Chapter 6. Integration

§1. Integrals of Nonnegative Functions

Let \((X, S, \mu)\) be a measure space. We denote by \(L^+\) the set of all measurable functions from \(X\) to \([0, \infty]\).

Let \(\phi\) be a simple function in \(L^+\). Suppose \(f(X) = \{a_1, \ldots, a_m\}\). Then \(\phi\) has the standard representation \(\phi = \sum_{j=1}^m a_j \chi_{E_j}\), where \(E_j := f^{-1}(\{a_j\})\), \(j = 1, \ldots, m\). We define the integral of \(\phi\) with respect to \(\mu\) to be

\[
\int_X \phi \, d\mu := \sum_{j=1}^m a_j \mu(E_j).
\]

For \(c \geq 0\), we have \(c\phi = \sum_{j=1}^m (ca_j) \chi_{E_j}\). It follows that

\[
\int_X c\phi \, d\mu = \sum_{j=1}^m (ca_j) \mu(E_j) = c \int_X \phi \, d\mu.
\]

Suppose that \(\phi = \sum_{j=1}^m a_j \chi_{E_j}\), where \(a_j \geq 0\) for \(j = 1, \ldots, m\) and \(E_1, \ldots, E_m\) are mutually disjoint measurable sets such that \(\bigcup_{j=1}^m E_j = X\). Suppose that \(\psi = \sum_{k=1}^n b_k \chi_{F_k}\), where \(b_k \geq 0\) for \(k = 1, \ldots, n\) and \(F_1, \ldots, F_n\) are mutually disjoint measurable sets such that \(\bigcup_{k=1}^n F_k = X\). Then we have

\[
\phi = \sum_{j=1}^m \sum_{k=1}^n a_j \chi_{E_j \cap F_k} \quad \text{and} \quad \psi = \sum_{k=1}^n \sum_{j=1}^m b_k \chi_{F_k \cap E_j}.
\]

If \(\phi \leq \psi\), then \(a_j \leq b_k\), provided \(E_j \cap F_k \neq \emptyset\). Consequently,

\[
\sum_{j=1}^m a_j \mu(E_j) = \sum_{j=1}^m \sum_{k=1}^n a_j \mu(E_j \cap F_k) \leq \sum_{j=1}^m \sum_{k=1}^n b_k \mu(E_j \cap F_k) = \sum_{k=1}^n b_k \mu(F_k).
\]

In particular, if \(\phi = \psi\), then

\[
\sum_{j=1}^m a_j \mu(E_j) = \sum_{k=1}^n b_k \mu(F_k).
\]

In general, if \(\phi \leq \psi\), then

\[
\int_X \phi \, d\mu = \sum_{j=1}^m a_j \mu(E_j) \leq \sum_{k=1}^n b_k \mu(F_k) = \int_X \psi \, d\mu.
\]
Furthermore,

$$\int_X (\phi + \psi) \, d\mu = \sum_{j=1}^m \sum_{k=1}^n (a_j + b_k) \mu(E_j \cap F_k) = \int_X \phi \, d\mu + \int_X \psi \, d\mu.$$ 

Suppose $\phi$ is a simple function in $L^+$ and $\sum_{j=1}^m a_j \chi_{E_j}$ is its standard representation. If $A \in S$, then $\phi \chi_A = \sum_{j=1}^m a_j \chi_{E_j \cap A}$. We define

$$\int_A \phi \, d\mu := \int_X \phi \chi_A \, d\mu = \sum_{j=1}^m a_j \mu(E_j \cap A).$$

The set function $\nu$ from $S$ to $[0, \infty]$ given by $\nu(A) := \int_A \phi \, d\mu$ is a measure. Indeed, $\nu(\emptyset) = 0$. To prove $\sigma$-additivity of $\nu$, we let $(A_k)_{k=1,2,\ldots}$ be a sequence of mutually disjoint measurable sets and $A := \cup_{k=1}^\infty A_k$. Then we have

$$\nu(A) = \int_A \phi \, d\mu = \sum_{j=1}^m a_j \mu(E_j \cap A) = \sum_{j=1}^m \sum_{k=1}^\infty a_j \mu(E_j \cap A_k) = \sum_{k=1}^\infty \sum_{j=1}^m a_j \mu(E_j \cap A_k) = \sum_{k=1}^\infty \int_{A_k} \phi \, d\mu = \sum_{k=1}^\infty \nu(A_k).$$

The **Lebesgue integral** of a function $f \in L^+$ is defined to be

$$\int_X f \, d\mu = \sup \left\{ \int_X \phi \, d\mu : 0 \leq \phi \leq f, \phi \text{ simple} \right\}.$$

When $f$ is a simple function, this definition agrees with the previous one. If $f, g \in L^+$ and $f \leq g$, then it follows from the definition that $\int_X f \, d\mu \leq \int_X g \, d\mu$. If $f \in L^+$ and $A \in S$, we define $\int_A f \, d\mu := \int_X (f \chi_A) \, d\mu$. Since $f \chi_A \leq f$, we have $\int_A f \, d\mu \leq \int_X f \, d\mu$.

**Theorem 1.1.** (The Monotone Convergence Theorem) Let $(f_n)_{n=1,2,\ldots}$ be an increasing sequence of functions in $L^+$. If $f = \lim_{n \to \infty} f_n$, then

$$\int_X f \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu.$$

**Proof.** Since each $f_n$ is measurable and $f = \lim_{n \to \infty} f_n$, the function $f$ is measurable. Moreover, $f_n \leq f_{n+1} \leq f$ for every $n \in \mathbb{N}$. Hence,

$$\int_X f_n \, d\mu \leq \int_X f_{n+1} \, d\mu \leq \int_X f \, d\mu \quad \forall n \in \mathbb{N}.$$
Thus, there exists an element \( \alpha \in [0, \infty] \) such that

\[
\alpha = \lim_{n \to \infty} \int_X f_n \, d\mu \leq \int_X f \, d\mu.
\]

Let \( \phi \) be any simple function on \((X, S)\) such that \( 0 \leq \phi \leq f \). Fix \( c \in (0, 1) \) for the time being and define

\[
E_n := \{ x \in X : f_n(x) \geq c\phi(x) \}, \quad n \in \mathbb{N}.
\]

Each \( E_n \) is measurable. For \( n \in \mathbb{N} \), we have

\[
\int_X f_n \, d\mu \geq \int_{E_n} f_n \, d\mu \geq \int_{E_n} (c\phi) \, d\mu = c \int_{E_n} \phi \, d\mu.
\]

But \( E_n \subseteq E_{n+1} \) for \( n \in \mathbb{N} \) and \( X = \bigcup_{n=1}^{\infty} E_n \). Hence, \( \lim_{n \to \infty} \int_{E_n} \phi \, d\mu = \int_X \phi \, d\mu \). It follows that

\[
\alpha = \lim_{n \to \infty} \int_X f_n \, d\mu \geq \lim_{n \to \infty} c \int_{E_n} \phi \, d\mu = c \int_X \phi \, d\mu.
\]

Thus, \( \alpha \geq c \int_X \phi \, d\mu \) for every \( c \in (0, 1) \). Consequently, \( \alpha \geq \int_X \phi \, d\mu \) for every simple function satisfying \( 0 \leq \phi \leq f \). Therefore, \( \alpha \geq \int_X f \, d\mu \) and

\[
\int_X f \, d\mu \leq \alpha = \lim_{n \to \infty} \int_X f_n \, d\mu \leq \int_X f \, d\mu.
\]

This completes the proof. \(\square\)

The monotone convergence theorem is still valid if the conditions \( f_n(x) \leq f_{n+1}(x) \) for all \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} f_n(x) = f(x) \) hold for almost every \( x \in X \). Indeed, under the stated conditions, there exists a null set \( N \) such that for every \( x \in X \setminus N \), \( f_n(x) \leq f_{n+1}(x) \) and \( \lim_{n \to \infty} f_n(x) = f(x) \). Note that \( \int_N g \, d\mu = 0 \) for all \( g \in L^+ \). With \( E := X \setminus N \) we have

\[
\int_X f \, d\mu = \int_X f_{\chi_E} \, d\mu = \lim_{n \to \infty} \int_X f_n \chi_E \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu.
\]

The hypothesis that the sequence \((f_n)_{n=1,2,\ldots}\) be increasing almost everywhere is essential for the monotone convergence theorem. For example, let \( \mu \) be the Lebesgue measure on \( X = \mathbb{R} \), and let \( f_n := \chi_{(n,n+1)} \) for \( n \in \mathbb{N} \). Then \((f_n)_{n=1,2,\ldots}\) converges to 0 pointwise, but \( \int_{\mathbb{R}} f_n \, d\mu = 1 \) for every \( n \in \mathbb{N} \).
**Theorem 1.2.** Let \( f, g \in L^+ \) and \( c \geq 0 \). Then

\[
\int_X (cf) \, d\mu = c \int_X f \, d\mu \quad \text{and} \quad \int_X (f + g) \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu.
\]

**Proof.** For \( f, g \in L^+ \), we can find increasing sequences \((f_n)_{n=1}^{\infty}\) and \((g_n)_{n=1}^{\infty}\) of simple functions in \( L^+ \) such that \( f = \lim_{n \to \infty} f_n \) and \( g = \lim_{n \to \infty} g_n \). By the monotone convergence theorem we have

\[
\int_X (cf) \, d\mu = \lim_{n \to \infty} \int_X (cf_n) \, d\mu = \lim_{n \to \infty} c \int_X f_n \, d\mu = c \int_X f \, d\mu
\]

and

\[
\int_X (f + g) \, d\mu = \lim_{n \to \infty} \int_X (f_n + g_n) \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu + \lim_{n \to \infty} \int_X g_n \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu.
\]

**Theorem 1.3.** If \((f_k)_{k=1}^{\infty}\) is a sequence of functions in \( L^+ \) and \( f = \sum_{k=1}^{\infty} f_k \), then

\[
\int_X f \, d\mu = \sum_{k=1}^{\infty} \int_X f_k \, d\mu.
\]

**Proof.** For \( n \in \mathbb{N} \), let \( g_n := \sum_{k=1}^{n} f_k \). Then \((g_n)_{n=1}^{\infty}\) is an increasing sequence of functions in \( L^+ \) such that \( \lim_{n \to \infty} g_n = f \). By the monotone convergence theorem we have

\[
\int_X f \, d\mu = \lim_{n \to \infty} \int_X g_n \, d\mu = \lim_{n \to \infty} \int_X \sum_{k=1}^{n} g_k \, d\mu = \sum_{k=1}^{\infty} \int_X f_k \, d\mu.
\]

**Theorem 1.4.** Let \( f \) be a function in \( L^+ \), and let \( \nu \) be the function from \( S \) to \([0, \infty]\) given by \( \nu(A) := \int_A f \, d\mu \), \( A \in S \). Then \( \nu \) is a measure on \( S \).

**Proof.** Clearly, \( \nu(\emptyset) = 0 \). Suppose that \((E_n)_{n=1}^{\infty}\) is a sequence of disjoint sets in \( S \) and \( E = \cup_{n=1}^{\infty} E_n \). Then we have \( f \chi_E = \sum_{n=1}^{\infty} f \chi_{E_n} \). By Theorem 1.3 we obtain

\[
\int_E f \, d\mu = \int_X f \chi_E \, d\mu = \sum_{n=1}^{\infty} \int_X f \chi_{E_n} \, d\mu = \sum_{n=1}^{\infty} \int_{E_n} f \, d\mu.
\]

This proves that \( \nu \) is \( \sigma \)-additive.

**Theorem 1.5.** Let \((f_n)_{n=1}^{\infty}\) be a sequence in \( L^+ \). Then

\[
\int_X \liminf_{n \to \infty} f_n \, d\mu \leq \liminf_{n \to \infty} \int_X f_n \, d\mu.
\]
Proof. Let \( f := \liminf_{n \to \infty} f_n \) and \( g_k := \inf_{n \geq k} f_n \) for \( k \in \mathbb{N} \). Then \((g_k)_{k=1,2,\ldots}\) is an increasing sequence and \( \lim_{k \to \infty} g_k = f \). By the monotone convergence theorem, we have

\[
\int_X f \, d\mu = \lim_{k \to \infty} \int_X g_k \, d\mu.
\]

For fixed \( k \), \( g_k \leq f_n \) for all \( n \geq k \); hence, \( \int_X g_k \, d\mu \leq \int_X f_n \, d\mu \) for all \( n \geq k \). It follows that

\[
\int_X g_k \, d\mu \leq \inf_{n \geq k} \int_X f_n \, d\mu.
\]

We conclude that

\[
\int_X f \, d\mu = \lim_{k \to \infty} \int_X g_k \, d\mu \leq \lim_{k \to \infty} \inf_{n \geq k} \int_X f_n \, d\mu = \liminf_{n \to \infty} \int_X f_n \, d\mu.
\]

The above theorem is often called Fatou's lemma.

§2. Integrable Functions

Let \((X, S, \mu)\) be a measure space. A measurable function \( f \) from \( X \) to \( \mathbb{R} \) is said to be integrable over a set \( E \in S \) if both \( \int_E f^+ \, d\mu \) and \( \int_E f^- \, d\mu \) are finite. In this case, we define

\[
\int_E f \, d\mu := \int_E f^+ \, d\mu - \int_E f^- \, d\mu.
\]

Clearly, \( f \) is integrable if and only if \( |f| \) is.

**Theorem 2.1.** Let \( f \) be an integrable function on a measure space \((X, S, \mu)\). Then \( f \) is finite almost everywhere. Moreover, if \( \int_X |f| \, d\mu = 0 \), then \( f = 0 \) almost everywhere.

**Proof.** In order to prove the theorem, it suffices to consider the case \( f \geq 0 \). For \( n \in \mathbb{N} \), let \( E_n := \{x \in X : f(x) \geq n\} \). For \( E := \cap_{n=1}^{\infty} E_n \) we have

\[
\int_X f \, d\mu \geq \int_{E_n} f \, d\mu \geq n\mu(E_n) \geq n\mu(E).
\]

It follows that \( \mu(E) \leq \int_X f \, d\mu/n \) for all \( n \in \mathbb{N} \). Since \( 0 \leq \int_X f \, d\mu < \infty \), we conclude that \( \mu(E) = 0 \). If \( x \in X \setminus E \), then \( x \notin E_n \) for some \( n \in \mathbb{N} \), that is, \( f(x) < n \). This shows that \( f(x) < \infty \) for all \( x \in X \setminus E \). In other words, \( f \) is finite almost everywhere.

For the second statement, let \( K_n := \{x \in X : f(x) \geq 1/n\} \) for \( n \in \mathbb{N} \), and let \( K := \cup_{n=1}^{\infty} K_n \). Then we have \( K = \{x \in X : f(x) > 0\} \) and

\[
0 = \int_X f \, d\mu \geq \int_{K_n} f \, d\mu \geq \mu(K_n)/n.
\]
It follows that $\mu(K_n) = 0$ for all $n \in \mathbb{N}$. Hence, $\mu(K) = \lim_{n \to \infty} \mu(K_n) = 0$. Thus, $f = 0$ almost everywhere.

Theorem 2.2. Let $f$ and $g$ be integrable functions on a measure space $(X, S, \mu)$. Then the following statements are true.

1. For each $c \in \mathbb{R}$, $cf$ is integrable and $\int_X (cf) \, d\mu = c \int_X f \, d\mu$.
2. The function $f + g$ is integrable, and $\int_X (f + g) \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu$.
3. If $f \leq g$ almost everywhere, then $\int_X f \, d\mu \leq \int_X g \, d\mu$.
4. If $A = \bigcup_{n=1}^{\infty} A_n$, where $A_n$ $(n \in \mathbb{N})$ are mutually disjoint sets in $S$, then

$$\int_A f \, d\mu = \sum_{n=1}^{\infty} \int_{A_n} f \, d\mu.$$

Proof. (1) If $c \geq 0$, then $(cf)^+ = cf^+$ and $(cf)^- = cf^-$. If $c < 0$, then $(cf)^+ = -cf^-$ and $(cf)^- = -cf^+$. In both cases, $cf$ is integrable and $\int_X (cf) \, d\mu = c \int_X f \, d\mu$.

(2) Since $|f + g| \leq |f| + |g|$, $f + g$ is integrable. Moreover, by Theorem 2.1, both $f$ and $g$ are finite almost everywhere. Hence, there exists a null set $E$ such that both $f(x)$ and $g(x)$ are finite for all $x \in X \setminus E$. Let $h(x) := f(x) + g(x)$ for $x \in X \setminus E$ and $h(x) := 0$ for $x \in E$. On $X \setminus E$ we have $h^+ - h^- = f^+ - f^- + g^+ - g^-$. It follows that $h^+ + f^- + g^- = h^- + f^+ + g^+$. Hence,

$$\int_X h^+ \, d\mu + \int_X f^- \, d\mu + \int_X g^- \, d\mu = \int_X h^- \, d\mu + \int_X f^+ \, d\mu + \int_X g^+ \, d\mu.$$

Consequently, $\int_X (f + g) \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu$.

(3) Let $h := f - g$. Then $h \geq 0$ almost everywhere. Hence, $\int_X h \, d\mu \geq 0$. It follows that $\int_X f \, d\mu \leq \int_X g \, d\mu$.

(4) By Theorem 1.4 we have

$$\int_A f \, d\mu = \int_A f^+ \, d\mu - \int_A f^- \, d\mu = \sum_{n=1}^{\infty} \int_{A_n} f^+ \, d\mu - \sum_{n=1}^{\infty} \int_{A_n} f^- \, d\mu.$$

Since $|f| = f^+ + f^-$ is integrable, we have $\sum_{n=1}^{\infty} \int_{A_n} f^+ \, d\mu + \int_{A_n} f^- \, d\mu < \infty$. Consequently,

$$\sum_{n=1}^{\infty} \int_{A_n} f^+ \, d\mu - \sum_{n=1}^{\infty} \int_{A_n} f^- \, d\mu = \sum_{n=1}^{\infty} \left( \int_{A_n} f^+ \, d\mu - \int_{A_n} f^- \, d\mu \right) = \sum_{n=1}^{\infty} \int_{A_n} f \, d\mu.$$

This completes the proof of the theorem.
Theorem 2.3. (The Lebesgue Dominated Convergence Theorem) Let \((f_n)_{n=1,2,...}\) be a sequence of integrable functions such that the sequence converges to \(f\) a.e. and there exists a nonnegative integrable function \(g\) such that \(|f_n| \leq g\) a.e. for all \(n \in \mathbb{N}\). Then
\[
\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu.
\]

Proof. The functions \(g - f_n\) \((n \in \mathbb{N})\) are nonnegative, and so by Fatou’s lemma,
\[
\int_X (g - f) \, d\mu \leq \liminf_{n \to \infty} \int_X (f - g_n) \, d\mu.
\]
Since \(|f| \leq g\), \(f\) is integrable, and we have
\[
\int_X g \, d\mu - \int_X f \, d\mu \leq \int_X g \, d\mu - \limsup_{n \to \infty} \int_X f_n \, d\mu.
\]
It follows that
\[
\int_X f \, d\mu \geq \limsup_{n \to \infty} \int_X f_n \, d\mu.
\]
Similarly, by considering \(g + f_n\) for \(n \in \mathbb{N}\), we get
\[
\int_X f \, d\mu \leq \liminf_{n \to \infty} \int_X f_n \, d\mu.
\]
This establishes the desired result.

§3. The Riemann and Lebesgue Integrals

We start by reviewing the definition of the Riemann integral. Suppose that \(a, b \in \mathbb{R}\) and \(a < b\). Let \(f\) be a bounded function from the interval \([a, b]\) to \(\mathbb{R}\). A collection of points \(P = \{x_0, x_1, \ldots, x_n\}\) is called a partition of \([a, b]\) if \(a = x_0 < x_1 < \cdots < x_n = b\). The norm of the partition is defined to be
\[
\|P\| := \max\{x_i - x_{i-1} : 1 \leq i \leq n\}.
\]
For \(i = 1, \ldots, n\), let
\[
m_i := \inf\{f(x) : x \in [x_{i-1}, x_i]\} \quad \text{and} \quad M_i := \sup\{f(x) : x \in [x_{i-1}, x_i]\}.
\]
Then the lower sum \(S_*(f, P)\) of \(f\) corresponding to the partition \(P\) is defined by
\[
S_*(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}),
\]
and similarly, the **upper sum** $S^*(f, P)$ of $f$ by

$$S^*(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}).$$

Clearly, $S_*(f, P) \leq S^*(f, P)$ holds for every partition $P$ of $[a, b]$.

Let $P$ and $Q$ be two partitions of $[a, b]$. We say that $P$ is **finer** than $Q$ if $Q \subseteq P$. If this is the case, then

$$S_*(f, Q) \leq S_*(f, P) \quad \text{and} \quad S^*(f, P) \leq S^*(f, Q).$$

To establish the inequalities, it suffices to assume that $P$ has only one more point than $Q$. Suppose that $Q = \{x_0, x_1, \ldots, x_n\}$ and $P = Q \cup \{t\}$. Then $x_{k-1} < t < x_k$ for some $k$, $1 \leq k \leq n$. Set $s_1 := \inf\{f(x) : x_{k-1} \leq x \leq t\}$ and $s_2 := \inf\{f(x) : t \leq x \leq x_k\}$. Observe that $m_k \leq s_1$ and $m_k \leq s_2$. Therefore,

$$S_*(f, Q) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \sum_{i\neq k} m_i(x_i - x_{i-1}) + m_k(x_k - x_{k-1})$$

$$\leq \sum_{i\neq k} m_i(x_i - x_{i-1}) + s_1(t - x_{k-1}) + s_2(x_k - t) = S_*(f, P).$$

The proof for $S^*(f, P) \leq S^*(f, Q)$ is similar. Consequently, for two arbitrary partitions $P$ and $Q$ of $[a, b]$ we have $S_*(f, P) \leq S^*(f, Q)$. Indeed,

$$S_*(f, P) \leq S_*(f, P \cup Q) \leq S^*(f, P \cup Q) \leq S^*(f, Q).$$

The **lower Riemann integral** of $f$ is defined by

$$I_*(f) := \sup\{S_*(f, P) : P \text{ is a partition of } [a, b]\},$$

and the **upper Riemann integral** of $f$ is defined by

$$I^*(f) := \inf\{S^*(f, P) : P \text{ is a partition of } [a, b]\}.$$

Clearly, $I_*(f) \leq I^*(f)$.

A bounded function $f : [a, b] \to \mathbb{R}$ is called **Riemann integrable** if $I_*(f) = I^*(f)$. In this case, the common value is called the **Riemann integral** of $f$ and is denoted by

$$\int_{a}^{b} f(x) \, dx.$$
A Lebesgue measurable function from \([a, b]\) to \(\mathbb{R}\) is said to be **Lebesgue integrable** if \(f\) is integrable with respect to the Lebesgue measure \(\lambda\) on \([a, b]\). In this case, the Lebesgue integral is denoted by

\[
\int_{[a,b]} f \, d\lambda.
\]

**Theorem 3.1.** If \(f : [a, b] \to \mathbb{R}\) is a bounded Riemann integrable function, then \(f\) is Lebesgue integrable and

\[
\int_a^b f(x) \, dx = \int_{[a,b]} f \, d\lambda.
\]

**Proof.** Let \(I := \int_a^b f(x) \, dx\). Since \(f\) is Riemann integrable, for each \(k \in \mathbb{N}\) there exists a partition \(Q_k\) of \([a, b]\) such that

\[
I - 1/k \leq S_*(f, Q_k) \leq S^*(f, Q_k) \leq I + 1/k.
\]

Let \(P_k := Q_1 \cup \cdots \cup Q_k\). Then \(P_k \subseteq P_{k+1}\) for \(k \in \mathbb{N}\) and

\[
I - 1/k \leq S_*(f, Q_k) \leq S_*(f, P_k) \leq S^*(f, P_k) \leq S^*(f, Q_k) \leq I + 1/k.
\]

It follows that \(\lim_{k \to \infty} (S^*(f, P_k) - S_*(f, P_k)) = 0\).

Suppose \(P_k = \{x_{k,j} : 0 \leq j \leq n_k\}\). Let \(m_{k,j} := \inf\{f(x) : x_{k,j-1} \leq x \leq x_{k,j}\}\) and \(M_{k,j} := \sup\{f(x) : x_{k,j-1} \leq x \leq x_{k,j}\}\) for \(1 \leq k \leq n_k\). Define

\[
l_k := \sum_{j=1}^{n_k} m_{k,j} \chi_{[x_{k,j-1}, x_{k,j}]} \quad \text{and} \quad u_k := \sum_{j=1}^{n_k} M_{k,j} \chi_{[x_{k,j-1}, x_{k,j}]}.
\]

Then \((l_k)_{k=1,2,\ldots}\) is an increasing sequence of bounded measurable functions on \([a, b]\), and \((u_k)_{k=1,2,\ldots}\) is a decreasing sequence of bounded measurable functions on \([a, b]\). Moreover,

\[
\int_{[a,b]} l_k \, d\lambda = S_*(f, P_k) \quad \text{and} \quad \int_{[a,b]} u_k \, d\lambda = S^*(f, P_k).
\]

Let \(l\) and \(u\) be the functions defined by \(l(x) := \lim_{k \to \infty} l_k(x)\) and \(u(x) := \lim_{k \to \infty} u_k(x)\), \(x \in [a, b]\). Then \(l\) and \(u\) are Lebesgue measurable. Furthermore, since \(l_k \leq f \leq u_k\) a.e. for every \(k \in \mathbb{N}\), we have \(l \leq f \leq u\) a.e. By the monotone convergence theorem, we obtain

\[
\int_{[a,b]} (u - l) \, d\lambda = \lim_{k \to \infty} \int_{[a,b]} (u_k - l_k) \, d\lambda = \lim_{k \to \infty} (S^*(f, P_k) - S_*(f, P_k)) = 0.
\]

Consequently, \(u - l = 0\) almost everywhere, by Theorem 2.1. It follows that \(f = u\) a.e. This shows that \(f\) is Lebesgue measurable. Finally, invoking the monotone convergence theorem once more, we arrive at the desired conclusion:

\[
\int_{[a,b]} f \, d\lambda = \lim_{k \to \infty} \int_{[a,b]} u_k \, d\lambda = \lim_{k \to \infty} S^*(f, P_k) = \int_a^b f(x) \, dx. \quad \square
\]
Theorem 3.2. A bounded function \( f : [a, b] \to \mathbb{R} \) is Riemann integrable if and only if there exists a subset \( N \) of \([a, b]\) such that \( \lambda(N) = 0 \) and \( f \) is continuous at every point \( x \in [a, b] \setminus N \).

Proof. Suppose that \( f \) is a bounded Riemann integrable function on \([a, b]\). There exists a sequence \((P_k)_{k=1,2,\ldots}\) of partitions of \([a, b]\) such that \( P_k \subset P_{k+1} \) for every \( k \in \mathbb{N} \) and \( \lim_{k\to\infty}(S^*(f, P_k) - S_*(f, P_k)) = 0 \). Let \( u_k \) and \( v_k \) \( (k \in \mathbb{N}) \) be as in the proof of Theorem 3.1. Set \( E := \bigcup_{k=1}^{\infty} P_k \). Then \( \lambda(E) = 0 \). Moreover, there exists a subset \( K \) of \([a, b]\) such that \( \lambda(K) = 0 \) and \( \lim_{n \to \infty}(u_k(x) - l_k(x)) = 0 \) for every \( x \in [a, b] \setminus K \). Let \( N := E \cup K \). Then \( \lambda(N) = 0 \). We claim that \( f \) is continuous at every point \( x \in [a, b] \setminus N \). Let \( x \) be a point in \([a, b] \setminus N \). For any \( \epsilon > 0 \), there exists some \( k \in \mathbb{N} \) such that \( u_k(x) - l_k(x) < \epsilon \).

Since \( x \in [a, b] \setminus P_k \), there exists an open interval \((\alpha, \beta)\) such that \( x \in (\alpha, \beta) \subset [a, b] \setminus P_k \).

For \( y \in (\alpha, \beta) \) we have \( l_k(y) = l_k(x) \) and \( u_k(y) = u_k(x) \). Hence,

\[
l_k(x) \leq f(x) \leq u_k(x) \quad \text{and} \quad l_k(x) = l_k(y) \leq f(y) \leq u_k(y) = u_k(x).
\]

Consequently, \(|f(y) - f(x)| \leq u_k(x) - l_k(x) < \epsilon\) for \( y \in (\alpha, \beta) \). This shows that \( f \) is continuous at every point \( x \in [a, b] \setminus N \).

Now suppose that \( f \) is a bounded function on \([a, b]\), \( N \subset [a, b] \) with \( \lambda(N) = 0 \), and \( f \) is continuous at each point in \([a, b] \setminus N \). Let \((P_k)_{k=1,2,\ldots}\) be a sequence of partitions of \([a, b]\) such that \( \lim_{k\to\infty}\|P_k\| = 0 \). For \( k \in \mathbb{N} \), let \( u_k \) and \( v_k \) be defined as in the proof of Theorem 3.1. Suppose \( x \in [a, b] \setminus N \). For any \( \epsilon > 0 \), since \( f \) is continuous at \( x \), there exists some \( \delta > 0 \) such that \( |f(y) - f(x)| < \epsilon \) whenever \( y \in [a, b] \) and \( |y - x| < \delta \). There exists \( k_0 \in \mathbb{N} \) such that \( \|P_k\| < \delta/2 \) for all \( k \geq k_0 \). Thus,

\[
u_k(x) - l_k(x) \leq \sup\{|f(y) - f(x)| : |y - x| < \delta\} < \epsilon
\]

whenever \( k > k_0 \). Hence, for every point \( x \in [a, b] \setminus N \),

\[
\lim_{k \to \infty} (u_k(x) - v_k(x)) = 0.
\]

By the Lebesgue dominated convergence theorem,

\[
\lim_{k \to \infty} (S^*(f, P_k) - S_*(f, P_k)) = \lim_{k \to \infty} \int_{[a,b]} (u_k - l_k) \, d\lambda = 0.
\]

This shows that \( f \) is Riemann integrable. \( \square \)