Chapter 3. Absolutely Continuous Functions

§1. Absolutely Continuous Functions

A function \( f : [a, b] \to \mathbb{R} \) is said to be **absolutely continuous** on \([a, b]\) if, given \( \varepsilon > 0 \), there exists some \( \delta > 0 \) such that

\[
\sum_{i=1}^{n} |f(y_i) - f(x_i)| < \varepsilon,
\]

whenever \( \{[x_i, y_i] : i = 1, \ldots, n\} \) is a finite collection of mutually disjoint subintervals of \([a, b]\) with \( \sum_{i=1}^{n} |y_i - x_i| < \delta \).

Clearly, an absolutely continuous function on \([a, b]\) is uniformly continuous. Moreover, a Lipschitz continuous function on \([a, b]\) is absolutely continuous. Let \( f \) and \( g \) be two absolutely continuous functions on \([a, b]\). Then \( f + g, f - g, \) and \( fg \) are absolutely continuous on \([a, b]\). If, in addition, there exists a constant \( C > 0 \) such that \( |g(x)| \geq C \) for all \( x \in [a, b] \), then \( f/g \) is absolutely continuous on \([a, b]\).

If \( f \) is integrable on \([a, b]\), then the function \( F \) defined by

\[
F(x) := \int_{a}^{x} f(t) \, dt, \quad a \leq x \leq b,
\]

is absolutely continuous on \([a, b]\).

**Theorem 1.1.** Let \( f \) be an absolutely continuous function on \([a, b]\). Then \( f \) is of bounded variation on \([a, b]\). Consequently, \( f'(x) \) exists for almost every \( x \in [a, b] \).

**Proof.** Since \( f \) is absolutely continuous on \([a, b]\), there exists some \( \delta > 0 \) such that \( \sum_{i=1}^{n} |f(y_i) - f(x_i)| < 1 \) whenever \( \{[x_i, y_i] : i = 1, \ldots, n\} \) is a finite collection of mutually disjoint subintervals of \([a, b]\) with \( \sum_{i=1}^{n} |y_i - x_i| < \delta \). Let \( N \) be the least integer such that \( N > (b - a)/\delta \), and let \( a_j := a + j(b - a)/N \) for \( j = 0, 1, \ldots, N \). Then \( a_j - a_{j-1} = (b - a)/N < \delta \). Hence, \( \bigvee_{a_{j-1}}^{a_j} f < 1 \) for \( j = 0, 1, \ldots, N \). It follows that

\[
\bigvee_{a}^{b} f = \sum_{j=1}^{N} \bigvee_{a_{j-1}}^{a_j} f < N.
\]

This shows that \( f \) is of bounded variation on \([a, b]\). Consequently, \( f'(x) \) exists for almost every \( x \in [a, b] \).
Theorem 1.2. If $f$ is absolutely continuous on $[a, b]$ and $f'(x) = 0$ for almost every $x \in [a, b]$, then $f$ is constant.

Proof. We wish to show $f(a) = f(c)$ for every $c \in [a, b]$. Let $E := \{x \in [a, c]: f'(x) = 0\}$. For given $\varepsilon > 0$, there exists some $\delta > 0$ such that $\sum_{i=1}^{n} |f(y_i) - f(x_i)| < \varepsilon$ whenever $\{[x_i, y_i] : i = 1, \ldots, n\}$ is a finite collection of mutually disjoint subintervals of $[a, b]$ with $\sum_{i=1}^{n} |y_i - x_i| < \delta$. For each $x \in E$, we have $f'(x) = 0$; hence there exists an arbitrary small interval $[a_x, c_x]$ such that $x \in [a_x, c_x] \subseteq [a, c]$ and

$$|f(c_x) - f(a_x)| < \varepsilon(c_x - a_x).$$

By the Vitali covering theorem we can find a finite collection $\{[x_k, y_k] : k = 1, \ldots, n\}$ of mutually disjoint intervals of this sort such that

$$\lambda(E \setminus \bigcup_{k=1}^{n} [x_k, y_k]) < \delta.$$

Since $\lambda([a, c] \setminus E) = 0$, we have

$$\lambda([a, c] \setminus \bigcup_{k=1}^{n} [x_k, y_k]) = \lambda(E \setminus \bigcup_{k=1}^{n} [x_k, y_k]) < \delta.$$

Suppose $a \leq x_1 < y_1 \leq x_2 < \cdots < y_n \leq c$. Let $y_0 := a$ and $x_{n+1} := c$. Then

$$\sum_{k=0}^{n} (x_{k+1} - y_k) = \lambda([a, c] \setminus \bigcup_{k=1}^{n} [x_k, y_k]) < \delta.$$

Consequently,

$$\sum_{k=0}^{n} |f(x_{k+1}) - f(y_k)| < \varepsilon.$$

Furthermore,

$$\sum_{k=1}^{n} |f(y_k) - f(x_k)| < \sum_{k=1}^{n} \varepsilon(y_k - x_k) \leq \varepsilon(c - a).$$

It follows from the above inequalities that

$$|f(c) - f(a)| \leq \sum_{k=0}^{n} |f(x_{k+1}) - f(y_k)| + \sum_{k=1}^{n} |f(y_k) - f(x_k)| < \varepsilon(c - a + 1).$$

This shows that $|f(c) - f(a)| \leq \varepsilon(c - a + 1)$ for all $\varepsilon > 0$. Therefore, $f(c) = f(a)$.  

§2. The Fundamental Theorem of Calculus

In this section we show that absolutely continuous functions are precisely those functions for which the fundamental theorem of calculus is valid.

**Theorem 2.1.** If \( f \) is integrable on \([a, b]\) and \[
\int_a^x f(t) \, dt = 0 \quad \forall \, x \in [a, b],
\]
then \( f(t) = 0 \) for almost every \( t \in [a, b] \).

**Proof.** By our assumption,
\[
\int_c^d f(t) \, dt = 0
\]
for all \( c, d \) with \( a \leq c < d \leq b \). If \( O \) is an open subset of \([a, b]\), then \( O \) is a countable union of mutually disjoint open intervals \((c_n, d_n) \) \( (n = 1, 2, \ldots) \); hence,
\[
\int_O f(t) \, dt = \sum_{n=1}^{\infty} \int_{c_n}^{d_n} f(t) \, dt = 0.
\]
It follows that for any closed subset \( K \) of \([a, b]\),
\[
\int_K f(t) \, dt = \int_{[a,b]} f(t) \, dt - \int_{[a,b]\setminus K} f(t) \, dt = 0.
\]
Let \( E_+ := \{x \in [a, b] : f(x) > 0\} \) and \( E_- := \{x \in [a, b] : f(x) < 0\} \). We wish to show that \( \lambda(E_+) = 0 \) and \( \lambda(E_-) = 0 \). If \( \lambda(E_+) > 0 \), then there exists some closed set \( K \subseteq E_+ \) such that \( \lambda(K) > 0 \). But \( \int_K f(t) \, dt = 0 \). It follows that \( f = 0 \) almost everywhere on \( K \). This contradiction shows that \( \lambda(E_+) = 0 \). Similarly, \( \lambda(E_-) = 0 \). Therefore, \( f(t) = 0 \) for almost every \( t \in [a, b] \).

**Theorem 2.2.** If \( f \) is integrable on \([a, b]\), and if \( F \) is defined by
\[
F(x) := \int_a^x f(t) \, dt, \quad a \leq x \leq b,
\]
then \( F'(x) = f(x) \) for almost every \( x \) in \([a, b]\).

**Proof.** First, we assume that \( f \) is bounded and measurable on \([a, b]\). For \( n = 1, 2, \ldots \), let
\[
g_n(x) := \frac{F(x + 1/n) - F(x)}{1/n}, \quad x \in [a, b].
\]
It follows that
\[ g_n(x) = n \int_x^{x + 1/n} f(t) \, dt, \quad x \in [a, b]. \]

Suppose \(|f(x)| \leq K\) for all \(x \in [a, b]\). Then \(|g_n(x)| \leq K\) for all \(x \in [a, b]\) and \(n \in \mathbb{N}\). Since \(\lim_{n \rightarrow \infty} g_n(x) = F'(x)\) for almost every \(x \in [a, b]\), by the Lebesgue dominated convergence theorem, we see that for each \(c \in [a, b]\),
\[
\int_a^c F'(x) \, dx = \lim_{n \rightarrow \infty} \int_a^c g_n(x) \, dx.
\]

But \(F\) is continuous; hence,\[
\lim_{n \rightarrow \infty} \int_a^c g_n(x) \, dx = \lim_{n \rightarrow \infty} n \left[ \int_c^{c + 1/n} F(x) \, dx - \int_a^{a + 1/n} F(x) \, dx \right] = F(c) - F(a).
\]

Consequently,
\[
\int_a^c F'(x) \, dx = \lim_{n \rightarrow \infty} \int_a^c g_n(x) \, dx = F(c) - F(a) = \int_a^c f(x) \, dx.
\]

It follows that
\[
\int_a^c [F'(x) - f(x)] \, dx = 0
\]
for every \(c \in [a, b]\). By Theorem 2.1, \(F'(x) = f(x)\) for almost every \(x \in [a, b]\).

Now let us assume that \(f\) is integrable on \([a, b]\). Without loss of any generality, we may assume that \(f \geq 0\). For \(n = 1, 2, \ldots\), let \(f_n\) be the function defined by
\[
f_n(x) := \begin{cases} f(x) & \text{if } 0 \leq f(x) \leq n, \\ 0 & \text{if } f(x) > n. \end{cases}
\]

It is easily seen that \(F = F_n + G_n\), where
\[
F_n(x) := \int_a^x f_n(t) \, dt \quad \text{and} \quad G_n(x) := \int_a^x [f(t) - f_n(t)] \, dt, \quad a \leq x \leq b.
\]

Since \(f(t) - f_n(t) \geq 0\) for all \(t \in [a, b]\), \(G_n\) is an increasing function on \([a, b]\). Moreover, by what has been proved, \(F'_n(x) = f_n(x)\) for almost every \(x \in [a, b]\). Thus, we have
\[
F'(x) = F'_n(x) + G'_n(x) \geq F'_n(x) = f_n(x) \quad \text{for almost every } x \in [a, b].
\]

Letting \(n \rightarrow \infty\) in the above inequality, we obtain \(F'(x) \geq f(x)\) for almost every \(x \in [a, b]\). It follows that
\[
\int_a^b F'(x) \, dx \geq \int_a^b f(x) \, dx = F(b) - F(a).
\]
On the other hand, 
\[ \int_a^b F'(x) \, dx \leq F(b) - F(a). \]

Consequently, 
\[ \int_a^b [F'(x) - f(x)] \, dx = 0. \]

But \( F'(x) \geq f(x) \) for almost every \( x \in [a, b] \). Therefore, \( F'(x) = f(x) \) for almost every \( x \) in \([a, b]\).

**Theorem 2.3.** A function \( F \) on \([a, b]\) is absolutely continuous if and only if 
\[ F(x) = F(a) + \int_a^x f(t) \, dt \]
for some integrable function \( f \) on \([a, b]\).

**Proof.** The sufficiency part has been established. To prove the necessity part, let \( F \) be an absolutely continuous function on \([a, b]\). Then \( F \) is differentiable almost everywhere and \( F' \) is integrable on \([a, b]\). Let 
\[ G(x) := F(a) + \int_a^x F'(t) \, dt, \quad x \in [a, b]. \]

By Theorem 2.2, \( G'(x) = F'(x) \) for almost every \( x \in [a, b] \). It follows that \((F - G)'(x) = 0\) for almost every \( x \in [a, b] \). By Theorem 1.2, \( F - G \) is constant. But \( F(a) = G(a) \). Therefore, \( F(x) = G(x) \) for all \( x \in [a, b] \). \( \square \)

§3. Change of Variables for the Lebesgue Integral

Let \( f \) be an absolutely continuous function on \([c, d]\), and let \( u \) be an absolutely continuous function on \([a, b]\) such that \( u([a, b]) \subseteq [c, d] \). Then the composition \( f \circ u \) is not necessarily absolutely continuous. However, we have the following result.

**Theorem 3.1.** Let \( f \) be a Lipschitz continuous function on \([c, d]\), and let \( u \) be an absolutely continuous function on \([a, b]\) such that \( u([a, b]) \subseteq [c, d] \). Then \( f \circ u \) is absolutely continuous. Moreover, 
\[ (f \circ u)'(t) = f'(u(t))u'(t) \quad \text{for almost every } t \in [a, b], \]
where \( f'(u(t))u'(t) \) is interpreted to be zero whenever \( u'(t) = 0 \) (even if \( f \) is not differentiable at \( u(t) \)).

**Proof.** Since \( f \) is a Lipschitz continuous function on \([c, d]\), there exists some \( M > 0 \) such that \( |f(x) - f(y)| \leq M|x - y| \) whenever \( x, y \in [c, d] \). Let \( \varepsilon > 0 \) be given. Since \( u \) is absolutely
continuous on \([a, b]\), there exists some \(\delta > 0\) such that \(\sum_{i=1}^{n} |u(t_i) - u(s_i)| < \varepsilon/M\), whenever 
\([s_i, t_i] : i = 1, \ldots, n\) is a finite collection of mutually disjoint subintervals of \([a, b]\) with 
\(\sum_{i=1}^{n} (t_i - s_i) < \delta\). Consequently,

\[
\sum_{i=1}^{n} |(f \circ u)(t_i) - (f \circ u)(s_i)| = \sum_{i=1}^{n} |f(u(t_i)) - f(u(s_i))| \leq \sum_{i=1}^{n} M|u(t_i) - u(s_i)| < \varepsilon.
\]

This shows that \(f \circ u\) is absolutely continuous on \([a, b]\).

Since both \(u\) and \(f \circ u\) are absolutely continuous on \([a, b]\), there exists a measurable subset \(E\) of \([a, b]\) such that \(\lambda(E) = 0\) and both \(u'(t)\) and \((f \circ u)'(t)\) exist for all \(t \in [a, b] \setminus E\).

Suppose \(t_0 \in [a, b] \setminus E\). If \((f \circ u)'(t_0) = 0\), then for given \(\varepsilon > 0\), there exists some \(h > 0\) such that 
\(|u(t) - u(t_0)| \leq \varepsilon |t - t_0|\) whenever \(t \in (t_0 - h, t_0 + h) \cap [a, b]\). It follows that

\[
|f \circ u(t) - f \circ u(t_0)| \leq M|u(t) - u(t_0)| \leq M\varepsilon|t - t_0|
\]

for all \(t \in (t_0 - h, t_0 + h) \cap [a, b]\). This shows that

\[
(f \circ u)'(t_0) = 0 = f'(u(t_0))u'(t_0).
\]

Now suppose \(t_0 \in [a, b] \setminus E\) and \((f \circ u)'(t_0) \neq 0\). Suppose \(u(t) \neq u(t_0)\). Then we have

\[
\frac{(f \circ u)(t) - (f \circ u)(t_0)}{t - t_0} = \frac{f(u(t)) - f(u(t_0))}{u(t) - u(t_0)} \frac{u(t) - u(t_0)}{t - t_0}.
\]

Since \(u'(t_0)\) and \((f \circ u)'(t_0)\) exist, we obtain

\[
\lim_{t \to t_0} \frac{(f \circ u)(t) - (f \circ u)(t_0)}{t - t_0} = (f \circ u)'(t_0) \quad \text{and} \quad \lim_{t \to t_0} \frac{u(t) - u(t_0)}{t - t_0} = u'(t_0) \neq 0.
\]

Consequently,

\[
\lim_{t \to t_0} \frac{f(u(t)) - f(u(t_0))}{u(t) - u(t_0)} = \frac{(f \circ u)'(t_0)}{u'(t_0)}.
\]

Let \(r := (f \circ u)'(t_0)/u'(t_0)\). For given \(\varepsilon > 0\), there exists some \(\delta > 0\) such that

\[
r - \varepsilon < \frac{f(u(t)) - f(u(t_0))}{u(t) - u(t_0)} < r + \varepsilon \quad \forall \ t \in (t_0 - \delta, t_0 + \delta) \cap [a, b].
\]

Since \((f \circ u)'(t_0) \neq 0\), there exists some \(\eta > 0\) such that any \(x \in (u(t_0) - \eta, u(t_0) + \eta) \cap [c, d]\) can be expressed as \(x = u(t)\) for some \(t \in (t_0 - \delta, t_0 + \delta) \cap [a, b]\). Therefore,

\[
r - \varepsilon < \frac{f(x) - f(u(t_0))}{x - u(t_0)} < r + \varepsilon \quad \forall \ x \in (u(t_0) - \eta, u(t_0) + \eta) \cap [c, d].
\]

This shows that \(f'(u(t_0))\) exists and \(f'(u(t_0)) = r = (f \circ u)'(t_0)/u'(t_0)\), as desired. □
**Theorem 3.2.** Let $g$ be a bounded and measurable function on $[c, d]$, and let $u$ be an absolutely continuous function on $[a, b]$ such that $u([a, b]) \subseteq [c, d]$. Then $(g \circ u)u'$ is integrable on $[a, b]$. Moreover, for any $\alpha, \beta \in [a, b],
\int_{u(\alpha)}^{u(\beta)} g(x) \, dx = \int_{\alpha}^{\beta} g(u(t))u'(t) \, dt.

**Proof.** Let

$$F(x) := \int_{c}^{x} g(t) \, dt, \quad x \in [c, d].$$

Since $g$ is bounded, $F$ is Lipschitz continuous. Moreover, $F'(x) = g(x)$ for almost every $x \in [a, b]$. By Theorem 3.1, $F \circ u$ is absolutely continuous on $[a, b]$ and, for almost every $t \in [a, b]$, $(F \circ u)'(t) = g(u(t))u'(t)$. Suppose $\alpha, \beta \in [a, b]$ and $\alpha < \beta$. By Theorem 2.3, we have

$$(F \circ u)(\beta) - (F \circ u)(\alpha) = F(u(\beta)) - F(u(\alpha)) = \int_{u(\alpha)}^{u(\beta)} F'(x) \, dx = \int_{u(\alpha)}^{u(\beta)} g(x) \, dx.$$

On the other hand,

$$(F \circ u)(\beta) - (F \circ u)(\alpha) = \int_{\alpha}^{\beta} (F \circ u)'(t) \, dt = \int_{\alpha}^{\beta} g(u(t))u'(t) \, dt.$$

This proves the desired formula for change of variables. \qed

**Theorem 3.3.** Let $g$ be an integrable function on $[c, d]$, and let $u$ be an absolutely continuous function on $[a, b]$ such that $u([a, b]) \subseteq [c, d]$. If $(g \circ u)u'$ is integrable on $[a, b]$, then

$$\int_{u(\alpha)}^{u(\beta)} g(x) \, dx = \int_{\alpha}^{\beta} g(u(t))u'(t) \, dt, \quad \alpha, \beta \in [a, b].$$

Moreover, $(g \circ u)u'$ is integrable if, in addition, $u$ is monotone.

**Proof.** Suppose that $g$ is integrable on $[a, b]$. Without loss of any generality, we may assume $g \geq 0$. For $n = 1, 2, \ldots$, let $g_n$ be the function defined by

$$g_n(x) := \begin{cases} 
  g(x) & \text{if } 0 \leq g(x) \leq n, \\
  0 & \text{if } g(x) > n.
\end{cases}$$

Then $g_n \leq g_{n+1}$ for all $n \in \mathbb{N}$. Suppose $\alpha, \beta \in [a, b]$ and $\alpha < \beta$. By Theorem 3.2 we have

$$\int_{u(\alpha)}^{u(\beta)} g_n(x) \, dx = \int_{\alpha}^{\beta} g_n(u(t))u'(t) \, dt.$$
If $u$ is monotone, then $u'(t) \geq 0$ for almost every $t \in [a, b]$. Letting $n \to \infty$ in the above equation, by the monotone convergence theorem we obtain

$$
\int_{u(\alpha)}^{u(\beta)} g(x) \, dx = \int_{\alpha}^{\beta} g(u(t))u'(t) \, dt.
$$

Since $g$ is integrable on $[c, d]$, it follows from the above equation that $(g \circ u)u'$ is integrable on $[a, b]$. More generally, we assume that $(g \circ u)u'$ is integrable on $[a, b]$ but $u$ is not necessarily monotone. Then $|g_n(u(t))u'(t)| \leq g(u(t))|u'(t)|$ for all $n \in \mathbb{N}$ and almost every $t \in [a, b]$. Thus, an application of the Lebesgue dominated convergence theorem gives the desired formula for change of variables. \qed

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