MATH 418 Assignment #4

1. For $a > 0$, let f_a be the function on $[0,1]$ defined by $f_a(0) := 0$ and

$$f_a(x) := x^a \cos\Big(\frac{\pi}{x^2}\Big), \quad 0 < x \le 1.$$

(a) Prove that for $a > 2$, f_a is absolutely continuous on $[0,1]$.

Proof. For $0 < x \le 1$, we have

$$f_a'(x) = ax^{a-1}\cos(\pi/x^2) + 2\pi x^{a-3}\sin(\pi/x^2).$$

For $a > 2$, f_a' is Lebesgue integrable on the interval $(0,1)$. Suppose $0 < c < 1$. Then f_a' is continuous on $[c,1]$. Hence,

$$f_a(x) = f_a(c) + \int_c^x f_a'(t)\,dt, \quad c \le x \le 1.$$

Letting $c \to 0+$ in the above equation, we obtain

$$f_a(x) = \int_a^x f_a'(t)\,dt, \quad 0 < x \le 1.$$

This shows that f_a is absolutely continuous on $[0,1]$.

(b) Prove that for $a \le 2$, f_a is not absolutely continuous on $[0,1]$.

Proof. For $N = 1, 2, \ldots$, we have

$$\bigvee_0^1 f_a \ge \sum_{j=1}^N \big|f_a(1/\sqrt{j}) - f_a(1/\sqrt{j+1})\big| \ge \sum_{j=1}^N \frac{1}{j^{a/2}}.$$

If $a \le 2$, then $\lim_{N\to\infty} \sum_{j=1}^N 1/j^{a/2} = \infty$. Therefore, for $a \le 2$, f_a is not absolutely continuous on $[0,1]$.

2. Suppose f and g are absolutely continuous functions on $[a,b]$. Prove that fg is also absolutely continuous and

$$\int_a^b \big[f(x)g'(x) + f'(x)g(x)\big]\,dx = f(b)g(b) - f(a)g(a).$$

Proof. Since both f and g are absolutely continuous on $[a,b]$, there exists a constant $M > 0$ such that $|f(x)| \le M$ and $|g(x)| \le M$ for all $x \in [a,b]$. Moreover, for given $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$\sum_{i=1}^n |f(y_i) - f(x_i)| < \frac{\varepsilon}{2M} \quad \text{and} \quad \sum_{i=1}^n |g(y_i) - g(x_i)| < \frac{\varepsilon}{2M},$$

whenever $\{[x_i, y_i] : i = 1, \ldots, n\}$ is a finite collection of mutually disjoint subintervals of $[a, b]$ with $\sum_{i=1}^{n} |y_i - x_i| < \delta$. Consequently, for such a collection of subintervals we have

$$\sum_{i=1}^{n} |f(y_i)g(y_i) - f(x_i)g(x_i)| \le \sum_{i=1}^{n} |f(y_i)[g(y_i) - g(x_i)]| + \sum_{i=1}^{n} |g(x_i)[f(y_i) - f(x_i)]| < \varepsilon.$$

This shows that fg is absolutely continuous. Furthermore, we have

$$f(b)g(b) - f(a)g(a) = \int_a^b (fg)'(x)\,dx = \int_a^b \big[f(x)g'(x) + f'(x)g(x)\big]\,dx.$$

3. Let f be a continuous function on $[a, b]$. Suppose that there is a partition

$$a = a_0 < a_1 < \cdots < a_m = b$$

such that the derivative f' is continuous on each interval (a_{k-1}, a_k), $k = 1, \ldots, m$. If f' is Lebesgue integrable on (a, b), prove that f is absolutely continuous on $[a, b]$.

Proof. The proof proceeds by induction on m. Suppose $m = 1$ and $a_0 < c < x < a_1$. Since f' is continuous on $[c, x]$, we have

$$f(x) = f(c) + \int_c^x f'(t)\,dt.$$

But f is continuous on $[a_0, a_1]$ and f' is Lebesgue integrable on (a_0, a_1). Letting $c \to a+$ in the above equation, we obtain

$$f(x) = f(a) + \int_a^x f'(t)\,dt.$$

The above equation is also valid for $x = a_1$ if we take the limit $x \to a_1-$.
Now suppose $m > 1$ and our assertion has been verified for $m - 1$. By the induction hypothesis,

$$f(x) = f(a) + \int_a^x f'(t)\,dt \quad \text{for } a < x \le a_{m-1}.$$

If $a_{m-1} < x \le a_m$, then by what has been proved we have

$$f(x) = f(a_{m-1}) + \int_{a_{m-1}}^x f'(t)\,dt.$$

Consequently,

$$f(x) = f(a) + \int_a^x f'(t)\,dt \quad \forall\, x \in [a, b].$$

This shows that f is absolutely continuous on $[a, b]$.

4. Let g be a Lebesgue integrable function on $[0,1]$, and let

$$f(x) := \int_0^x g(t)\,dt, \quad x \in [0,1].$$

(a) Show that for each closed interval $[a,b] \subseteq [0,1]$,

$$\lambda\big(f([a,b])\big) \le \int_a^b |g(t)|\,dt.$$

Proof. Let $m := \min\{f(x) : x \in [a,b]\}$ and $M := \max\{f(x) : x \in [a,b]\}$. Then there exist points $c, d \in [a,b]$ such that $f(c) = m$ and $f(d) = M$. Hence,

$$\lambda\big(f([a,b])\big) = M - m = \int_c^d g(t)\,dt \le \int_a^b |g(t)|\,dt.$$

(b) Prove that if E is a subset of $[0,1]$ with $\lambda(E) = 0$, then $\lambda(f(E)) = 0$.

Proof. Let E be a subset of $[0,1]$ such that $\lambda(E) = 0$. We wish to show $\lambda(f(E)) < \varepsilon$ for all $\varepsilon > 0$. Let $\varepsilon > 0$ be given. There exists some $\delta > 0$ such that $\int_A |g|\,d\lambda < \varepsilon$ whenever $\lambda(A) < \delta$. Without loss of any generality, we may assume that $E \subset (0,1)$. Since $\lambda(E) = 0$, there exists an open set G such that $E \subset G \subset (0,1)$ and $\lambda(G) < \delta$. As an open set, G can be expressed as $\cup_{j\in J}(a_j, b_j)$, where J is a countable set and $(a_j, b_j) \cap (a_k, b_k) = \emptyset$ for $j, k \in J$ with $j \ne k$. It follows that

$$f(E) \subseteq f(G) \subseteq \cup_{j\in J} f((a_j, b_j)).$$

Consequently, by part (a) we have

$$\lambda(f(E)) \le \sum_{j\in J} \lambda(f(a_j, b_j)) \le \sum_{j\in J} \int_{a_j}^{b_j} |g(t)|\,dt = \int_G |g|\,d\lambda < \varepsilon.$$

This shows $\lambda(f(E)) = 0$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(z) := \begin{cases} \sqrt{|z|} & \text{if } |z| < 1, \\ 1 & \text{if } |z| \ge 1, \end{cases}$$

and let $u : [-1,1] \to \mathbb{R}$ be the function given by

$$u(x) := \begin{cases} x^2 \sin^2(1/x) & \text{if } x \ne 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Prove that f and u are absolutely continuous functions but their composition $f \circ u$ is not absolutely continuous.

Proof. We have $f'(z) = 0$ for $|z| > 1$, $f'(z) = -1/(2\sqrt{|z|})$ for $z \in (-1, 0)$ and $f'(z) = 1/(2\sqrt{z})$ for $z \in (0, 1)$. Thus, f' is continuous and Lebesgue integrable on $(-1, 0)$ and $(0, 1)$. Moreover, f is continuous on $\mathbb{R}$. Hence, by the results in Problem 3, f is absolutely continuous.

Clearly, u is continuous. For $x \neq 0$, we have $u'(x) = 2x \sin^2(1/x) - \sin(2/x)$. Hence u' is continuous and Lebesgue integrable on $(-1, 0)$ and $(0, 1)$. By the results in Problem 3, u is absolutely continuous.

We have $f \circ u(x) = 0$ for $x = 0$ and $f \circ u(x) = x \sin(1/x)$ for $x \in [-1, 0) \cup (0, 1]$. For $j \in \mathbb{N}$ let $x_j := 1/((j + 1/2)\pi)$. Then we have

$$\bigvee_{-1}^{1} f \circ u \geq \sum_{j=1}^{n} |(f \circ u)(x_j) - (f \circ u)(x_{j+1})| \geq \sum_{j=1}^{n} \frac{1}{(j + 1/2)\pi}.$$

The above inequalities are valid for all $n \in \mathbb{N}$. But

$$\lim_{n \to \infty} \sum_{j=1}^{n} \frac{1}{(j + 1/2)\pi} = \infty.$$

Hence, $f \circ u$ is not of bounded variation on $[-1, 1]$.