ is a Riesz basis for $L_2(\mathbb{R})$.

Example 6.2. Suppose N is an even integer. Let

$$\psi_N := \sum_{j=0}^{N+2} \frac{(-1)^j}{4} \big[M_{N+2}(j-1) + 2M_{N+2}(j) + M_{N+2}(j+1)\big] M_N(2\cdot - j).$$

For $r = 0, 1, \ldots, N-1$, the set

$$\big\{2^{j/2}\psi_N^{(r)}(2^j \cdot - k):\ j, k \in \mathbb{Z}\big\}$$

is a Riesz basis for $L_2(\mathbb{R})$. Moreover, ψ_N is supported on $[0, N+1]$, and ψ_N is symmetric about $(N+1)/2$.

To verify our assertions, we choose $\phi = M_N$ and $\tilde{\phi} = M_2(\cdot + 1)$. For $j \in \mathbb{Z}$ we have

$$\big\langle \phi, \tilde{\phi}(\cdot - j)\big\rangle = \int_{\mathbb{R}} M_N(x) M_2(x+1-j)\,\mathrm{d}x = \int_{\mathbb{R}} M_N(x) M_2(1+j-x)\,\mathrm{d}x = M_{N+2}(1+j).$$

Since $N+2$ is an even integer, we obtain

$$[\phi, \tilde{\phi}](\xi) = \sum_{j \in \mathbb{Z}} M_{N+2}(j+1)\mathrm{e}^{-ij\xi} \neq 0 \quad \forall \xi \in \mathbb{R}.$$

Recall that $\mu(j) = \langle \tilde{\phi}, \phi(2\cdot - j)\rangle$, $j \in \mathbb{Z}$. Hence,

$$\begin{aligned}
\mu(j) &= \int_{\mathbb{R}} M_2(x+1) M_N(2x-j)\,\mathrm{d}x \\
&= \frac{1}{4} \int_{\mathbb{R}} \big[M_2(x+2) + 2M_2(x+1) + M_2(x)\big] M_N(x-j)\,\mathrm{d}x \\
&= \frac{1}{4} \big[M_{N+2}(-j) + 2M_{N+2}(1-j) + M_{N+2}(2-j)\big].
\end{aligned}$$

Therefore,

$$\psi_N = \sum_{j \in \mathbb{Z}} (-1)^j \overline{\mu(1-j)}\, \phi(2 \cdot - j).$$

By Theorems 4.4 and 5.1, for $r = 0, 1, \ldots, N-1$, $\{2^{j/2}\psi_N^{(r)}(2^j \cdot - k):\ j, k \in \mathbb{Z}\}$ is a Riesz basis for $L_2(\mathbb{R})$. Clearly, ψ_N is supported on $[0, N+1]$. Furthermore, ψ_N is symmetric about $(N+1)/2$.

For the special case $N = 4$ we have

$$\begin{aligned}
\psi_4(x) = \frac{1}{120}\big[&M_4(2x) - 28M_4(2x-1) + 119M_4(2x-2) - 184M_4(2x-3) \\
&+ 119M_4(2x-4) - 28M_4(2x-5) + M_4(2x-6)\big], \quad x \in \mathbb{R}.
\end{aligned}$$

For $r = 0, 1, 2, 3$, $\{2^{j/2}\psi_4^{(r)}(2^j \cdot - k):\ j, k \in \mathbb{Z}\}$ is a Riesz basis for $L_2(\mathbb{R})$.

References

[1] F. Bastin, C. Boigelot, Biorthogonal wavelets in $H^m(\mathbb{R})$, J. Fourier Anal. Appl. 4 (1998) 749–768.
[2] C. de Boor, R. DeVore, A. Ron, The structure of finitely generated shift-invariant subspaces in $L_2(\mathbb{R}^d)$, J. Funct. Anal. 119 (1994) 37–78.
[3] C. de Boor, R. DeVore, A. Ron, On the construction of multivariate (pre) wavelets, Constr. Approx. 9 (1993) 123–166.
[4] C.K. Chui, J.Z. Wang, On compactly supported spline wavelets and a duality principle, Trans. Amer. Math. Soc. 330 (1992) 903–916.
[5] C.K. Chui, J.Z. Wang, A general framework of compactly supported splines and wavelets, J. Approx. Theory 71 (1993) 263–304.
[6] A. Cohen, I. Daubechies, A stability criterion for biorthogonal wavelet bases and their related subband coding scheme, Duke Math. J. 68 (1992) 313–335.
[7] A. Cohen, I. Daubechies, J.-C. Feauveau, Biorthogonal bases of compactly supported wavelets, Comm. Pure Appl. Math. 45 (1992) 485–560.
[8] I. Daubechies, Ten Lectures on Wavelets, SIAM, Philadelphia, 1992.
[9] H. Helson, Lectures on Invariant Subspaces, Academic Press, New York, 1964.
[10] R.-Q. Jia, Subdivision schemes in L_p spaces, Adv. Comput. Math. 3 (1995) 309–341.
[11] R.-Q. Jia, Shift-invariant spaces and linear operator equations, Israel J. Math. 103 (1998) 259–288.
[12] R.-Q. Jia, Cascade algorithms in wavelet analysis, in: D.X. Zhou (Ed.), Wavelet Analysis: Twenty Years' Developments, World Scientific, Singapore, 2002, pp. 196–230.
[13] R.-Q. Jia, K.-S. Lau, D.-X. Zhou, L_p solutions of refinement equations, J. Fourier Anal. Appl. 7 (2001) 143–167.
[14] R.-Q. Jia, C.A. Micchelli, Using the refinement equations for the construction of pre-wavelets II: Powers of two, in: P.J. Laurent, A. Le Méhauté, L.L. Schumaker (Eds.), Curves and Surfaces, Academic Press, New York, 1991, pp. 209–246.
[15] Y. Meyer, Wavelets and Operators, Cambridge Univ. Press, Cambridge, 1992.
[16] C.A. Micchelli, Using the refinement equations for the construction of pre-wavelets, Numer. Algorithms 1 (1991) 75–116.
[17] C.A. Micchelli, On a family of filters in wavelet construction, Appl. Comput. Harmon. Anal. 4 (1997) 38–50.
[18] I.J. Schoenberg, Cardinal Spline Interpolation, SIAM, Philadelphia, 1973.
[19] L.F. Villemoes, Sobolev regularity of wavelets and stability of iterated filter banks, in: Y. Meyer, S. Roques (Eds.), Wavelet Analysis and Applications, Frontiéres, 1993, pp. 243–251.
[20] L.F. Villemoes, Wavelet analysis of refinement equations, SIAM J. Math. Anal. 25 (1994) 1433–1460.