Applied and Computational Harmonic Analysis **12**, 128–149 (2002)
doi:10.1006/acha.2001.0363, available online at http://www.idealibrary.com on IDEAL®

# Convergence of Vector Subdivision Schemes in Sobolev Spaces [1]

Di-Rong Chen

*Department of Applied Mathematics, Beijing University of Aeronautics and Astronautics, Beijing 100083, People's Republic of China*

Rong-Qing Jia

*Department of Mathematical Sciences, University of Alberta, Edmonton, T6G 2G1, Canada*

S. D. Riemenschneider

*Department of Mathematics, West Virginia University, Morgantown, West Virginia, 26506-6310*

*Communicated by Charles K. Chui*

Received July 14, 2000

In this paper we consider functional equations of the form

$$\Phi = \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\Phi(M\cdot-\alpha),$$

where $\Phi = (\phi_1, \ldots, \phi_r)^{\mathrm{T}}$ is an $r \times 1$ vector of functions on the $s$-dimensional Euclidean space, $a(\alpha)$, $\alpha \in \mathbb{Z}^s$, is a finitely supported sequence of $r \times r$ complex matrices, and $M$ is an $s\times s$ isotropic integer matrix such that $\lim_{n\to\infty} M^{-n} = 0$. We are interested in the question, for which sequences $a$ will there exist a solution to the functional equation with each function $\phi_j$, $j = 1, \ldots, r$, belonging to the Sobolev space $W_p^k(\mathbb{R}^s)$? Our approach will be to consider the convergence of the cascade algorithm. The cascade operator $Q_a$ associated with the sequence $a$ is defined by

$$Q_a F := \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)F(M\cdot-\alpha), \qquad F \in (W_p^k(\mathbb{R}^s))^r.$$

Let $\Phi_0$ be a nontrivial $r \times 1$ vector of compactly supported functions in $W_p^k(\mathbb{R}^s)$. The iteration scheme $\Phi_n = Q_a\Phi_{n-1}$, $n = 1, 2, \ldots$, is called a cascade algorithm, or a subdivision scheme. Under natural assumptions on $a$, a feasible set of initial vectors is identified from the conditions on an initial vector implied by the convergence of the subdivision scheme. These conditions are determined by the matrix $A(0) = m^{-1}\sum_{\alpha\in\mathbb{Z}^s} a(\alpha)$, $m = |\det M|$, and are related to polynomial reproducibility and the classical Strang–Fix conditions.

[1] Research supported in part by the Natural Sciences and Engineering Research Council Canada under grants OGP 121336 and A7687.



1063-5203/02 $35.00

The formal definition of convergence in the Sobolev norm for the subdivision scheme is that the scheme will converge for any choice of initial vector from the feasible set (to the same solution $\Phi$). We give a characterization for this concept of convergence in terms of the $p$-norm joint spectral radius of a finite collection of transition operators determined by the sequence $a$ restricted to a certain invariant subspace. The invariant subspace is intimately connected to the Strang–Fix type conditions that determine the feasible set of initial vectors. 

*Key Words:* refinement equations; multiple refinable functions; vector subdivision schemes; cascade algorithm; joint spectral radii; transition operators.

## 1. INTRODUCTION

We are concerned with functional equations of the form

$$\Phi = \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\Phi(M\cdot - \alpha), \tag{1.1}$$

where $\Phi = (\phi_1, \ldots, \phi_r)^{\mathrm{T}}$ is an $r \times 1$ vector of functions on the $s$-dimensional Euclidean space $\mathbb{R}^s$, each $a(\alpha)$ ($\alpha \in \mathbb{Z}^s$) is an $r \times r$ complex matrix, and $M$ is an $s \times s$ integer matrix such that $\lim_{n\to\infty} M^{-n} = 0$. The equation (1.1) is called a (vector) refinement equation, $M$ is called a dilation matrix and the sequence $a$ is called a refinement mask. The transpose of the matrix $M$ is denoted by $M^{\mathrm{T}}$. Throughout this paper we assume that the mask $a$ is finitely supported, i.e., $a(\alpha) = 0$ except for finitely many $\alpha$.

For $1 \le p \le \infty$, by $L_p(\mathbb{R}^s)$ we denote the Banach space of all complex-valued measurable functions $f$ on $\mathbb{R}^s$ such that $\|f\|_p < \infty$, where

$$\|f\|_p := \left(\int_{\mathbb{R}^s} |f(x)|^p\,dx\right)^{1/p} \qquad \text{for } 1 \le p < \infty,$$

and $\|f\|_\infty$ is the essential supremum of $|f|$ on $\mathbb{R}^s$. The Fourier transform of a function $f \in L_1(\mathbb{R}^s)$ is defined by

$$\hat{f}(\omega) := \int_{\mathbb{R}^s} f(x)e^{-ix\cdot\omega}\,dx, \qquad \omega \in \mathbb{R}^s,$$

where $x \cdot \omega$ denotes the inner product of two vectors $x$ and $\omega$ in $\mathbb{R}^s$. The Fourier transform is naturally extended to the space of all compactly supported distributions.

With the use of the Fourier transform, (1.1) can be rewritten as

$$\hat{\Phi}(M^{\mathrm{T}}\omega) = A(\omega)\hat{\Phi}(\omega), \qquad \omega \in \mathbb{R}^s, \tag{1.2}$$

where

$$A(\omega) := \frac{1}{|\det M|}\sum_{\alpha\in\mathbb{Z}^s} a(\alpha)e^{-i\alpha\cdot\omega}, \qquad \omega \in \mathbb{R}^s.$$

Clearly, $A$ is $2\pi$-periodic. If $\hat{\Phi}(0) \ne 0$, then $\hat{\Phi}(0)$ is an eigenvector of the matrix $A(0)$ corresponding to eigenvalue 1.

We denote the set of all positive integers by $\mathbb{N}$ and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. A multi-index is an $s$-tuple $\mu = (\mu_1, \dots, \mu_s) \in \mathbb{N}_0^s$. The length of $\mu$ is $|\mu| = \mu_1 + \cdots + \mu_s$. Let $\sigma_1, \dots, \sigma_s$ be the eigenvalues of the matrix $M$. For $\mu = (\mu_1, \dots, \mu_s) \in \mathbb{Z}^s$, we write $\sigma^\mu$ for $\sigma_1^{\mu_1} \cdots \sigma_s^{\mu_s}$. Throughout this paper, we assume that $A(0)$ has 1 as its simple eigenvalue and has no other eigenvalues of the form $\sigma^\mu$, $\mu = (\mu_1, \dots, \mu_s) \in \mathbb{N}_0^s$. If this is the case, then there exists a nontrivial $r \times 1$ vector $\Phi$ of compactly supported distributions on $\mathbb{R}^s$ such that $\Phi$ is a solution of the refinement equation (1.1) (see [10] and [20]). Moreover, if $\Phi_1$ and $\Phi_2$ are two nontrivial solutions of the refinement equation (1.1), then $\Phi_1 = c\Phi_2$ for some constant $c$.

The partial derivative of a differentiable function $f$ with respect to the $j$th coordinate is denoted by $D_j f$, $j = 1, \dots, s$, and for $\mu = (\mu_1, \dots, \mu_s) \in \mathbb{N}_0^s$, $D^\mu$ is the differential operator $D_1^{\mu_1} \cdots D_s^{\mu_s}$. For $1 \le p \le \infty$ and an integer $k \ge 0$, we use $W_p^k(\mathbb{R}^s)$ to denote the Sobolev space that consists of all distributions $f$ such that $D^\mu f \in L_p(\mathbb{R}^s)$ for all multi-indices $\mu$, with $|\mu| \le k$. Equipped with the norm defined by

$$\|f\|_{W_p^k(\mathbb{R}^s)} := \sum_{|\mu| \le k} \|D^\mu f\|_p,$$

$W_p^k(\mathbb{R}^s)$ becomes a Banach space. We denote by $C^k(\mathbb{R}^s)$ the space of all functions on $\mathbb{R}^s$ possessing continuous partial derivatives of order up to $k$. The norm in $C^k(\mathbb{R}^s)$ is given by

$$\|f\|_{C^k(\mathbb{R}^s)} := \sum_{|\mu| \le k} \|D^\mu f\|_\infty.$$

For $\mu = (\mu_1, \dots, \mu_s) \in \mathbb{N}_0^s$ and $x = (x_1, \dots, x_s) \in \mathbb{R}^s$, define

$$x^\mu := x_1^{\mu_1} \cdots x_s^{\mu_s}.$$

The function $x \mapsto x^\mu$ $(x \in \mathbb{R}^s)$ is called a monomial, and its (total) degree is $|\mu|$. A polynomial is a linear combination of monomials. The degree of a polynomial $q = \sum_\mu c_\mu x^\mu$ is defined to be $\deg q := \max\{|\mu| : c_\mu \ne 0\}$. By $q(D)$ we denote the differential operator $\sum_\mu c_\mu D^\mu$. Let $\Pi$ denote the linear space of all polynomials, and let $\Pi_k$ denote the linear space of all polynomials of degree at most $k$. By convention, $\Pi_{-1} = \{0\}$.

We use $\ell(\mathbb{Z}^s)$ to denote the linear space of all (complex) sequences on $\mathbb{Z}^s$. A sequence $u$ on $\mathbb{Z}^s$ is called a polynomial sequence, if there is a polynomial $q$ such that $u(\alpha) = q(\alpha)$ for all $\alpha \in \mathbb{Z}^s$. The degree of $u$ is the same as the degree of $q$. We use $P(\mathbb{Z}^s)$ to denote the linear space of all polynomial sequences on $\mathbb{Z}^s$, and use $P_k(\mathbb{Z}^s)$ to denote the linear space of all polynomial sequences of degree at most $k$. For $1 \le p \le \infty$, by $\ell_p(\mathbb{Z}^s)$ we denote the Banach space of all complex sequences $u$ on $\mathbb{Z}^s$ such that $\|u\|_p < \infty$, where

$$\|u\|_p := \left( \sum_{\alpha \in \mathbb{Z}^s} |u(\alpha)|^p \right)^{1/p} \qquad \text{for } 1 \le p < \infty,$$

and $\|u\|_\infty$ is the supremum of $\{|u(\alpha)| : \alpha \in \mathbb{Z}^s\}$. The support of a sequence $v$ on $\mathbb{Z}^s$ is defined to be $\operatorname{supp} v := \{\alpha \in \mathbb{Z}^s : v(\alpha) \ne 0\}$. If $\operatorname{supp} v$ is a finite set, then we say that $v$ is finitely supported. By $\ell_0(\mathbb{Z}^s)$ we denote the linear space of all finitely supported sequences on $\mathbb{Z}^s$. Clearly, $\ell_0(\mathbb{Z}^s)$ is a subspace of $\ell_p(\mathbb{Z}^s)$ for any $p$ with $1 \le p \le \infty$.

For a given linear space $H$, we use $H^r = H^{r\times 1}$ to denote the linear space of all $r \times 1$ vectors $F = (f_1, \dots, f_r)^{\mathrm{T}}$, where $f_1, \dots, f_r \in H$. Similarly, we use $H^{1\times r}$ to denote the linear space of all $1 \times r$ vectors whose components are elements of $H$. If $H$ is a normed linear space, then $H^r$ can be equipped with the norm given by

$$\|F\|_{H^r} := \sum_{j=1}^{r} \|f_j\|_H, \qquad F = (f_1, \dots, f_r)^{\mathrm{T}}.$$

Let $F = (f_1, \dots, f_r)^{\mathrm{T}}$ be an $r \times 1$ vector of compactly supported functions in $L_p(\mathbb{R}^s)$ $(1 \le P \le \infty)$. We say that the shifts of $f_1, \dots, f_r$ are stable if there are two positive constants $C_1$ and $C_2$ such that

$$C_1\|u\|_{(\ell_p(\mathbb{Z}^s))^r} \le \left\| \sum_{\alpha\in\mathbb{Z}^s} u(\alpha)F(\cdot - \alpha) \right\|_{L_p(\mathbb{R}^s)} \le C_2\|u\|_{(\ell_p(\mathbb{Z}^s))^r}.$$

It was proved in [16] and [14] that the shifts of $f_1, \dots, f_r$ are stable if and only if, for every $\omega \in \mathbb{R}^s$,

$$\operatorname{span}\{\hat{F}(\omega + 2\pi\beta) : \beta \in \mathbb{Z}^s\} = \mathbb{C}^r.$$

The cascade operator $Q_a$ associated with the refinement mask $a$ is defined by

$$Q_a F := \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)F(M\cdot - \alpha), \qquad F \in (W_p^k(\mathbb{R}^s))^r.$$

Let $\Phi_0$ be a nontrivial $r \times 1$ vector of compactly supported functions in $W_p^k(\mathbb{R}^s)$. The iteration scheme $\Phi_n = Q_a\Phi_{n-1}$, $n = 1, 2, \dots$, is called a cascade algorithm, or a subdivision scheme. Suppose the subdivision scheme converges in $W_p^k(\mathbb{R}^s)$, that is, there exists some $\Phi \in (W_p^k(\mathbb{R}^s))^r$ such that

$$\lim_{n\to\infty} \|\Phi_n - \Phi\|_{(W_p^k(\mathbb{R}^s))^r} = 0.$$

Then $\Phi$ is a solution of the refinement equation (1.1).

For the sequence $(\Phi_n)_{n=1,2,\dots}$ to converge in the Sobolev space, the initial vector $\Phi_0$ of functions must satisfy certain conditions. In Section 2 we clarify those conditions and give a formal definition for the convergence of subdivision schemes in Sobolev spaces. This study is related to polynomial reproducibility of $\Phi$. In Section 3 we introduce the subdivision and transition operators associated with the refinement equation in (1.1) and consider invariant subspaces of the transition operators. Finally, in Section 4, we give a characterization for the convergence of the subdivision scheme in the Sobolev space in terms of the $p$-norm joint spectral radius of a finite collection of the transition operators restricted to a certain invariant subspace.

A comprehensive study of stationary subdivision schemes was given in [3]. In [11] and [9], the $p$-norm joint spectral radius was employed to give a characterization of the $L_P$ convergence of subdivision schemes. Vector subdivision schemes were investigated in [18] and [22]. The present paper is closely related to [8, 15, 23], in which (scalar) subdivision schemes in Sobolev spaces were discussed.

## 2. POLYNOMIAL REPRODUCIBILITY

The purpose of this section is to find appropriate conditions on $\Phi_0$ such that the subdivision scheme $\Phi_n = Q_a^n \Phi_0$ $(n = 1, 2, \ldots)$ converges in $W_p^k(\mathbb{R}^s)$. This problem is related to polynomial reproducibility of $\Phi$.

We assume that the dilation matrix $M$ is isotropic, that is, there exists an invertible $s \times s$ matrix $\Lambda$ such that

$$\Lambda M \Lambda^{-1} = \operatorname{diag}(\sigma_1, \ldots, \sigma_s),$$

where $\sigma_1, \ldots, \sigma_s$ are complex numbers and

$$|\sigma_1| = \cdots = |\sigma_s| = \rho,$$

where $\rho$ is the spectral radius of $M$. For any given vector norm $\|\cdot\|$ on $\mathbb{R}^s$, there exist two positive constants $C_1$ and $C_2$ such that the inequalities

$$C_1 \rho^n \|v\| \le \|M^n v\| \le C_2 \rho^n \|v\| \tag{2.1}$$

hold for every positive integer $n$ and every vector $v \in \mathbb{R}^s$.

Let $f$ be a smooth function on $\mathbb{R}^s$. By using the chain rule for differentiation, we have

$$\begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_s \end{bmatrix} (f \circ M^{\mathrm{T}})(x) = M \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_s \end{bmatrix} f(M^{\mathrm{T}} x), \qquad x \in \mathbb{R}^s.$$

It follows that

$$\Lambda \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_s \end{bmatrix} (f \circ M^{\mathrm{T}})(x) = \Lambda M \Lambda^{-1} \Lambda \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_s \end{bmatrix} f(M^{\mathrm{T}} x), \qquad x \in \mathbb{R}^s.$$

To a multi-index $\mu$ we associate the polynomial $q_\mu$ given by

$$q_\mu(x) := (\Lambda x)^\mu, \qquad x \in \mathbb{R}^s.$$

Since $\Lambda M \Lambda^{-1} = \operatorname{diag}(\sigma_1, \ldots, \sigma_s)$, from the above discussion we obtain the following identity:

$$q_\mu(D)(f \circ M^{\mathrm{T}})(x) = \sigma^\mu q_\mu(D) f(M^{\mathrm{T}} x), \qquad x \in \mathbb{R}^s. \tag{2.2}$$

The factorial of a multi-index $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$ is defined to be $\mu! := \mu_1! \cdots \mu_s!$. Let $\mu = (\mu_1, \ldots, \mu_s)$ and $\nu = (\nu_1, \ldots, \nu_s)$ be two multi-indices. Then $\nu \le \mu$ means $\nu_j \le \mu_j$ for $j = 1, \ldots, s$. By $\nu < \mu$ we mean $\nu \le \mu$ and $\nu \ne \mu$. For $\nu \le \mu$, define

$$\binom{\mu}{\nu} := \frac{\mu!}{\nu!(\mu - \nu)!}.$$

Then we have the following Leibniz rule for differentiation:

$$q_\mu(D)(fg) = \sum_{\nu \le \mu} \binom{\mu}{\nu} [q_{\mu-\nu}(D)f][q_\nu(D)g]. \tag{2.3}$$

Suppose $\phi_1, \dots, \phi_r$ are compactly supported functions in $L_1(\mathbb{R}^s)$ such that the $r \times 1$ vector $\Phi := (\phi_1, \dots, \phi_r)^{\mathrm{T}}$ is a solution of the refinement equation (1.1). It was proved in [5] and [18] that $A(0)$ has 1 as its simple eigenvalue, and its other eigenvalues are less than 1 in modulus, provided $\hat{\Phi}(0) \neq 0$ and $\operatorname{span}\{\hat{\Phi}(2\beta\pi) : \beta \in \mathbb{Z}^s\} = \mathbb{C}^r$. The following lemma extends this result.

LEMMA 2.1. *Let $\phi_1, \dots, \phi_r$ be compactly supported functions in $W_p^k(\mathbb{R}^s)$ such that the $r \times 1$ vector $\Phi := (\phi_1, \dots, \phi_r)^{\mathrm{T}}$ is a solution of the refinement equation*

$$\Phi = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\Phi(M \cdot - \alpha).$$

*If, in addition, $\hat{\Phi}(0) \neq 0$ and $\operatorname{span}\{\hat{\Phi}(2\beta\pi) : \beta \in \mathbb{Z}^s\} = \mathbb{C}^r$, then 1 is a simple eigenvalue of $A(0)$ and its other eigenvalues are less than $\rho^{-k}$ in modulus, where $\rho$ denotes the spectral radius of the dilation matrix $M$.*

*Proof.* A repeated use of (1.2) gives

$$\hat{\Phi}\big((M^{\mathrm{T}})^n\omega\big) = A\big((M^{\mathrm{T}})^{n-1}\omega\big) \cdots A(M^{\mathrm{T}}\omega)A(\omega)\hat{\Phi}(\omega)$$

for $n = 1, 2, \dots$ and $\omega \in \mathbb{R}^s$. Setting $\omega = 2\beta\pi$ for $\beta \in \mathbb{Z}^s$ in the above equation, we obtain

$$\hat{\Phi}\big((M^{\mathrm{T}})^n 2\beta\pi\big) = A(0)^n \hat{\Phi}(2\beta\pi). \tag{2.4}$$

Since $\Phi$ lies in $(W_1^k(\mathbb{R}^s))^r$, $D^\mu\Phi$ is in $(L_1(\mathbb{R}^s))^r$ for $|\mu| = k$. The Fourier transform of $D^\mu\Phi$ is $(D^\mu\Phi)\hat{}\,(\omega) = (i\omega)^\mu\hat{\Phi}(\omega)$, $\omega \in \mathbb{R}^s$. By the Riemann–Lebesgue lemma we obtain

$$\lim_{n\to\infty} \big((M^{\mathrm{T}})^n 2\beta\pi\big)^\mu \hat{\Phi}\big((M^{\mathrm{T}})^n 2\beta\pi\big) = 0 \qquad \forall \beta \in \mathbb{Z}^s\backslash\{0\} \text{ and } |\mu| = k. \tag{2.5}$$

Note that $M^{\mathrm{T}}$ is an isotropic matrix with a spectral radius of $\rho$. Thus, (2.5) together with (2.1) gives

$$\lim_{n\to\infty} \rho^{nk}\hat{\Phi}\big((M^{\mathrm{T}})^n 2\beta\pi\big) = 0 \qquad \forall \beta \in \mathbb{Z}^s\backslash\{0\}.$$

This in connection with (2.4) yields

$$\lim_{n\to\infty} \big(\rho^k A(0)\big)^n \hat{\Phi}(2\beta\pi) = 0 \qquad \forall \beta \in \mathbb{Z}^s\backslash\{0\}. \tag{2.6}$$

Let $V$ be the linear subspace of $\mathbb{C}^r$ spanned by $\hat{\Phi}(2\beta\pi)$, $\beta \in \mathbb{Z}^s\backslash\{0\}$. By our assumption, $\operatorname{span}\{\hat{\Phi}(2\beta\pi) : \beta \in \mathbb{Z}^s\} = \mathbb{C}^r$; hence, either $V = \mathbb{C}^r$ or $\dim V = r - 1$. If $V = \mathbb{C}^r$, then we would have $\hat{\Phi}(0) \in V$ and $\lim_{n\to\infty}(\rho^k A(0))^n\hat{\Phi}(0) = 0$, which is impossible since $A(0)\hat{\Phi}(0) = \hat{\Phi}(0)$. Therefore, $\dim V = r - 1$ and $V$ is an invariant subspace of $A(0)$. We deduce from (2.6) that the spectral radius of $\rho^k A(0)|_V$ is less than 1 in modulus. This completes the proof of the lemma. ■

From Lemma 2.1, we make the following assumption on the matrix $A(0)$:

*Eigenvalue Condition*. The matrix $A(0)$ has 1 as a simple eigenvalue and the other eigenvalues are of modulus less than $\rho^{-k}$, where $\rho$ denotes the spectral radius of the dilation matrix $M$.

Let $A(0)$ satisfy the eigenvalue condition, and choose $B_0$ to be a left $1 \times r$ eigenvector of the matrix $A(0)$ corresponding to eigenvalue 1. Let $B_\mu (|\mu| \le k)$ be the $1 \times r$ vectors given by the recursive relation

$$B_\mu = \sum_{\nu \le \mu} \binom{\mu}{\nu} \sigma^{\mu-\nu} B_{\mu-\nu} q_\nu(-iD)A(0). \tag{2.7}$$

This equation can be rewritten as

$$B_\mu \big(I_r - \sigma^\mu A(0)\big) = \sum_{0 \ne \nu \le \mu} \binom{\mu}{\nu} \sigma^{\mu-\nu} B_{\mu-\nu} q_\nu(-iD)A(0),$$

where $I_r$ denotes the $r \times r$ identity matrix. By Lemma 2.1, the matrix $I_r - \sigma^\mu A(0)$ is invertible for any multi-index $\mu$ with $0 < |\mu| \le k$. Therefore, the vectors $B_\mu$ $(0 < |\mu| \le k)$ are uniquely determined by (2.7).

For $|\mu| \le k$ and $F \in (W_1^k(\mathbb{R}^s))^r$, define

$$J_{\mu,F}(\omega) := \sum_{\nu \le \mu} \binom{\mu}{\nu} B_{\mu-\nu} q_\nu(-iD)\hat{F}(\omega), \qquad \omega \in \mathbb{R}^s.$$

Note that $q_\nu(-iD)\hat{F}$ is the Fourier transform of $q_\nu F$.

LEMMA 2.2. *The following identity holds for* $|\mu| \le k$*:*

$$\sigma^\mu J_{\mu,Q_a F}(M^{\mathrm{T}} 2\pi\beta) = J_{\mu,F}(2\pi\beta) \qquad \forall \beta \in \mathbb{Z}^s.$$

*Proof.* Let $G := Q_a F$. Then $\hat{G}(\omega) = A((M^{\mathrm{T}})^{-1}\omega)\hat{F}((M^{\mathrm{T}})^{-1}\omega)$, $\omega \in \mathbb{R}^s$. By using (2.2) and (2.3) we obtain

$$q_\nu(-iD)\hat{G}(\omega) = \sum_{\lambda \le \nu} \binom{\nu}{\lambda} \big[\sigma^{-(\nu-\lambda)} q_{\nu-\lambda}(-iD)A((M^{\mathrm{T}})^{-1}\omega)\big] \times \big[\sigma^{-\lambda} q_\lambda(-iD)\hat{F}((M^{\mathrm{T}})^{-1}\omega)\big].$$

It follows that

$$J_{\mu,G}(\omega) = \sum_{\nu \le \mu} \sum_{\lambda \le \nu} \binom{\mu}{\nu} B_{\mu-\nu} \binom{\nu}{\lambda} \sigma^{-(\nu-\lambda)} q_{\nu-\lambda}(-iD)A(\xi)\sigma^{-\lambda} q_\lambda(-iD)\hat{F}(\xi),$$

where $\xi := (M^{\mathrm{T}})^{-1}\omega$. Using new indices $\gamma = \mu - \lambda$ and $\tau = \nu - \lambda$ in the above double sum gives

$$J_{\mu,G}(\omega) = \sum_{\gamma \le \mu} \binom{\mu}{\gamma} \sigma^{-\mu} \Bigg[\sum_{\tau \le \gamma} \binom{\gamma}{\tau} B_{\gamma-\tau} \sigma^{\gamma-\tau} q_\tau(-iD)A(\xi)\Bigg] q_{\mu-\gamma}(-iD)\hat{F}(\xi).$$

Note that $A$ is $2\pi$-periodic. For $\omega = M^{\mathrm{T}} 2\pi\beta$, $\beta \in \mathbb{Z}^s$, by (2.7) we have

$$\sum_{\tau \le \gamma} \binom{\gamma}{\tau} B_{\gamma-\tau} \sigma^{\gamma-\tau} q_\tau(-iD) A(2\pi\beta) = B_\gamma .$$

Consequently,

$$J_{\mu,G}(M^{\mathrm{T}} 2\pi\beta) = \sigma^{-\mu} \sum_{\gamma \le \mu} \binom{\mu}{\gamma} B_\gamma q_{\mu-\gamma}(-iD) \hat{F}(2\pi\beta).$$

This completes the proof of the lemma. ■

Let $\Phi_n := Q_a^n \Phi_0$, $n = 0, 1, 2, \ldots$. A repeated application of Lemma 2.2 gives

$$\sigma^{\mu n} J_{\mu,\Phi_n}((M^{\mathrm{T}})^n 2\pi\beta) = J_{\mu,\Phi_0}(2\pi\beta) \qquad \forall \beta \in \mathbb{Z}^s. \tag{2.8}$$

LEMMA 2.3. *Let $\phi_1, \ldots, \phi_r$ be compactly supported functions in $W_1^k(\mathbb{R}^s)$ such that the $r \times 1$ vector $\Phi := (\phi_1, \ldots, \phi_r)^{\mathrm{T}}$ is a solution of the refinement equation* (1.1). *Suppose $A(0)$ satisfies the eigenvalue condition, $B_0\hat{\Phi}(0) = 1$, and $B_\mu$ $(|\mu| \le k)$ are the $1 \times r$ vectors determined by* (2.7). *Then $J_{\mu,\Phi}(0) = 0$ for all $0 < |\mu| \le k$ and $J_{\mu,\Phi}(2\beta\pi) = 0$ for all $\beta \in \mathbb{Z}^s \backslash \{0\}$ and all $|\mu| \le k$.*

*Proof.* Since $Q_a\Phi = \Phi$, by Lemma 2.2 we have $\sigma^\mu J_{\mu,\Phi}(0) = J_{\mu,\Phi}(0)$. But $|\sigma^\mu| > 1$ for $0 < |\mu| \le k$. Hence, $J_{\mu,\Phi}(0) = 0$ for $0 < |\mu| \le k$. Moreover, by (2.8) we have

$$J_{\mu,\Phi}(2\beta\pi) = \sigma^{\mu n} J_{\mu,\Phi}\big((M^{\mathrm{T}})^n 2\beta\pi\big) \qquad \forall n \in \mathbb{N},\ \beta \in \mathbb{Z}^s.$$

Since $\Phi$ is compactly supported and lies in $(W_1^k(\mathbb{R}^s))^r$, each $q_\nu\Phi$ is also, $|\nu| \le k$, and by the Riemann–Lebesgue lemma we obtain

$$\lim_{n\to\infty} \big((M^{\mathrm{T}})^n 2\beta\pi\big)^\mu J_{\mu,\Phi}\big((M^{\mathrm{T}})^n 2\beta\pi\big) = 0 \qquad \forall \beta \in \mathbb{Z}^s \backslash \{0\}.$$

Thus, by (2.1),

$$\lim_{n\to\infty} \sigma^{\mu n} J_{\mu,\Phi}\big((M^{\mathrm{T}})^n 2\beta\pi\big) = 0 \qquad \forall \beta \in \mathbb{Z}^s \backslash \{0\}. \tag{2.9}$$

This shows $J_{\mu,\Phi}(2\beta\pi) = 0$ for all $\beta \in \mathbb{Z}^s \backslash \{0\}$. The proof of the lemma is complete. ■

Let $\psi_\mu := B_\mu\Phi/\mu!$, $|\mu| \le k$. It follows from Lemma 2.3 that $\hat{\psi}_0(0) = 1$, $\hat{\psi}_0(2\beta\pi) = 0$ for $\beta \in \mathbb{Z}^s \backslash \{0\}$, and

$$\sum_{\nu \le \mu} \frac{q_\nu(-iD)}{\nu!} \hat{\psi}_{\mu-\nu}(2\beta\pi) = 0 \qquad \forall 1 \le |\mu| \le k,\ \beta \in \mathbb{Z}^s.$$

These conditions are called the Strang–Fix conditions of order $k+1$ (see [25, Theorem II]). By using the Poisson summation formula one can easily prove that the above conditions are equivalent to the following conditions:

$$\sum_{\alpha\in\mathbb{Z}^s} \sum_{\nu \le \mu} \binom{\mu}{\nu} (\Lambda\alpha)^\nu B_{\mu-\nu} \Phi(x-\alpha) = (\Lambda x)^\mu \qquad \text{for all } |\mu| \le k \text{ and a.e. } x \in \mathbb{R}^s. \tag{2.10}$$

Thus, $\Phi$ reproduces all polynomials in $\Pi_k$. Note that $\Phi$ might reproduce polynomials of higher order. For a detailed discussion on polynomial reproducibility of a refinable vector of functions, the reader is referred to [1, 2, 19].

THEOREM 2.4. *Under the assumptions of Lemma 2.3 if $\Phi_0$ is an $r \times 1$ vector of compactly supported functions in $W_p^k(\mathbb{R}^s)$ such that*

$$\lim_{n\to\infty} \|Q_a^n \Phi_0 - \Phi\|_{(W_p^k(\mathbb{R}^s))^r} = 0, \tag{2.11}$$

*then $B_0\hat{\Phi}_0(0) = 1$ and*

$$\sum_{\nu\le\mu} \binom{\mu}{\nu} B_{\mu-\nu} q_\nu(-iD)\hat{\Phi}_0(2\pi\beta) = 0 \qquad \forall|\mu| \le k \text{ and } \beta \in \mathbb{Z}^s\backslash\{0\}.$$

*Proof.* For $n = 1, 2, \ldots$, let $\Phi_n := Q_a^n \Phi_0$. Given multi-indices $\nu$ and $\mu$, the Fourier transform of $D^\mu(q_\nu\Phi_n - q_\nu\Phi)$ is $(i\omega)^\mu q_\nu(-iD)(\hat{\Phi}_n(\omega) - \hat{\Phi}(\omega))$, $\omega \in \mathbb{R}^s$. From (2.11) and the compactness of the union of all of the supports of $\Phi, \Phi_0, \Phi_1, \ldots$, it follows that

$$\lim_{n\to\infty} \|D^\mu(q_\nu\Phi_n - q_\nu\Phi)\|_{(L_1(\mathbb{R}^s))^r} = 0 \qquad \forall|\mu| \le k.$$

Hence, for all $|\mu| \le k$ we have

$$\lim_{n\to\infty} |\omega^\mu| \left|q_\nu(-iD)(\hat{\Phi}_n(\omega) - \hat{\Phi}(\omega))\right| = 0,$$

uniformly in $\omega$. Using this for $\omega = (M^{\mathrm{T}})^n 2\beta\pi$, $\beta \in \mathbb{Z}^s\backslash\{0\}$, from (2.1) we have

$$\lim_{n\to\infty} \rho^{|\mu|n} \left|q_\nu(-iD)\big(\hat{\Phi}_n((M^{\mathrm{T}})^n 2\beta\pi) - \hat{\Phi}((M^{\mathrm{T}})^n 2\beta\pi)\big)\right| = 0 \qquad \forall\beta \in \mathbb{Z}^s\backslash\{0\},$$

where $\rho$ is the spectral radius of the dilation matrix $M$. Consequently,

$$\lim_{n\to\infty} \left|\sigma^{\mu n} J_{\mu,\Phi_n}\big((M^{\mathrm{T}})^n 2\beta\pi\big) - \sigma^{\mu n} J_{\mu,\Phi}\big((M^{\mathrm{T}})^n 2\beta\pi\big)\right| = 0 \qquad \forall\beta \in \mathbb{Z}^s\backslash\{0\}.$$

Thus, by (2.9) and Lemma 2.2 for all $\beta \in \mathbb{Z}^s\backslash\{0\}$, we have

$$J_{\mu,\Phi_0}(2\beta\pi) = \lim_{n\to\infty} \sigma^{\mu n} J_{\mu,\Phi_n}\big((M^{\mathrm{T}})^n 2\beta\pi\big) = \lim_{n\to\infty} \sigma^{\mu n} J_{\mu,\Phi}\big((M^{\mathrm{T}})^n 2\beta\pi\big) = 0.$$

■

Let $Y_k$ denote the class of all $r \times 1$ vectors $F$ of compactly supported functions in $W_p^k(\mathbb{R}^s)$ such that

$$B_0\hat{F}(0) = 1 \tag{2.12}$$

and

$$\sum_{\nu\le\mu} \binom{\mu}{\nu} B_{\mu-\nu} q_\nu(-iD)\hat{F}(2\beta\pi) = 0 \qquad \forall|\mu| \le k \text{ and } \beta \in \mathbb{Z}^s\backslash\{0\}. \tag{2.13}$$

By Theorem 2.4 we conclude that $\Phi_0 \in Y_k$ is a necessary condition for $(Q_a^n \Phi_0)_{n=1,2,\dots}$ to converge in $W_p^k(\mathbb{R}^s)$. Moreover, (2.12) and (2.13) imply that for each $\mu$ ($|\mu| \le k$) there exists some $g_\mu \in \Pi_{|\mu|-1}$ such that

$$\sum_{\alpha\in\mathbb{Z}^s} \sum_{\nu\le\mu} \binom{\mu}{\nu} (\Lambda\alpha)^\nu B_{\mu-\nu} F(x-\alpha) = (\Lambda x)^\mu + g_\mu(x), \qquad x \in \mathbb{R}^s. \tag{2.14}$$

This fact can be established by using the Poisson summation formula.

We are in a position to give a formal definition of the convergence of subdivision schemes in Sobolev spaces. We say that the subdivision scheme associated with mask $a$ converges in the Sobolev space $W_p^k(\mathbb{R}^s)$ if there exists a compactly supported $\Phi \in (W_p^k(\mathbb{R}^s))^r$ so that, for any $\Phi_0 \in Y_k$,

$$\lim_{n\to\infty} \| Q_a^n \Phi_0 - \Phi \|_{(W_p^k(\mathbb{R}^s))^r} = 0.$$

Clearly, if $1 \le q \le p \le \infty$ and if the subdivision scheme associated with mask $a$ converges in the Sobolev space $W_p^k(\mathbb{R}^s)$, then it also converges in $W_q^k(\mathbb{R}^s)$.

When $p = \infty$, we often discuss convergence in the space $C^k(\mathbb{R}^s)$. Let $Y_k$ be the class of all $r \times 1$ vectors $F$ of compactly supported functions in $C^k(\mathbb{R}^s)$ such that (2.12) and (2.13) hold. We say that the subdivision scheme associated with mask $a$ converges in $C^k(\mathbb{R}^s)$ if, for any $\Phi_0 \in Y_k$,

$$\lim_{n\to\infty} \| Q_a^n \Phi_0 - \Phi \|_{(C^k(\mathbb{R}^s))^r} = 0.$$

## 3. THE SUBDIVISION AND TRANSITION OPERATORS

The subdivision operator $S_a$ is the linear operator on $(\ell(\mathbb{Z}^s))^{1\times r}$ defined by

$$S_a u(\alpha) = \sum_{\beta\in\mathbb{Z}^s} u(\beta) a(\alpha - M\beta), \qquad \alpha \in \mathbb{Z}^s,\ u \in (\ell(\mathbb{Z}^s))^{1\times r}.$$

The transition operator $T_a$ is the linear operator on $(\ell_0(\mathbb{Z}^s))^{r\times 1}$ defined by

$$T_a v(\alpha) = \sum_{\beta\in\mathbb{Z}^s} a(M\alpha - \beta) v(\beta), \qquad \alpha \in \mathbb{Z}^s,\ v \in (\ell_0(\mathbb{Z}^s))^{r\times 1}.$$

The subdivision and transition operators were used in [4, 12, 17] to study refinement equations.

For $u \in (\ell(\mathbb{Z}^s))^{1\times r}$ and $v \in (\ell_0(\mathbb{Z}^s))^{r\times 1}$, we define the bilinear form $\langle u, v\rangle$ as follows:

$$\langle u, v\rangle := \sum_{\alpha\in\mathbb{Z}^s} u(-\alpha) v(\alpha).$$

Clearly, $S_a$ is the algebraic adjoint of $T_a$ with respect to the bilinear form given above. Indeed,

$$\langle S_a u, v\rangle = \sum_{\alpha\in\mathbb{Z}^s} \sum_{\beta\in\mathbb{Z}^s} u(\beta) a(\alpha - M\beta) v(-\alpha) = \langle u, T_a v\rangle.$$

LEMMA 3.1. *Let $U$ be a finite dimensional subspace of $(\ell(\mathbb{Z}^s))^{1\times r}$, and let*

$$V := \left\{v \in (\ell_0(\mathbb{Z}^s))^{r\times 1} : \langle u, v\rangle = 0\ \forall u \in U\right\}.$$

*Then $U$ is invariant under the subdivision operator $S_a$ if and only if $V$ is invariant under the transition operator $T_a$.*

*Proof.* Suppose $U$ is invariant under $S_a$. Let $v \in V$. Then for any $u \in U$, $S_a u$ lies in $U$. Hence,

$$\langle u, T_a v\rangle = \langle S_a u, v\rangle = 0.$$

This shows $T_a v \in V$ for $v \in V$. In other words, $V$ is invariant under $T_a$.

Now suppose $V$ is invariant under $T_a$. For $u \in U$, we have

$$\langle S_a u, v\rangle = \langle u, T_a v\rangle = 0 \qquad \forall v \in V.$$

Since $U$ is finite dimensional, $U$ is spanned by finitely many elements, say, $u_1, \ldots, u_m$. Thus, the null space of $S_a u$ contains the intersection of the null spaces of $u_1, \ldots, u_m$. By the Theorem on Linear Dependence (see [21, p. 7]), $S_a u$ lies in $U$. This shows that $U$ is invariant under $S_a$. ■

For a given mask $a$ for which $A(0)$ satisfies the eigenvalue condition, we define $U_k \subset (P(\mathbb{Z}^s))^{1\times r}$ to be the linear span of $u_\mu$, $|\mu| \le k$, where each $u_\mu$ is given by

$$u_\mu(\alpha) = \sum_{\nu\le\mu} \binom{\mu}{\nu} (\Lambda\alpha)^\nu B_{\mu-\nu}, \qquad \alpha \in \mathbb{Z}^s. \tag{3.1}$$

In view of (2.10), for any solution $\Phi \in (W_p^k(\mathbb{R}^s))^r$ of the refinement equation (1.1) for $a$, we have

$$\sum_{\alpha\in\mathbb{Z}^s} u(\alpha)\Phi(\cdot - \alpha) \in \Pi_k, \qquad \forall u \in U_k. \tag{3.2}$$

Suppose $F \in Y_k$. From (2.14) we see that $u \in U_k$ implies $\sum_{\alpha\in\mathbb{Z}^s} u(\alpha)F(\cdot - \alpha) \in \Pi_k$. Conversely, for $q \in \Pi_k$, there exists some $u \in U_k$ such that $\sum_{\alpha\in\mathbb{Z}^s} u(\alpha)F(\cdot - \alpha) = q$.

Let

$$V_k := \left\{v \in (\ell_0(\mathbb{Z}^s))^{r\times 1} : \langle u, v\rangle = 0\ \forall u \in U_k\right\}. \tag{3.3}$$

The main purpose of this section is to establish the following result.

THEOREM 3.2. *If the subdivision scheme associated with mask $a$ converges in $W_p^k(\mathbb{R}^s)$ $(1 \le p \le \infty)$, then $U_k$ is invariant under the subdivision operator $S_a$ and $V_k$ is invariant under the transition operator $T_a$.*

The proof of this theorem requires two auxiliary lemmas.

Let $\Gamma$ be a complete set of representatives of the cosets of $\mathbb{Z}^s/M^{\mathrm{T}}\mathbb{Z}^s$, and let $E$ be a complete set of representatives of the cosets of $\mathbb{Z}^s/M\mathbb{Z}^s$. Clearly, $\#\Gamma = \#E = |\det M|$. Without loss of generality we assume $0 \in E$ and $0 \in \Gamma$.

LEMMA 3.3. *There exists some $F$ in $Y_k$ such that for every $\gamma \in \Gamma$,*

$$\mathrm{span}\left\{\hat F\left((M^{\mathrm{T}})^{-1}2\pi\gamma + 2\pi\beta\right) : \beta \in \mathbb{Z}^s\right\} = \mathbb{C}^r. \tag{3.4}$$

*Proof.* Choose compactly supported functions $h_1, \dots, h_r \in C^k(\mathbb{R}^s)$ such that the shifts of $h_1, \dots, h_r$ are linearly independent. Let

$$g(\omega) := \prod_{\gamma \in \Gamma \setminus \{0\}} \frac{e^{i\omega\cdot\eta} - e^{i(M^{\mathrm{T}})^{-1}2\pi\gamma\cdot\eta}}{1 - e^{i(M^{\mathrm{T}})^{-1}2\pi\gamma\cdot\eta}}, \qquad \omega \in \mathbb{R}^s,$$

where $\eta \in \mathbb{R}^s$ is chosen so that $e^{i(M^{\mathrm{T}})^{-1}2\pi\gamma\cdot\eta} \neq 1$ for all $\gamma \in \Gamma\setminus\{0\}$. By our choice of $g$, $g(\omega)$ is a trigonometric polynomial of $\omega$. Moreover, $g(0) = 1$ and $g((M^{\mathrm{T}})^{-1}2\pi\gamma) = 0$ for all $\gamma \in \Gamma\setminus\{0\}$. Set

$$\hat{F}(\omega) := [1 - (1 - g(\omega))^{k+1}]\hat{\Phi}(\omega) + (1 - g(\omega))^{k+1}\hat{H}(\omega), \qquad \omega \in \mathbb{R}^s, \tag{3.5}$$

where $H = (h_1, \dots, h_r)^{\mathrm{T}}$. Since $g(2\pi\beta) = 1$ for all $\beta \in \mathbb{Z}^s$, we have

$$D^\mu[(1-g)^{k+1}](2\pi\beta) = 0 \qquad \forall |\mu| \le k \text{ and } \beta \in \mathbb{Z}^s.$$

Consequently, $D^\mu \hat{F}(2\pi\beta) = D^\mu\hat{\Phi}(2\pi\beta)$ for all $|\mu| \le k$ and $\beta \in \mathbb{Z}^s$. Hence, $F$ lies in $Y_k$ and

$$\operatorname{span}\{\hat{F}(2\pi\beta) : \beta \in \mathbb{Z}^s\} = \operatorname{span}\{\hat{\Phi}(2\pi\beta) : \beta \in \mathbb{Z}^s\} = \mathbb{C}^r.$$

This verifies (3.4) for $\gamma = 0$. Moreover, for $\gamma \in \Gamma\setminus\{0\}$ and $\beta \in \mathbb{Z}^s$, it follows from (3.5) that

$$\hat{F}\big((M^{\mathrm{T}})^{-1}2\pi\gamma + 2\pi\beta\big) = \hat{H}\big((M^{\mathrm{T}})^{-1}2\pi\gamma + 2\pi\beta\big).$$

Since the shifts of $h_1, \dots, h_r$ are linearly independent, we have

$$\operatorname{span}\left\{\hat{H}\big((M^{\mathrm{T}})^{-1}2\pi\gamma + 2\pi\beta\big) : \beta \in \mathbb{Z}^s\right\} = \mathbb{C}^r.$$

Therefore, (3.4) is valid for every $\gamma \in \Gamma$. ■

It would be interesting to know whether there always exists some $F = (f_1, \dots, f_r)^{\mathrm{T}}$ in $Y_k$ such that the shifts of $f_1, \dots, f_r$ are stable.

LEMMA 3.4. *Let $F = (f_1, \dots, f_r)^{\mathrm{T}}$ be an $r \times 1$ vector of compactly supported functions in $L_1(\mathbb{R}^s)$. Suppose $w$ is an element in $(\ell(\mathbb{Z}^s))^{1\times r}$ such that $\sum_{\alpha\in\mathbb{Z}^s} w(\alpha)\times F(x-\alpha) = 0$ for a.e. $x \in \mathbb{R}^s$. If $w(\alpha + M\eta) = w(\alpha)$ for all $\alpha, \eta \in \mathbb{Z}^s$, and if*

$$\operatorname{span}\left\{\hat{F}\big((M^{\mathrm{T}})^{-1}2\pi\gamma + 2\pi\beta\big) : \beta \in \mathbb{Z}^s\right\} = \mathbb{C}^r, \tag{3.6}$$

*then $w = 0$.*

*Proof.* For $\beta \in \mathbb{Z}^s$ and $\gamma \in \Gamma$, let

$$c_{\beta\gamma} := \int_{M([0,1)^s)} \sum_{\alpha\in\mathbb{Z}^s} w(\alpha)F(x-\alpha)e^{-ix\cdot(2\pi\beta + (M^{\mathrm{T}})^{-1}2\pi\gamma)}\,dx.$$

Then we have

$$c_{\beta\gamma} = \int_{M([0,1)^s)} \sum_{\varepsilon\in E}\sum_{\alpha\in\mathbb{Z}^s} w(M\alpha+\varepsilon)F(x - M\alpha - \varepsilon)e^{-ix\cdot(2\pi\beta + (M^{\mathrm{T}})^{-1}2\pi\gamma)}\,dx.$$

By our assumption, $w(M\alpha+\varepsilon)=w(\varepsilon)$ for all $\alpha\in\mathbb{Z}^s$. Hence

$$\begin{aligned} c_{\beta\gamma} &= \sum_{\varepsilon\in E} w(\varepsilon)\int_{\mathbb{R}^s} F(x-\varepsilon)e^{-ix\cdot(2\pi\beta+(M^{\mathrm{T}})^{-1}2\pi\gamma)}\,dx \\ &= \sum_{\varepsilon\in E} w(\varepsilon)e^{-i\varepsilon\cdot(M^{\mathrm{T}})^{-1}2\pi\gamma}\hat{F}\big((M^{\mathrm{T}})^{-1}2\pi\gamma+2\pi\beta\big). \end{aligned}$$

Since $\sum_{\alpha\in\mathbb{Z}^s} w(\alpha)F(\cdot-\alpha)=0$, it follows that $c_{\beta\gamma}=0$ i.e.,

$$\left(\sum_{\varepsilon\in E} w(\varepsilon)e^{-i\varepsilon\cdot(M^{\mathrm{T}})^{-1}2\pi\gamma}\right)\hat{F}\big((M^{\mathrm{T}})^{-1}2\pi\gamma+2\pi\beta\big)=0 \qquad \forall\beta\in\mathbb{Z}^s.$$

This in connection with (3.6) gives

$$\sum_{\varepsilon\in E} w(\varepsilon)e^{-i\varepsilon\cdot(M^{\mathrm{T}})^{-1}2\pi\gamma}=0 \qquad \forall\gamma\in\Gamma.$$

But the matrix $(e^{-i\varepsilon\cdot(M^{\mathrm{T}})^{-1}2\pi\gamma})_{\varepsilon\in E,\ \gamma\in\Gamma}$ is invertible (see [12, Lemma 3.2]). Therefore, we obtain $w(\varepsilon)=0$ for all $\varepsilon\in E$. This completes the proof. ■

*Proof of Theorem* 3.2. By Lemma 3.3, there exists an element $F$ in $Y_k$ such that

$$\operatorname{span}\left\{\hat{F}\big((M^{\mathrm{T}})^{-1}2\pi\gamma+2\pi\beta\big):\beta\in\mathbb{Z}^s\right\}=\mathbb{C}^r \qquad \forall\gamma\in\Gamma.$$

Since $(Q_a^nF)_{n=1,2,\ldots}$ converges to $\Phi$ in $W_p^k(\mathbb{R}^s)$, $(Q_a^n(Q_aF))_{n=1,2,\ldots}$ also converges to $\Phi$ in $W_p^k(\mathbb{R}^s)$. By Theorem 2.4, $Q_aF$ belongs to $Y_k$.

Let us prove that $U_k$ is invariant under the subdivision operator $S_a$. Pick an element $u$ from $U_k$. Since $Q_aF\in Y_k$,

$$p:=\sum_{\alpha\in\mathbb{Z}^s} u(\alpha)Q_aF(\cdot-\alpha) \tag{3.7}$$

is a polynomial in $\Pi_k$. Let $q(x)=p(M^{-1}x)$, $x\in\mathbb{R}^s$. Then $q$ also lies in $\Pi_k$. Hence,

$$q=\sum_{\alpha\in\mathbb{Z}^s} v(\alpha)F(\cdot-\alpha)$$

for some $v\in U_k$. Consequently,

$$p(x)=q(Mx)=\sum_{\alpha\in\mathbb{Z}^s} v(\alpha)F(Mx-\alpha), \qquad x\in\mathbb{R}^s. \tag{3.8}$$

On the other hand, it follows from (3.7) that

$$p(x)=\sum_{\alpha\in\mathbb{Z}^s}(S_au)(\alpha)F(Mx-\alpha), \qquad x\in\mathbb{R}^s. \tag{3.9}$$

Let $w:=S_au-v$. Comparing (3.8) with (3.9) gives

$$\sum_{\alpha\in\mathbb{Z}^s} w(\alpha)F(x-\alpha)=0 \qquad \text{for a.e. } x\in\mathbb{R}^s. \tag{3.10}$$

Suppose $w = (w_1, \ldots, w_r)$. Note that

$$S_a u(\varepsilon + M\alpha) = \sum_{\beta \in \mathbb{Z}^s} u(\beta) a(\varepsilon + M\alpha - M\beta) = \sum_{\beta \in \mathbb{Z}^s} u(\alpha - \beta) a(\varepsilon + M\beta), \qquad \varepsilon \in E.$$

Hence, since $u$ and $v$ are polynomial sequences, for each $j = 1, \ldots, r$ and each $\varepsilon \in E$, there exists a polynomial $p_{j,\varepsilon} \in \Pi_k$ such that $w_j(\varepsilon + M\beta) = p_{j,\varepsilon}(\beta)$ for all $\beta \in \mathbb{Z}^s$. We shall show $w = 0$. For this purpose we employ the difference operator $\nabla_\gamma$ for each $\gamma \in \mathbb{Z}^s$ defined by $\nabla_\gamma h = h - h(\cdot - \gamma)$ for $h \in \ell(\mathbb{Z}^s)$. Let $e_j$ $(j = 1, \ldots, s)$ denote the $j$th coordinate unit vector. If $w \neq 0$, then there exists a multi-index $\mu = (\mu_1, \ldots, \mu_s)$ with $|\mu| \le k$ such that $\nabla_{Me_1}^{\mu_1} \cdots \nabla_{Me_s}^{\mu_s} w \neq 0$ and for each $j = 1, \ldots, r$ and $\varepsilon \in E$,

$$\nabla_{Me_1}^{\mu_1} \cdots \nabla_{Me_s}^{\mu_s} w_j(\varepsilon + M\beta) = c_{j,\varepsilon} \qquad \forall \beta \in \mathbb{Z}^s,$$

where each $c_{j,\varepsilon}$ is a complex constant. It follows from (3.10) that

$$\sum_{\alpha \in \mathbb{Z}^s} \nabla_{Me_1}^{\mu_1} \cdots \nabla_{Me_s}^{\mu_s} w(\alpha) F(x - \alpha) = 0 \qquad \text{for a.e. } x \in \mathbb{R}^s.$$

By Lemma 3.4 we deduce that $c_{j,\varepsilon} = 0$ for all $j = 1, \ldots, r$ and all $\varepsilon \in E$. This is a contradiction. Therefore, $S_a u - v = w = 0$. In other words, $S_a u = v$ lies in $U_k$. This shows that $U_k$ is invariant under $S_a$.

Finally, by Lemma 3.1, $V_k$ is invariant under $T_a$. ■

## 4. CHARACTERIZATION OF CONVERGENCE

The uniform joint spectral radius was introduced in [24]. This concept was employed in [6] to investigate the regularity of refinable functions. The joint spectral radius was introduced in [26] for $p = 1$ and in [11] for $1 < p < \infty$, where it was applied to the $L_p$-convergence of subdivision schemes.

Let us recall the definition of the $p$-norm joint spectral radius. Let $V$ be a *finite-dimensional* vector space equipped with a vector norm $\|\cdot\|$. For a linear operator $A$ on $V$, define

$$\|A\| := \max_{\|v\|=1} \{\|Av\|\}.$$

Let $\mathcal{A}$ be a finite multiset of linear operators on $V$. For a positive integer $n$ we denote by $\mathcal{A}^n$ the $n$th Cartesian power of $\mathcal{A}$:

$$\mathcal{A}^n = \{(A_1, \ldots, A_n) : A_1, \ldots, A_n \in \mathcal{A}\}.$$

For $1 \le p \le \infty$, let

$$\|\mathcal{A}^n\|_p := \left( \sum_{(A_1,\ldots,A_n) \in \mathcal{A}^n} \|A_1 \cdots A_n\|^p \right)^{1/p},$$

and, for $p = \infty$, define

$$\|\mathcal{A}^n\|_\infty := \max\{\|A_1 \cdots A_n\| : (A_1, \ldots, A_n) \in \mathcal{A}^n\}.$$

For $1 \le p \le \infty$, the $p$-norm joint spectral radius of $\mathcal{A}$ is defined to be

$$\rho_p(\mathcal{A}) := \lim_{n\to\infty} \|\mathcal{A}^n\|_p^{1/n}.$$

It is easily seen that this limit indeed exists, and

$$\lim_{n\to\infty} \|\mathcal{A}^n\|_p^{1/n} = \inf_{n\ge 1} \|\mathcal{A}^n\|_p^{1/n}.$$

Clearly, $\rho_p(\mathcal{A})$ is independent of the choice of the vector norm on $V$.

Recall that $E$ is a complete set of representatives of the cosets $\mathbb{Z}^s/M\mathbb{Z}^s$ . It is assumed that $0 \in E$. For $\varepsilon \in E$, let $A_\varepsilon$ be the linear operator on $(\ell_0(\mathbb{Z}^s))^r$ defined by

$$A_\varepsilon v(\alpha) = \sum_{\beta\in\mathbb{Z}^s} a(\varepsilon + M\alpha - \beta)v(\beta), \qquad \alpha \in \mathbb{Z}^s,\ v \in (\ell_0(\mathbb{Z}^s))^r.$$

Let

$$K := \sum_{n=1}^{\infty} M^{-n} G,$$

where $G$ is the set given by

$$G := \big(\operatorname{supp} a \cup \{0\}\big) - E + [-1, 1]^s.$$

Let $\ell(K)$ denote the linear space of all sequences supported in $K$. It is easily seen that $(\ell(K))^r$ is invariant under every $A_\varepsilon$, $\varepsilon \in E$.

Recall that $V_k$ is the linear space defined in (3.3). Let

$$V := V_k \cap (\ell(K))^r.$$

Then $V$ is a finite-dimensional vector space.

We are in a position to give a characterization of the convergence of a subdivision scheme in Sobolev spaces.

THEOREM 4.1. *The subdivision scheme associated with a mask a and a dilation matrix M converges in the Sobolev space* $W_p^k(\mathbb{R}^s)$ $(1 \le p < \infty)$ *if and only if the following two conditions are satisfied:*

(a) $V_k$ *is invariant under* $A_\varepsilon$ *for every* $\varepsilon \in E$;
(b) $\rho_p(\{A_\varepsilon|_V : \varepsilon \in E\}) < m^{-k/s+1/p}$, *where* $m := |\det M|$.

*Furthermore, the subdivision scheme associated with mask a converges in* $C^k(\mathbb{R}^s)$ *if and only if* $V_k$ *is invariant under* $A_\varepsilon$ *for every* $\varepsilon \in E$ *and* $\rho_\infty(\{A_\varepsilon|_V : \varepsilon \in E\}) < m^{-k/s}$.

We shall establish the theorem for the case $1 \le p < \infty$ only. The proof for the case $p = \infty$ is similar.

For $v = (v_1, \ldots, v_r)^{\mathrm{T}} \in (\ell_p(\mathbb{Z}^s))^r$ we define

$$\|v\|_p = \left(\sum_{j=1}^{r} \|v_j\|_p^p\right)^{1/p}, \qquad 1 \le p < \infty,$$

and $\|v\|_\infty = \max_{1\le j\le r} \|v_j\|_\infty$. Let $\mathcal{A} := \{A_\varepsilon : \varepsilon \in E\}$. We define, for $1 \le p < \infty$,

$$\|\mathcal{A}^n v\|_p := \left( \sum_{(\varepsilon_1,\dots,\varepsilon_n)\in E^n} \|A_{\varepsilon_1}\cdots A_{\varepsilon_n} v\|_p^p \right)^{1/p},$$

and for $p = \infty$,

$$\|A^n v\|_\infty := \max\big\{ \|A_{\varepsilon_1}\cdots A_{\varepsilon_n} v\|_\infty : (\varepsilon_1,\dots,\varepsilon_n) \in E^n \big\}.$$

LEMMA 4.2. *Suppose $V_k$ is invariant under $A_\varepsilon$ for every $\varepsilon \in E$. Then, given a compact $K_0 \subset \mathbb{R}^s$, $\eta > 0$, and $1 \le p \le \infty$, there exist a constant $C = C(K_0)$ and an integer $N_0 = N_0(\eta)$ such that for any $v \in V_k \cap (\ell(K_0))^r$, we have*

$$\|\mathcal{A}^n v\|_p \le C(\rho_p(\{\mathcal{A}|_V\}) + \eta)^n \|v\|_p, \qquad \forall n \ge N_0.$$

*Consequently,*

$$\lim_{n\to\infty} \|\mathcal{A}^n v\|_p^{1/n} \le \rho_p(\{\mathcal{A}|_V\}).$$

*Proof.* For $\varepsilon \in E$ we have

$$\operatorname{supp}(A_\varepsilon v) \subseteq M^{-1}(\operatorname{supp} a - E) + M^{-1}(K_0).$$

Hence, for $\varepsilon_1,\dots,\varepsilon_N \in E$,

$$\operatorname{supp}(A_{\varepsilon_N}\cdots A_{\varepsilon_1} v) \subseteq \sum_{j=1}^{N} M^{-j}(\operatorname{supp} a - E) + M^{-N}(K_0).$$

There exists a positive integer $N$ such that

$$\sum_{j=1}^{N} M^{-j}(\operatorname{supp} a - E) + M^{-N}(K_0) \subseteq \sum_{j=1}^{N} M^{-j} G \subseteq K.$$

Therefore,

$$A_{\varepsilon_N}\cdots A_{\varepsilon_1} v \in (\ell(K))^r \cap V_k = V.$$

Suppose $1 \le p < \infty$. For $n \ge N$ we have

$$\begin{aligned}
\|\mathcal{A}^n v\|_p^p &= \sum_{\varepsilon_1,\dots,\varepsilon_n\in E} \|A_{\varepsilon_n}\cdots A_{\varepsilon_1} v\|_p^p \\
&= \sum_{\varepsilon_n,\dots,\varepsilon_{N+1}\in E}\ \sum_{\varepsilon_N,\dots,\varepsilon_1\in E} \|(A_{\varepsilon_n}\cdots A_{\varepsilon_{N+1}})(A_{\varepsilon_N}\cdots A_{\varepsilon_1} v)\|_p^p \\
&\le \|\mathcal{A}^{n-N}|_V\|_p^p (C^N \|v\|_p)^p,
\end{aligned}$$

where

$$C := \max_{\varepsilon\in E}\{\|A_\varepsilon\|_p\}.$$

The first result now follows by choosing $N_0$ sufficiently large. The second result also follows by taking the $n$th root of the last inequality to obtain

$$\|\mathcal{A}^n v\|_p^{1/n} \le \|\mathcal{A}^{n-N}|_V\|_p^{1/n}(C^N\|v\|_p)^{1/n}.$$

Therefore,

$$\lim_{n\to\infty} \|\mathcal{A}^n v\|_p^{1/n} \le \rho_p(\{\mathcal{A}|_V\}).$$

The proof for the case $p = \infty$ is analogous. ■

For $n = 1, 2, \ldots$, let $a_1 := a$ and

$$a_n(\alpha) = \sum_{\beta\in\mathbb{Z}^s} a_{n-1}(\beta)a(\alpha - M\beta), \qquad \alpha \in \mathbb{Z}^s.$$

For $v \in (\ell_0(\mathbb{Z}^s))^r$, we define $a_n * v$ to be the element in $(\ell_0(\mathbb{Z}^s))^r$ given by

$$a_n * v(\alpha) := \sum_{\beta\in\mathbb{Z}^s} a_n(\alpha - \beta)v(\beta), \qquad \alpha \in \mathbb{Z}^s.$$

Suppose $\alpha = \varepsilon_1 + M\varepsilon_2 + \cdots + M^{n-1}\varepsilon_n + M^n\gamma$, where $\varepsilon_1, \ldots, \varepsilon_n \in E$ and $\gamma \in \mathbb{Z}^s$. Then

$$a_n * v(\alpha) = A_{\varepsilon_n} \cdots A_{\varepsilon_1} v(\gamma).$$

See [9] for its proof.

We claim that

$$\|a_n * v\|_p = \|\mathcal{A}^n v\|_p, \qquad v \in (\ell_0(\mathbb{Z}^s))^r,\ 1 \le p \le \infty. \tag{4.1}$$

Indeed, for $1 \le p < \infty$, we have

$$\|a_n * v\|_p^p = \sum_{\alpha\in\mathbb{Z}^s} \|(a_n * v)(\alpha)\|_p^p = \sum_{\varepsilon_1,\ldots,\varepsilon_n\in E}\sum_{\gamma\in\mathbb{Z}^s} \|A_{\varepsilon_n}\cdots A_{\varepsilon_1} v(\gamma)\|_p^p = \|\mathcal{A}^n v\|_p^p.$$

Similarly, for $p = \infty$, we have

$$\|a_n * v\|_\infty = \sup_{\varepsilon_1,\ldots,\varepsilon_n\in E}\ \sup_{\gamma\in\mathbb{Z}^s} \left\{|A_{\varepsilon_n}\cdots A_{\varepsilon_1} v(\gamma)|\right\} = \|\mathcal{A}^n v\|_\infty$$

Recall that the matrix $\Lambda$ given in Section 2 satisfies $\Lambda M \Lambda^{-1} = \operatorname{diag}(\sigma_1, \ldots, \sigma_s)$. Taking the transpose of both sides of this equality, we obtain

$$(\Lambda^{\mathrm{T}})^{-1} M^{\mathrm{T}} \Lambda^{\mathrm{T}} = \operatorname{diag}(\sigma_1, \ldots, \sigma_s).$$

For a multi-index $\mu$, let $\tilde{q}_\mu$ be the polynomial given by

$$\tilde{q}_\mu(x) := ((\Lambda^{\mathrm{T}})^{-1}x)^\mu, \qquad x \in \mathbb{R}^s.$$

Suppose $f$ and $g$ are smooth functions on $\mathbb{R}^s$. The following two identities are similar to (2.2) and (2.3):

$$\tilde{q}_\mu(D)(f \circ M)(x) = \sigma^\mu \tilde{q}_\mu(D) f(Mx), \qquad x \in \mathbb{R}^s,$$

$$\tilde{q}_\mu(D)(fg) = \sum_{\nu \le \mu} \binom{\mu}{\nu} [\tilde{q}_{\mu-\nu}(D) f][\tilde{q}_\nu(D) g].$$

LEMMA 4.3. *If the subdivision scheme associated with a mask a and a dilation matrix M converges in the Sobolev space* $W_p^k(\mathbb{R}^s)$ $(1 \le p < \infty)$, *then for any* $v \in V_k$,

$$\lim_{n\to\infty} m^{n(k/s-1/p)} \|a_n * v\|_p = 0,$$

*where* $m := |\det M|$.

*Proof.* Pick a nontrivial function $f$ in $C^k(\mathbb{R}^s)$ such that $f$ is supported on the unit cube $[0, 1]^s$. For $v \in V_k$, let

$$\Psi := \sum_{\alpha \in \mathbb{Z}^s} v(\alpha) f(\cdot - \alpha).$$

Since $v$ is finitely supported, $\Psi$ is compactly supported. The Fourier transform of $\Psi$ is

$$\hat{\Psi}(\omega) = \sum_{\alpha \in \mathbb{Z}^s} v(\alpha) e^{-i\alpha\cdot\omega} \hat{f}(\omega), \qquad \omega \in \mathbb{R}^s.$$

Hence, for $|\mu| \le k$ we have

$$\begin{aligned} J_{\mu,\Psi}(\omega) &:= \sum_{\nu \le \mu} \binom{\mu}{\nu} B_{\mu-\nu} q_\nu(-iD) \hat{\Psi}(\omega) \\ &= \sum_{\nu \le \mu} \binom{\mu}{\nu} B_{\mu-\nu} q_\nu(-iD)(\hat{v}(\omega) \hat{f}(\omega)) \\ &= \sum_{\nu \le \mu} \binom{\mu}{\nu} B_{\mu-\nu} \sum_{\lambda \le \nu} \binom{\nu}{\lambda} q_{\nu-\lambda}(-iD)\hat{v}(\omega) q_\lambda(-iD)\hat{f}(\omega) \\ &= \sum_{\gamma \le \mu} \binom{\mu}{\gamma} \left[ \sum_{\tau \le \gamma} \binom{\gamma}{\tau} B_{\gamma-\tau} q_\tau(-iD)\hat{v}(\omega) \right] q_{\mu-\gamma}(-iD)\hat{f}(\omega), \end{aligned}$$

where the last equation resulted from the change of indices $\gamma = \mu - \lambda$ and $\tau = \nu - \lambda$. Setting $\omega = 2\beta\pi$ for $\beta \in \mathbb{Z}^s$, we obtain

$$\begin{aligned} J_{\mu,\Psi}(2\beta\pi) &:= \sum_{\gamma \le \mu} \binom{\mu}{\gamma} \left[ \sum_{\alpha \in \mathbb{Z}^s} \sum_{\tau \le \gamma} \binom{\gamma}{\tau} B_{\gamma-\tau} (-\Lambda\alpha)^\tau v(\alpha) \right] q_{\mu-\gamma}(-iD)\hat{f}(2\beta\pi) \\ &= \sum_{\gamma \le \mu} \binom{\mu}{\gamma} \langle u_\gamma, v \rangle q_{\mu-\gamma}(-iD)\hat{f}(2\beta\pi), \end{aligned}$$

where $u_\gamma$ is the element in $U_k$ given in (3.1). Since $\langle u_\mu, v\rangle = 0$, we obtain $J_{\mu,\Psi}(2\beta\pi) = 0$ for all $|\mu| \le k$ and all $\beta \in \mathbb{Z}^s$. Recall that $F \in (W_p^k(\mathbb{R}^s))^r$ lies in $Y_k$ if and only if $F$ satisfies (2.12) and (2.13). Let $\Phi_0$ be an element in $Y_k$. Then the preceding discussion tells us that $\Phi_0 + \Psi$ is also an element in $Y_k$. Hence, there exists $\Phi \in (W_p^k(\mathbb{R}^s))^r$ such that

$$\lim_{n\to\infty} \|Q_a^n \Phi_0 - \Phi\|_{(W_p^k(\mathbb{R}^s))^r} = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \|Q_a^n(\Phi_0 + \Psi) - \Phi\|_{(W_p^k(\mathbb{R}^s))^r} = 0.$$

It follows that

$$\lim_{n\to\infty} \|Q_a^n \Psi\|_{(W_p^k(\mathbb{R}^s))^r} = 0. \tag{4.2}$$

We have

$$Q_a^n \Psi = \sum_{\alpha\in\mathbb{Z}^s} (a_n * v)(\alpha) f(M^n \cdot - \alpha).$$

For $|\mu| = k$, applying the differential operator $\tilde{q}_\mu(D)$ to both sides of the above equation, we obtain

$$\tilde{q}_\mu(D)(Q_a^n \Psi) = \sum_{\alpha\in\mathbb{Z}^s} (a_n * v)(\alpha)\sigma^{\mu n} \tilde{q}_\mu(D) f(M^n \cdot - \alpha).$$

Since $f$ is supported on $[0, 1]^s$, there exists a positive constant $C$ independent of $n$ such that

$$m^{kn/s} m^{-n/p} \|a_n * v\|_p \le C \|\tilde{q}_\mu(D)(Q_a^n \Psi)\|_p. \tag{4.3}$$

Combining (4.2) and (4.3), we obtain

$$\lim_{n\to\infty} m^{n(k/s-1/p)} \|a_n * v\|_p = 0 \qquad \forall v \in V_k,$$

as desired. ■

*Proof of Theorem* 4.1. We first establish the necessity part of the theorem. Suppose the subdivision scheme associated with mask $a$ converges to $\Phi$ in the Sobolev space $W_p^k(\mathbb{R}^s)$. It was proved in Theorem 3.2 that $V_k$ is invariant under the transition operator $T_a = A_0$. Let $\varepsilon \in E$ and $v \in V_k$. We have

$$A_\varepsilon v(\alpha) = \sum_{\beta\in\mathbb{Z}^s} a(\varepsilon + M\alpha - \beta) v(\beta) = \sum_{\beta\in\mathbb{Z}^s} a(M\alpha - \beta) v(\beta + \varepsilon) = T_a\big(v(\varepsilon + \cdot)\big).$$

Since $V_k$ is shift-invariant, we have $v(\varepsilon + \cdot) \in V_k$. Hence, $A_\varepsilon v = T_a(v(\varepsilon + \cdot))$ is in $V_k$. This shows that $V_k$ is invariant under $A_\varepsilon$ for each $\varepsilon \in E$.

We write $\mathcal{A}$ for $\{A_\varepsilon : \varepsilon \in E\}$. Suppose $X$ is a basis for the vector space $V$. There exists a positive constant $C$ independent of $n$ such that

$$\|\mathcal{A}^n|_V\|_p \le C \max_{v\in X} \|\mathcal{A}^n v\|_p.$$

By Lemma 4.3 we have

$$\lim_{n\to\infty} m^{n(k/s-1/p)} \|a_n * v\|_p = 0 \qquad \forall v \in V.$$

But $\|\mathcal{A}^n v\|_p = \|a_n * v\|_p$, by (4.1). Therefore,

$$\lim_{n\to\infty} \big(m^{k/s-1/p} \|\mathcal{A}^n|_V\|_p^{1/n}\big)^n = 0. \tag{4.4}$$

Note that

$$\rho_p\big(\{\mathcal{A}|_V\}\big) = \lim_{n\to\infty} \|\mathcal{A}^n|_V\|_p^{1/n} = \inf_{n\ge 1} \|\mathcal{A}^n|_V\|_p^{1/n}.$$

Thus, for (4.4) to hold we must have

$$m^{k/s-1/p}\rho_p\big(\{\mathcal{A}|_V\}\big) < 1.$$

In other words, $\rho_p(\{\mathcal{A}|_V\}) < m^{-k/s+1/p}$. This finishes the proof for the necessity part.

Next, we establish the sufficiency part of the theorem. Suppose $V_k$ is invariant under $A_\varepsilon$ for each $\varepsilon \in E$ and $\rho_p(\{\mathcal{A}|_V\}) < m^{-k/s+1/p}$. Let $\Phi_0$ be an element in $Y_k$. Then $Q_a^{n+1}\Phi_0 - Q_a^n\Phi_0 = Q_a^n\Psi_0$, where $\Psi_0 := Q_a\Phi_0 - \Phi_0$. We have

$$Q_a^n\Psi_0 = \sum_{\alpha\in\mathbb{Z}^s} a_n(\alpha)\Psi_0(M^n\cdot - \alpha).$$

For $|\mu| = k$, applying the differential operator $\tilde{q}_\mu(D)$ to both sides of the above equation, we obtain

$$\tilde{q}_\mu(D)(Q_a^n\Psi_0) = \sum_{\alpha\in\mathbb{Z}^s} a_n(\alpha)\sigma^{\mu n}\tilde{q}_\mu(D)\Psi_0(M^n\cdot - \alpha). \tag{4.5}$$

We observe that

$$\begin{aligned}\|\tilde{q}_\mu(D)(Q_a^n\Psi_0)\|_p^p &= \int_{\mathbb{R}^s} |\tilde{q}_\mu(D)(Q_a^n\Psi_0)(y)|^p\,dy\\ &= \sum_{\beta\in\mathbb{Z}^s}\int_{M^{-n}([0,1)^s+\beta)} |\tilde{q}_\mu(D)(Q_a^n\Psi_0)(y)|^p\,dy.\end{aligned}$$

By making the substitution $y = M^{-n}(x+\beta)$ in the above integral and using (4.5), we obtain

$$\|\tilde{q}_\mu(D)(Q_a^n\Psi_0)\|_p^p = m^{pn(k/s-1/p)}\int_{[0,1)^s}\sum_{\beta\in\mathbb{Z}^s}\left|\sum_{\alpha\in\mathbb{Z}^s} a_n(\alpha)f_\mu(x+\beta-\alpha)\right|^p dx, \tag{4.6}$$

where $f_\mu := \tilde{q}_\mu(D)\Psi_0$ lies in $L_p(\mathbb{R}^s)$. Let $v_x(\alpha) := f_\mu(x+\alpha)$ for $\alpha \in \mathbb{Z}^s$ and $x \in [0,1)^s$. Since $f_\mu \in L_p(\mathbb{R}^s)$, we have $v_x \in \ell_p(\mathbb{Z}^s)$ for almost every $x \in [0,1)^s$ and

$$\int_{[0,1)^s}\|v_x\|_p^p\,dx = \int_{[0,1)^s}\sum_{\alpha\in\mathbb{Z}^s}|f_\mu(x+\alpha)|^p\,dx = \int_{\mathbb{R}^s}|f_\mu(x)|^p\,dx = \|f_\mu\|_p^p. \tag{4.7}$$

Thus, (4.6) can be rewritten as

$$\|\tilde{q}_\mu(D)(Q_a^n\Psi_0)\|_p^p = m^{pn(k/s-1/p)}\int_{[0,1)^s}\|a_n * v_x\|_p^p\,dx. \tag{4.8}$$

We claim that $v_x$ lies in $V_k$ for almost every $x \in [0,1)^s$. Since both $\Phi_0$ and $Q_a\Phi_0$ belong to $Y_k$, in light of (2.10) and (2.14) we have

$$\sum_{\alpha\in\mathbb{Z}^s} u_\mu(\alpha)\Psi_0(\cdot-\alpha) = \sum_{\alpha\in\mathbb{Z}^s} u_\mu(\alpha)(Q_a\Phi_0 - \Phi_0)(\cdot-\alpha) \in \Pi_{|\mu|-1}.$$

By definition $U_k$ is spanned by $u_\mu$, $|\mu| \le k$. Therefore,

$$\sum_{\alpha\in\mathbb{Z}^s} u(\alpha)\Psi_0(\cdot-\alpha) \in \Pi_{k-1} \qquad \forall u \in U_k.$$

Consequently, for $|\mu| = k$ and $u \in U_k$ we have

$$\sum_{\alpha\in\mathbb{Z}^s} u(\alpha) f_\mu(x-\alpha) = \tilde{q}_\mu(D)\left(\sum_{\alpha\in\mathbb{Z}^s} u(\alpha)\Psi_0(x-\alpha)\right) = 0.$$

In other words, for almost every $x \in [0, 1)^s$,

$$\langle u, v_x\rangle = \sum_{\alpha\in\mathbb{Z}^s} u(\alpha) v_x(-\alpha) = \sum_{\alpha\in\mathbb{Z}^s} u(\alpha) f_\mu(x-\alpha) = 0 \qquad \forall u \in U_k.$$

This shows $v_x \in V_k$ for almost every $x \in [0, 1)^s$.

Write $\rho$ for $\rho_p(\mathcal{A}|_V)$. Each $v_x$ is supported in the compact set $K_0 := \operatorname{supp} f_\mu - [0, 1]^s$. Therefore, by Lemma 4.2, for $\varepsilon > 0$ and all sufficiently large $n$, we have

$$\|a_n * v_x\|_p = \|\mathcal{A}^n v_x\|_p \le C(\rho+\varepsilon)^n \|v_x\|_p,$$

where $C$ is a constant independent of $n$ and $x$. This, together with (4.7) and (4.8), gives

$$\|\tilde{q}_\mu(D)(Q_a^n\Psi_0)\|_p^p \le C^p m^{pn(k/s-1/p)}(\rho+\varepsilon)^{np}\|f_\mu\|_p^p.$$

It follows that

$$\|\tilde{q}_\mu(D)(Q_a^n\Psi_0)\|_p \le Ct^n\|f_\mu\|_p,$$

where $t := m^{k/s-1/p}(\rho+\varepsilon)$. Since $\rho < m^{-k/s+1/p}$, for sufficiently small $\varepsilon > 0$, we have $t < 1$. Therefore, for $|\mu| = k$, the sequence $(\tilde{q}_\mu(D)Q_a^n\Psi_0)_{n=1,2,\ldots}$ converges in $L_p(\mathbb{R}^s)$.

We observe that there exists a compact subset of $\mathbb{R}^s$ such that $Q_a^n\Psi_0$ are supported in it for all $n$. Thus, for $|\nu| \le k$, by Poincare's inequality (see, e.g., [7, p. 276]), there exists a constant $C > 0$ such that

$$\|\tilde{q}_\nu(D)(Q_a^n\Psi_0)\|_p \le C \max_{|\mu|=k}\|\tilde{q}_\mu(D)(Q_a^n\Psi_0)\|_p$$

holds for all $n$. This shows that the sequence $(\tilde{q}_\nu(D)(Q_a^n\Psi_0))_{n=1,2,\ldots}$ converges in $L_p(\mathbb{R}^s)$. Let $g_\nu$ be the corresponding limit function. In particular, $\Phi = g_0$. It follows that $\tilde{q}_\nu(D)\Phi = g_\nu$ in the distributional sense. Hence, $\Phi \in (W_p^k(\mathbb{R}^s))^r$. Moreover,

$$\lim_{n\to\infty}\|Q_a^n\Psi_0 - \Phi\|_{(W_p^k(\mathbb{R}^s))^r} = 0.$$

The proof of the theorem is complete. ■

## ACKNOWLEDGMENT

The authors are grateful to Dr. Qingtang Jiang for his valuable comments on this paper.

## REFERENCES

1. C. de Boor, R. DeVore, and A. Ron, Approximation orders of FSI spaces in $L_2(\mathbb{R}^d)$, *Constr. Approx.* **14** (1998), 631–652.
2. C. Cabrelli, C. Heil, and U. Molter, Accuracy of lattice translates of several multidimensional refinable functions, *J. Approx. Theory* **95** (1998), 5–52.

3. A. S. Cavaretta, W. Dahmen, and C. A. Micchelli, Stationary subdivision, *Mem. Amer. Math. Soc.* **93**, No. 453, (1991). pp. i–186.
4. D. R. Chen, Algebraic properties of subdivision operators with matrix mask and their applications, *J. Approx. Theory* **97** (1999), 294–310.
5. W. Dahmen and C. A. Micchelli, Biorthogonal wavelet expansions, *Constr. Approx.* **13** (1997), 293–328.
6. I. Daubechies and J. Lagarias, Two-scale difference equations: II. Local regularity, infinite products of matrices, and fractals, *SIAM J. Math. Anal.* **23** (1992), 1031–1079.
7. L. C. Evans, "Partial Differential Equations," Graduate Studies in Mathematics, Vol. 19, Amer. Math. Soc., Providence, RI, 1998.
8. T. N. T. Goodman and S. L. Lee, Convergence of nonstationary cascade algorithms, *Numer. Math.* **84** (1999), 1–33.
9. B. Han and R. Q. Jia, Multivariate refinement equations and convergence of subdivision schemes, *SIAM J. Math. Anal.* **29** (1998), 1177–1199.
10. C. Heil and D. Colella, Matrix refinement equations: existence and uniqueness, *J. Fourier Anal. Appl.* **2** (1996), 363–377.
11. R. Q. Jia, Subdivision schemes in $L_p$ spaces, *Adv. Comput. Math.* **3** (1995), 309–341.
12. R. Q. Jia, Approximation properties of multivariate wavelets, *Math. Comp.* **67** (1998), 647–665.
13. R. Q. Jia, Shift-invariant spaces and linear operator equations, *Israel J. Math.* **103** (1998), 259–288.
14. R. Q. Jia, Stability of the shifts of a finite number of functions, *J. Approx. Theory* **95** (1998), 194–202.
15. R. Q. Jia, Q. T. Jiang, and S. L. Lee, Convergence of cascade algorithms in Sobolev spaces and integrals of wavelets, *Numer. Math.*, to appear.
16. R. Q. Jia and C. A. Micchelli, On linear independence of integer translates of a finite number of functions, *Proc. Edinburgh Math. Soc.* **36** (1992), 69–85.
17. R. Q. Jia, S. D. Riemenschneider, and D. X. Zhou, Approximation by multiple refinable functions, *Canad. J. Math.* **49** (1997), 944–962.
18. R. Q. Jia, S. D. Riemenschneider, and D. X. Zhou, Vector subdivision schemes and multiple wavelets, *Math. Comp.* **67** (1998), 1533–1563.
19. Q. T. Jiang, Multivariate matrix refinable functions with arbitrary matrix dilation, *Trans. Amer. Math. Soc.* **351** (1999), 2407–2438.
20. Q. T. Jiang and Z. W. Shen, On existence and weak stability of matrix refinable functions, *Constr. Approx.* **15** (1999), 337–353.
21. J. L. Kelley and I. Namioka, "Linear Topological Spaces," Springer-Verlag, New York, 1963.
22. C. A. Micchelli and T. Sauer, Regularity of multiwavelets, *Adv. Comput. Math.* **7** (1997), 455–545.
23. C. A. Micchelli and T. Sauer, Sobolev norm convergence of stationary subdivision schemes. In "Surface Fitting and Multiresolution Methods" (A. Le Méhauté, C. Rabut, and L. L. Schumaker, Eds.), pp. 245–260, Vanderbilt University Press, Nashville, 1997.
24. G.-C. Rota and G. Strang, A note on the joint spectral radius, *Indag. Math.* **22** (1960), 379–381.
25. G. Strang and G. Fix, A Fourier analysis of the finite-element variational method, *in* "Constructive Aspects of Functional Analysis" (G. Geymonat, Ed.), pp. 793–840, Centro Internationale Matematico Estivo, Rome, Italy, 1973.
26. Y. Wang, Two-scale dilation equations and the mean spectral radius, *Random Comput. Dynam.* **4** (1996), 49–72.