Chapter 2. Functions of Bounded Variation

§1. Monotone Functions

Let I be an interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is said to be **increasing** (**strictly increasing**) if $f(x) \le f(y)$ ($f(x) < f(y)$) whenever $x, y \in I$ and $x < y$. A function $f : I \to \mathbb{R}$ is said to be **decreasing** (**strictly decreasing**) if $f(x) \ge f(y)$ ($f(x) > f(y)$) whenever $x, y \in I$ and $x < y$. A function $f : I \to \mathbb{R}$ is said to be **monotone** if f is either increasing or decreasing.

In this section we will show that a monotone function is differentiable almost everywhere. For this purpose, we first establish Vitali covering lemma.

Let E be a subset of $\mathbb{R}$, and let Γ be a collection of closed intervals in $\mathbb{R}$. We say that Γ covers E in the sense of Vitali if for each $\delta > 0$ and each $x \in E$, there exists an interval $I \in \Gamma$ such that $x \in I$ and $\ell(I) < \delta$.

Theorem 1.1. *Let E be a subset of $\mathbb{R}$ with $\lambda^*(E) < \infty$ and Γ a collection of closed intervals that cover E in the sense of Vitali. Then, for given $\varepsilon > 0$, there exists a finite disjoint collection $\{I_1, \ldots, I_N\}$ of intervals in Γ such that*

$$\lambda^*\big(E \setminus \cup_{n=1}^N I_n\big) < \varepsilon.$$

Proof. Let G be an open set containing E such that $\lambda(G) < \infty$. Since Γ is a Vitali covering of E, we may assume that each $I \in \Gamma$ is contained in G.

We choose a sequence $(I_n)_{n=1,2,\ldots}$ of *disjoint* intervals from Γ recursively as follows. Let I_1 be any interval in Γ. Suppose $I_1, \ldots, I_n$ have been chosen. Let k_n be the supremum of the lengths of the intervals in Γ that do not meet any of the intervals $I_1, \ldots, I_n$. Choose I_{n+1} such that $\ell(I_{n+1}) > k_n/2$ and I_{n+1} is disjoint from $I_1, \ldots, I_n$. We have

$$\sum_{n=1}^{\infty} \ell(I_n) \le \lambda(G) < \infty.$$

Hence, $\lim_{n\to\infty} k_n = 0$. Moreover, for given $\varepsilon > 0$, we can find an integer $N > 0$ such that

$$\sum_{n=N+1}^{\infty} \ell(I_n) < \frac{\varepsilon}{5}.$$

Let $R := E \setminus \cup_{n=1}^N I_n$. The theorem will be proved if we can show that $\lambda^*(R) < \varepsilon$. For this purpose, let

$$J_n := I_n + 2\ell(I_n)[-1, 1], \quad n = 1, 2, \ldots.$$

Then $\lambda^*(R) < \varepsilon$ if we can prove $R \subseteq \cup_{n=N+1}^{\infty} J_n$.

To prove $R \subseteq \cup_{n=N+1}^{\infty} J_n$, let $x \in R = E \setminus \cup_{n=1}^{N} I_n$. Since Γ covers E in the sense of Vitali, we can find an interval $I \in \Gamma$ such that $x \in I$ and $I \subset G \setminus \cup_{n=1}^{N} I_n$. Then $I \cap I_n \neq \emptyset$ for some $n \in \mathbb{N}$, for otherwise we would have $\ell(I) \leq k_n$ for all $n \in \mathbb{N}$, which contradicts the fact that $\lim_{n\to\infty} k_n = 0$. Let n_0 be the smallest integer such that $I \cap I_{n_0} \neq \emptyset$. Then $n_0 > N$ and $\ell(I) \leq 2\ell(I_{n_0})$. It follows that $I \subseteq J_{n_0}$, as desired. $\square$

Let f be a function from an interval I to $\mathbb{R}$. For $x \in I$, we define the four derivatives of f at x in the following way:

$$D^+ f(x) := \limsup_{h\to 0+} \frac{f(x+h) - f(x)}{h} \quad \text{and} \quad D^- f(x) := \limsup_{h\to 0-} \frac{f(x+h) - f(x)}{h},$$

$$D_+ f(x) := \liminf_{h\to 0+} \frac{f(x+h) - f(x)}{h} \quad \text{and} \quad D_- f(x) := \liminf_{h\to 0-} \frac{f(x+h) - f(x)}{h}.$$

Suppose f is a real-valued function defined on $[a, b]$, where $a, b \in \mathbb{R}$ and $a < b$. Let

$$A := \{x \in [a, b] : D^+ f(x) > D_- f(x)\} \quad \text{and} \quad B := \{x \in [a, b] : D^- f(x) > D_+ f(x)\}.$$

It is easily seen that f is differentiable at each point $x \in [a, b] \setminus (A \cup B)$.

Theorem 1.2. *An increasing real-valued function f on an interval $[a, b]$ is differentiable almost everywhere. The derivative f' is measurable and*

$$\int_a^b f'(x)\, dx \leq f(b) - f(a).$$

Proof. The existence of f' is proved by showing that $\lambda(A) = 0$ and $\lambda(B) = 0$. We carry out the proof for A. The proof for B is similar.

For each pair of rational numbers s and t with $s > t$, let

$$A_{s,t} := \{x \in [a, b] : D^+ f(x) > s > t > D_- f(x)\}.$$

Then $A = \cup_{s>t} A_{s,t}$, so it suffices to prove that $\lambda^*(A_{s,t}) = 0$ for all $s, t \in \mathbb{Q}$ with $s > t$. Let $a := \lambda^*(A_{s,t})$. For given $\varepsilon > 0$, there exists an open set O such that $A_{s,t} \subset O$ and $\lambda(O) < a + \varepsilon$.

In light of the definition of $D_- f(x)$, for each $x \in A_{s,t}$ there exists an arbitrary small interval $[x - h, x]$ contained in O such that $f(x) - f(x - h) < th$. The collection of such intervals covers $A_{s,t}$ in the sense of Vitali. By Theorem 1.1, there exists a finite disjoint collection of such intervals $\{I_1, \ldots, I_M\}$ such that

$$\lambda^*\big(A_{s,t} \setminus \cup_{j=1}^{M} I_j\big) < \varepsilon.$$

If $I_j = [x_j - h_j, x_j]$ for $j = 1, \ldots, M$, we have

$$\sum_{j=1}^{M}[f(x_j) - f(x_j - h_j)] < t\sum_{j=1}^{M} h_j < t\lambda(O) < t(a+\varepsilon).$$

Let $G := A_{s,t} \cap \big(\cup_{j=1}^{M}(x_j - h_j, x_j)\big)$. In light of the definition of $D^+f(x)$, for each $y \in G$ there exists an arbitrary small interval $[y, y+k]$ contained in some I_j such that $f(y+k) - f(y) > sk$. By Theorem 1.1, there exists a finite disjoint collection of such intervals $\{J_1, \ldots, J_K\}$ such that

$$\lambda^*\big(G \setminus \cup_{i=1}^{K} J_i\big) < \varepsilon.$$

It follows that $\lambda^*(\cup_{i=1}^{K} J_i) > \lambda^*(G) - \varepsilon$. But $A_{s,t} \setminus G = A_{s,t} \setminus \cup_{j=1}^{M}(x_j - h_j, x_j)$. Hence,

$$\lambda^*(A_{s,t}) \le \lambda^*(G) + \lambda^*(A_{s,t} \setminus G) = \lambda^*(G) + \lambda^*\big(A_{s,t} \setminus \cup_{j=1}^{M} I_j\big) < \lambda^*(G) + \varepsilon.$$

Consequently,

$$\lambda^*\big(\cup_{i=1}^{K} J_i\big) > \lambda^*(G) - \varepsilon > \lambda^*(A_{s,t}) - 2\varepsilon = a - 2\varepsilon.$$

Suppose $J_i = [y_i, y_i + k_i]$, $i = 1, \ldots, K$. Each J_i was so chosen to be contained in some interval I_j. If we some over those i for which $J_i \subseteq I_j$ we find that

$$\sum_{J_i \subseteq I_j} [f(y_i + k_i) - f(y_i)] \le f(x_j) - f(x_j - h_j),$$

because f is an increasing function. Therefore,

$$s(a - 2\varepsilon) < s\sum_{i=1}^{K} k_i < \sum_{i=1}^{K}[f(y_i + k_i) - f(y_i)] \le \sum_{j=1}^{M}[f(x_j) - f(x_j - h_j)] < t(a+\varepsilon).$$

Thus, $s(a - 2\varepsilon) \le t(a + \varepsilon)$ for every $\varepsilon > 0$. It follows that $sa \le ta$. But $s > t$. So we must have $a = 0$, as desired.

We have shown that the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists for almost every $x \in [a, b]$. Define $g(x)$ to be the value of this limit if it exists and 0 otherwise. Set $f(x) := f(b)$ for $x > b$ and define

$$g_n(x) := n\big[f(x + 1/n) - f(x)\big], \quad a \le x \le b.$$

Then each g_n is nonnegative because f is increasing and $(g_n)_{n=1,2,\ldots}$ converges to f' almost everywhere. Moreover,

$$\int_a^b g_n(x)\,dx = n\Big[\int_b^{b+1/n} f(x)\,dx - \int_a^{a+1/n} f(x)\,dx\Big] \le f(b) - f(a).$$

Hence, by Fatou's lemma we obtain

$$\int_a^b f'(x)\,dx \le \liminf_{n\to\infty} \int_a^b g_n(x)\,dx \le f(b) - f(a).$$

This completes the proof of the theorem. □

§2. The Cantor Function

The Cantor set is a subset of the interval $[0,1]$ constructed as follows. Let $I_{1,0} := [0,1]$ and $J_{1,0} = (1/3, 2/3)$. For $n \ge 2$ and $0 \le j \le 2^{n-1} - 1$, we may express j uniquely as $j = t_1 2^{n-2} + t_2 2^{n-3} + \cdots + t_{n-1}$, where $t_1, t_2, \ldots, t_{n-1} \in \{0,1\}$. Set

$$I_{n,j} := \left[\sum_{k=1}^{n-1} \frac{2t_k}{3^k}, \sum_{k=1}^{n-1} \frac{2t_k}{3^k} + \frac{1}{3^{n-1}}\right] \quad \text{and} \quad J_{n,j} := \left(\sum_{k=1}^{n-1} \frac{2t_k}{3^k} + \frac{1}{3^n}, \sum_{k=1}^{n-1} \frac{2t_k}{3^k} + \frac{2}{3^n}\right).$$

It is easily seen that $I_{n,j}$ is the disjoint union $I_{n+1,2j} \cup J_{n,j} \cup I_{n+1,2j+1}$. For $n \in \mathbb{N}$, let

$$F_n := \cup_{j=0}^{2^{n-1}-1} I_{n,j} \quad \text{and} \quad G_n := \cup_{j=0}^{2^{n-1}-1} J_{n,j}.$$

Then $G_n \subset F_n$ and $F_{n+1} = F_n \setminus G_n$ for all $n \in \mathbb{N}$. It follows that $F_{n+1} = [0,1] \setminus \cup_{k=1}^{n} G_k$. Consequently, $G_m \cap G_n = \emptyset$ for $m \ne n$. The Cantor set is defined to be

$$C := \cap_{n=1}^{\infty} F_n = [0,1] \setminus \cup_{n=1}^{\infty} G_n.$$

Theorem 2.1. *The Cantor set C has the following properties:*
(a) *C is compact;*
(b) *$\lambda(C) = 0$;*
(c) *$x \in C$ if and only if there exist $t_k \in \{0,1\}$ for $k \in \mathbb{N}$ such that $x = \sum_{k=1}^{\infty} 2t_k/3^k$;*
(d) *there exists a one-to-one and onto mapping from $\{0,1\}^{\mathbb{N}}$ to C.*

Proof. (a) The Cantor set C is a closed subset of $[0,1]$. Hence, C is compact.
(b) Since $G_m \cap G_n = \emptyset$ for $m \ne n$, we have

$$\lambda(C) = \lambda([0,1]) - \sum_{n=1}^{\infty} \lambda(G_n) = 1 - \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 0.$$

(c) Suppose $x \in [0,1]$. Then $x = \sum_{k=1}^{\infty} s_k/3^k$, where $s_k \in \{0,1,2\}$ for $k \in \mathbb{N}$. Let $x_0 := 0$ and $x_n := \sum_{k=1}^{n} s_k/3^k$ for $n \in \mathbb{N}$. If $s_k \in \{0,2\}$ for all $k \in \mathbb{N}$, then

$$x \in [x_{n-1}, x_{n-1} + 1/3^n] \cup [x_{n-1} + 2/3^n, x_{n-1} + 1/3^{n-1}] \subseteq F_{n+1}.$$

Consequently, $x \in \cap_{n=1}^{\infty} F_{n+1} = C$. Now suppose $x \in C$. If $s_m = 1$ for some $m \in \mathbb{N}$, then there exists some $n \in \mathbb{N}$ such that $s_n = 1$ and $s_k \in \{0,2\}$ for all $k < n$. It follows that $x \in [x_{n-1} + 1/3^n, x_{n-1} + 2/3^n]$. But $(x_{n-1} + 1/3^n, x_{n-1} + 2/3^n) \subseteq G_n$. Therefore, $x = x_{n-1} + 1/3^n$ or $x = x_{n-1} + 2/3^n$. In the former case, we have $x = \sum_{k=1}^{n-1} s_k/3^k + \sum_{k=n+1}^{\infty} 2/3^k$. In the latter case, we have $x = \sum_{k=1}^{n-1} s_k/3^k + 2/3^n$. This verifies our assertion.

(d) Let φ be the mapping from $\{0,1\}^{\mathbb{N}}$ to C that sends $(t_k)_{k=1,2,\ldots}$ to $\sum_{k=1}^{\infty} 2t_k/3^k$. By (c), the mapping φ is onto. In order to prove that φ is one-to-one, let $(s_k)_{k=1,2,\ldots}$ and $(t_k)_{k=1,2,\ldots}$ be two different elements in $\{0,1\}^{\mathbb{N}}$. Then there exists some $n \in \mathbb{N}$ such that $s_n \neq t_n$ and $s_k = t_k$ for all $k < n$. Without loss of any generality, we may assume that $s_n < t_n$, *i.e.*, $s_n = 0$ and $t_n = 1$. Then we have

$$\sum_{k=1}^{\infty} \frac{2s_k}{3^k} \leq \sum_{k=1}^{n-1} \frac{2s_k}{3^k} + \frac{1}{3^n} < \sum_{k=1}^{n-1} \frac{2t_k}{3^k} + \frac{2}{3^n} \leq \sum_{k=1}^{\infty} \frac{2t_k}{3^k}.$$

This shows that φ is a one-to-one mapping. □

We are in a position to define the Cantor-Lebesgue function f on $[0,1]$. For each $x = \sum_{k=1}^{\infty} 2t_k/3^k \in C$, where $t_k \in \{0,1\}$ for $k \in \mathbb{N}$, define

$$f(x) := \sum_{k=1}^{\infty} \frac{t_k}{2^k}.$$

For $x \in [0,1] \setminus C$, we have $x \in J_{n,j}$ for some $n \in \mathbb{N}$ and $j \in \{0, \ldots, 2^{n-1} - 1\}$. Suppose $J_{n,j} = (c_{n,j} + 1/3^n, c_{n,j} + 2/3^n)$, where $c_{n,j} = \sum_{k=1}^{n-1} 2t_k/3^k$ with $t_k \in \{0,1\}$, $1 \leq k \leq n-1$. Then both $c_{n,j} + 1/3^n$ and $c_{n,j} + 2/3^n$ belong to C. For $x \in J_{n,j}$, we define

$$f(x) := f(c_{n,j} + 1/3^n) = f(c_{n,j} + 2/3^n) = \sum_{k=1}^{n-1} \frac{t_k}{2^k} + \frac{1}{2^n}.$$

Theorem 2.2. *The Cantor-Lebesgue function f is a continuous and increasing function from $[0,1]$ onto $[0,1]$. Moreover, $f'(x) = 0$ for each $x \in [0,1] \setminus C$.*

Proof. Suppose $x, y \in [0,1]$ and $x < y$. Then there exist $a, b \in C$ such that $a \leq x < y \leq b$, $f(a) = f(x)$ and $f(y) = f(b)$. For $a, b \in C$ and $a < b$, we have $f(a) \leq f(b)$. It follows that $f(x) \leq f(y)$. This shows that f is increasing.

Let us show that f is continuous on $[0,1]$. Suppose $a \in [0,1]$. If $a \in [0,1] \setminus C$, then a lies in some open interval $J_{n,j}$ and f is a constant on $J_{n,j}$. Hence, f is continuous at a. Suppose $a \in C$. For $0 < \varepsilon < 1$, let n be the least integer such that $0 < 1/2^n < \varepsilon$. Let $\delta := 1/3^n$. Suppose $x \in C \cap (a - \delta, a + \delta)$. Then a and x have the following ternary expansions:

$$a = \sum_{k=1}^{\infty} \frac{2t_k}{3^k} \quad \text{and} \quad x = \sum_{k=1}^{\infty} \frac{2s_k}{3^k},$$

where $s_k, t_k \in \{0,1\}$ for all $k \in \mathbb{N}$. Since $|x - a| < \delta = 1/3^n$, $s_k = t_k$ for $k = 1, \ldots, n$. It follows that

$$|f(x) - f(a)| = \left| \sum_{k=n+1}^{\infty} \frac{s_k - t_k}{2^k} \right| \le \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^n} < \varepsilon.$$

Now suppose $y \in [0,1] \cap (a - \delta, a + \delta)$. Then there exists some $x \in C \cap (a - \delta, a + \delta)$ such that $f(y) = f(x)$. Consequently, $|f(y) - f(a)| < \varepsilon$. This shows that f is continuous on $[0,1]$.

Note that $f(0) = 0$ and $f(1) = 1$. Since f is a continuous and increasing function on $[0,1]$, we have $f([0,1]) = [0,1]$.

Finally, if $x \in [0,1] \setminus C$, then $x \in J_{n,j}$ for some $n \in \mathbb{N}$ and $j \in \{0, \ldots, 2^{n-1} - 1\}$. Since f is a constant on the open interval $J_{n,j}$, we have $f'(x) = 0$. $\square$

§3. Functions of Bounded Variation

Let f be a function from a closed interval $I = [a,b]$ to $\mathbb{R}$. The **total variation** of f over I, denoted $\bigvee_a^b f$, is the quantity

$$\bigvee_a^b f := \sup \left\{ \sum_{i=1}^{n} \bigl| f(x_i) - f(x_{i-1}) \bigr| \right\},$$

where the supremum is taken over all possible partitions $a = x_0 < x_1 < \cdots < x_n = b$ of I. If $\bigvee_a^b f$ is finite, we say that f is of **bounded variation**.

We see that a function of bounded variation is bounded. A monotone function on a closed interval is of bounded variation. Let f and g be two functions of bounded variation on a closed interval I. Then $f + g$, $f - g$, and fg are of bounded variation on I. If, in addition, there exists a constant $C > 0$ such that $|g(x)| \ge C$ for all $x \in I$, then f/g is of bounded variation on I.

If $a < c < b$, then it is easily verified that

$$\bigvee_a^b f = \bigvee_a^c f + \bigvee_c^b f.$$

Theorem 3.1. *A function $f : [a, b] \to \mathbb{R}$ is of bounded variation if and only if f is the difference of two monotone functions on $[a, b]$. Consequently, if f is a function of bounded variation on $[a, b]$, then $f'(x)$ exists for almost all $x \in [a, b]$.*

Proof. Any monotone function on $[a, b]$ is of bounded variation, so the sufficiency part is obvious.

To prove the necessity, we let f be a function of bounded variation on $[a, b]$ and set

$$g(x) := \bigvee_a^x f \quad \text{for } a \le x \le b.$$

Then for $a \le x < y \le b$, we have

$$g(y) - g(x) = \bigvee_x^y f \ge 0.$$

Hence, g is an increasing function on $[a, b]$. Moreover,

$$g(y) - g(x) \ge |f(y) - f(x)| \ge f(y) - f(x).$$

Let $h := g - f$. Then $h(y) \ge h(x)$ for $a \le x < y \le b$. Thus, h is also an increasing function on $[a, b]$. This shows that $f = g - h$ is the difference of two increasing functions on $[a, b]$. By Theorem 1.2, $g'(x)$ and $h'(x)$ exist for almost all $x \in [a, b]$. Consequently, $f'(x)$ exists for almost all $x \in [a, b]$. $\square$

Theorem 3.2. *Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation, and let $g(x) := \vee_a^x f$ for $a \le x \le b$. If f is continuous at some point $x_0 \in [a, b]$, then g is also continuous at x_0. Consequently, a continuous function of bounded variation is the difference of two continuous monotone functions.*

Proof. Suppose that f is continuous at $x_0 \in [a, b]$. For given $\varepsilon > 0$, there exists a partition $x_0 < x_1 < \ldots < x_n = b$ of $[x_0, b]$ such that $|f(x_1) - f(x_0)| < \varepsilon$ and

$$\bigvee_{x_0}^b f \le \sum_{i=1}^n \bigl|f(x_i) - f(x_{i-1})\bigr| + \varepsilon.$$

It follows that

$$\bigvee_{x_0}^b f \le |f(x_1) - f(x_0)| + \sum_{i=2}^n \bigl|f(x_i) - f(x_{i-1})\bigr| + \varepsilon < 2\varepsilon + \bigvee_{x_1}^b f.$$

Hence,

$$g(x_1) - g(x_0) = \bigvee_{x_0}^{x_1} f = \bigvee_{x_0}^{b} f - \bigvee_{x_1}^{b} f < 2\varepsilon.$$

But g is an increasing function. Thus, $0 \le g(x) - g(x_0) < 2\varepsilon$ for all $x \in (x_0, x_1)$. This shows that g is right-continuous at x_0. A similar argument shows that g is left-continuous at x_0.

If f is a continuous function of bounded variation on $[a, b]$, then both g and $h := g - f$ are continuous increasing functions. Therefore, $f = g - h$ is the difference of two continuous increasing functions. □

A function $f : [a, b] \to \mathbb{R}$ is said to be **Lipschitz continuous** if there exists a positive constant M such that $|f(x) - f(y)| \le M|x - y|$ for all $x, y \in [a, b]$. Clearly, a Lipschitz continuous function on $[a, b]$ is of bounded variation.

If f is a continuous function from $[a, b]$ to $\mathbb{R}$, and if f is differentiable on (a, b) with $|f'(x)| \le M$ for all $x \in (a, b)$, then $|f(x) - f(y)| \le M|x - y|$ for all $x, y \in [a, b]$, by the mean-value theorem. Hence, in this case, f is a Lipschitz continuous function on $[a, b]$.

§4. Curve Length

A curve in the Euclidean plane $\mathbb{R}^2$ is represented by a continuous mapping γ from a closed interval $[a, b]$ to $\mathbb{R}^2$. Suppose $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ for $t \in [a, b]$, where γ_1 and γ_2 are real-valued continuous functions on $[a, b]$. Let $P := \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a, b]$, that is, $a = t_0 < t_1 < \cdots < t_n = b$. Let

$$L(\gamma, P) := \sum_{j=1}^{n} \sqrt{[\gamma_1(t_j) - \gamma_1(t_{j-1})]^2 + [\gamma_2(t_j) - \gamma_2(t_{j-1})]^2}.$$

The **length** of the curve γ is defined to be

$$L(\gamma) := \sup\{L(\gamma, P) : P \text{ is a partition of } [a, b]\}.$$

If $L(\gamma) < \infty$, then γ is said to be **rectifiable**.

Theorem 4.1. *A continuous curve $\gamma = (\gamma_1, \gamma_2) : [a, b] \to \mathbb{R}^2$ is rectifiable if and only if γ_1 and γ_2 are of bounded variation on $[a, b]$.*

Proof. Suppose that γ_1 and γ_2 are real-valued functions of bounded variation on $[a, b]$. Let $P := \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a, b]$. Then

$$\begin{aligned} L(\gamma, P) &= \sum_{j=1}^{n} \sqrt{[\gamma_1(t_j) - \gamma_1(t_{j-1})]^2 + [\gamma_2(t_j) - \gamma_2(t_{j-1})]^2} \\ &\le \sum_{j=1}^{n} \big|\gamma_1(t_j) - \gamma_1(t_{j-1})\big| + \sum_{j=1}^{n} \big|\gamma_2(t_j) - \gamma_1(t_{j-1})\big| \le \vee_a^b \gamma_1 + \vee_a^b \gamma_2. \end{aligned}$$

Hence, γ is rectifiable.

Conversely, suppose that γ is rectifiable. Then for any partition $P := \{t_0, t_1, \ldots, t_n\}$ of $[a, b]$ we have

$$\sum_{j=1}^{n} \left|\gamma_k(t_j) - \gamma_k(t_{j-1})\right| \leq L(\gamma), \quad k = 1, 2.$$

Consequently, γ_1 and γ_2 are of bounded variation on $[a, b]$. $\square$