Chapter 5. Measurable Functions

§1. Measurable Functions

Let X be a nonempty set, and let S be a σ-algebra of subsets of X. Then (X, S) is a measurable space. A subset E of X is said to be measurable if $E \in S$.

In this chapter, we will consider functions from X to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ is the set of **extended real numbers**. For simplicity, we write ∞ for $+\infty$. The set $\overline{\mathbb{R}}$ is an ordered set:

$$-\infty < x < \infty \quad \text{for} \quad x \in \mathbb{R}.$$

A function f from X to $\overline{\mathbb{R}}$ is called **measurable** if, for each $a \in \mathbb{R}$, $\{x \in X : f(x) > a\}$ is a measurable set.

Theorem 1.1. *Let f be a function from a measurable space (X, S) to $\overline{\mathbb{R}}$. Then the following conditions are equivalent:*

(1) f is measurable;
(2) for each $a \in \mathbb{R}$, $f^{-1}([a, \infty])$ is measurable;
(3) for each $a \in \mathbb{R}$, $f^{-1}([-\infty, a))$ is measurable;
(4) for each $a \in \mathbb{R}$, $f^{-1}([-\infty, a])$ is measurable;
(5) the sets $f^{-1}(\{-\infty\})$ and $f^{-1}(\{\infty\})$ are measurable, and for each pair of real numbers a and b with $a < b$, $f^{-1}((a, b))$ is measurable.

Proof. (1) $\Rightarrow$ (2): $f^{-1}([a, \infty]) = \cap_{n=1}^{\infty} f^{-1}((a - 1/n, \infty])$.

(2) $\Rightarrow$ (3): $f^{-1}([-\infty, a)) = X \setminus f^{-1}([a, \infty])$.

(3) $\Rightarrow$ (4): $f^{-1}([-\infty, a]) = \cap_{n=1}^{\infty} f^{-1}([-\infty, a + 1/n))$.

(4) $\Rightarrow$ (5): If (4) is true, then $f^{-1}([-\infty, b)) = \cup_{n=1}^{\infty} f^{-1}([-\infty, b - 1/n])$ is measurable. It follows that the set $f^{-1}((a, b)) = f^{-1}([-\infty, b)) \setminus f^{-1}([-\infty, a])$ is measurable. Moreover, $f^{-1}(\{-\infty\}) = \cap_{n=1}^{\infty} f^{-1}([-\infty, -n])$ and $f^{-1}(\{\infty\}) = X \setminus \cup_{n=1}^{\infty} f^{-1}([-\infty, n])$ are measurable.

(5) $\Rightarrow$ (1): $f^{-1}((a, \infty]) = f^{-1}(\{\infty\}) \cup \big(\cup_{n=1}^{\infty} f^{-1}((a, a + n))\big)$. $\square$

As a consequence of the above theorem, we see that a continuous function from $\mathbb{R}$ to $\mathbb{R}$ is Lebesgue measurable.

For two functions f and g from a nonempty set X to $\overline{\mathbb{R}}$, define

$$f \vee g(x) := \max\{f(x), g(x)\} \quad \text{and} \quad f \wedge g(x) := \min\{f(x), g(x)\}, \quad x \in X.$$

Furthermore, let $f^+ := f \vee 0$ and $f^- := (-f) \vee 0$. We call f^+ the **positive part** of f and f^- the **negative part** of f, respectively.

Theorem 1.2. *If f and g are measurable functions, then the three sets*

$$\{x \in X : f(x) > g(x)\}, \quad \{x \in X : f(x) \geq g(x)\} \quad \text{and} \;\; \{x \in X : f(x) = g(x)\}$$

are all measurable. Moreover, the functions $f \vee g$, $f \wedge g$, f^+, f^-, and $|f|$ are all measurable.

Proof. The set

$$\{x \in X : f(x) > g(x)\} = \cup_{r \in \mathbf{Q}} \big(\{x \in X : f(x) > r\} \cap \{x \in X : g(x) < r\}\big)$$

is measurable, since it is a countable union of measurable sets. It follows that the set

$$\{x \in X : f(x) \geq g(x)\} = X \setminus \{x \in X : g(x) > f(x)\}$$

is measurable. Consequently, the set

$$\{x \in X : f(x) = g(x)\} = \{x \in X : f(x) \geq g(x)\} \cap \{x \in X : g(x) \geq f(x)\}$$

is measurable.

For $a \in \mathbb{R}$, we have

$$\{x \in X : f \vee g(x) > a\} = \{x \in X : f(x) > a\} \cup \{x \in X : g(x) > a\}$$

and

$$\{x \in X : f \wedge g(x) > a\} = \{x \in X : f(x) > a\} \cap \{x \in X : g(x) > a\}.$$

Hence, $f \vee g$ and $f \wedge g$ are measurable. In particular, f^+ and f^- are measurable. For $a \leq 0$, we have $\{x \in X : |f(x)| \geq a\} = X$. For $a > 0$ we have

$$\{x \in X : |f(x)| \geq a\} = \{x \in X : f(x) \geq a\} \cup \{x \in X : f(x) \leq -a\}.$$

Hence, $|f|$ is measurable. □

The arithmetical operations on $\mathbb{R}$ can be partially extended to $\overline{\mathbb{R}}$. For $x \in \mathbb{R}$, define

$$x + (\pm\infty) = (\pm\infty) + x = \pm\infty \quad \text{and} \quad x - (\pm\infty) = (\mp\infty) - x = \mp\infty.$$

Moreover, we define

$$(+\infty) + (+\infty) = (+\infty) - (-\infty) = +\infty \quad \text{and} \quad (-\infty) + (-\infty) = (-\infty) - (+\infty) = -\infty.$$

But

$$(+\infty)+(-\infty), \quad (+\infty)-(+\infty), \quad \text{and } \ (-\infty)-(-\infty)$$

are not defined. Thus, for $x, y \in \overline{\mathbb{R}}$, $x+y$ is well defined if and only if $\{x, y\} \neq \{-\infty, +\infty\}$. Multiplication on $\overline{\mathbb{R}}$ is defined as follows:

$$(\pm\infty)(\pm\infty) = +\infty, \quad (\pm\infty)(\mp\infty) = -\infty,$$

and for $x \in \mathbb{R}$,

$$x(\pm\infty) = (\pm\infty)x = \begin{cases} \pm\infty, & \text{for } x > 0, \\ 0, & \text{for } x = 0, \\ \mp\infty, & \text{for } x < 0. \end{cases}$$

Let f and g be two functions from a nonempty set X to $\overline{\mathbb{R}}$. The product fg is defined to be the function that maps $x \in X$ to $f(x)g(x) \in \overline{\mathbb{R}}$. If $\{f(x), g(x)\} \neq \{-\infty, +\infty\}$ for every $x \in X$, then the sum $f+g$ is defined to be the function that maps $x \in X$ to $f(x)+g(x) \in \overline{\mathbb{R}}$. The difference $f-g$ is defined to be $f+(-g)$. It is well defined if $\{f(x), -g(x)\} \neq \{-\infty, +\infty\}$ for every $x \in X$. For example, for a function $f : X \to \overline{\mathbb{R}}$, $f^+(x) = \infty$ implies $f^-(x) = 0$, and $f^-(x) = \infty$ implies $f^+(x) = 0$. Hence, $f^+ + f^-$ and $f^+ - f^-$ are well defined. In fact, $|f| = f^+ + f^-$ and $f = f^+ - f^-$.

Theorem 1.3. *Let f and g be two measurable functions from a measurable space (X, S) to $\overline{\mathbb{R}}$. Then $f+g$ is a measurable function, provided $\{f(x), g(x)\} \neq \{-\infty, +\infty\}$ for every $x \in X$. Moreover, fg is also a measurable function.*

Proof. For $a \in \mathbb{R}$, the function $a - g$ is measurable. Moreover, we have

$$\{x \in X : f(x) + g(x) > a\} = \{x \in X : f(x) > a - g(x)\}.$$

By Theorem 1.2. the set $\{x \in X : f(x) > a - g(x)\}$ is measurable. Hence, for every $a \in \mathbb{R}$, $\{x \in X : f(x) + g(x) > a\}$ is measurable. This shows that $f+g$ is measurable.

Suppose that $f \geq 0$ and $g \geq 0$. Let $\mathbb{Q}_+ := \{r \in \mathbb{Q} : r > 0\}$. For $a < 0$, we have $\{x \in X : f(x)g(x) > a\} = X$. For $a \geq 0$, we have

$$\{x \in X : f(x)g(x) > a\} = \cup_{r \in \mathbb{Q}_+} \big(\{x \in X : f(x) > r\} \cap \{x \in X : g(x) > a/r\}\big).$$

As a countable union of measurable sets, the set $\{x \in X : f(x)g(x) > a\}$ is measurable. This shows that fg is measurable whenever $f \geq 0$ and $g \geq 0$.

Now let us consider the general case. Write $f = f^+ - f^-$ and $g = g^+ - g^-$. By Theorem 1.2, f^+, f^-, g^+, and g^- are all measurable. It can be easily verifies that, for every $x \in X$, $f(x)g(x) = h_1(x) - h_2(x)$, where $h_1 := f^+g^+ + f^-g^-$ and $h_2 := f^+g^- + f^-g^+$. By what has been proved, h_1 and h_2 are measurable. Consequently, $fg = h_1 - h_2$ is measurable. $\square$

§2. Limits of Measurable Functions

Let A be a nonempty subset of $\overline{\mathbb{R}}$. Then $\sup A$ and $\inf A$ exist as elements of $\overline{\mathbb{R}}$. For a sequence $(x_n)_{n=1,2,\ldots}$ of elements in $\overline{\mathbb{R}}$, define

$$s_k := \sup\{x_n : n \ge k\} \quad \text{and} \quad t_k := \inf\{x_n : n \ge k\}, \quad k \in \mathbb{N}.$$

Furthermore, define

$$\limsup_{n\to\infty} x_n := \inf\{s_k : k \in \mathbb{N}\} \quad \text{and} \quad \liminf_{n\to\infty} x_n := \sup\{s_k : k \in \mathbb{N}\}.$$

If $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = x$, then we write $\lim_{n\to\infty} x_n = x$. Let $(x_n)_{n=1,2,\ldots}$ and $(y_n)_{n=1,2,\ldots}$ be two sequences of real numbers such that $\lim_{n\to\infty} x_n = x \in \overline{\mathbb{R}}$ and $\lim_{n\to\infty} y_n = y \in \overline{\mathbb{R}}$. If $x + y$ is well defined, then $\lim_{n\to\infty}(x_n + y_n) = x + y$.

Theorem 2.1. *Let $(f_n)_{n=1,2,\ldots}$ be a sequence of measurable functions from a measurable space (X, S) to $\overline{\mathbb{R}}$. For $x \in X$, define*

$$g_1(x) := \sup\{f_n(x) : n \in \mathbb{N}\}, \quad g_2(x) := \inf\{f_n(x) : n \in \mathbb{N}\},$$
$$g_3(x) := \limsup_{n\to\infty} f_n(x), \quad g_4(x) := \liminf_{n\to\infty} f_n(x).$$

Then the functions g_1, g_2, g_3, and g_4 are all measurable. Moreover, if $f(x) = \lim_{n\to\infty} f_n(x)$ exists for every $x \in X$, then f is measurable.

Proof. For each $a \in \mathbb{R}$, the sets

$$g_1^{-1}((a,\infty]) = \cup_{n=1}^{\infty} f_n^{-1}((a,\infty]) \quad \text{and} \quad g_2^{-1}([-\infty,a)) = \cup_{n=1}^{\infty} f_n^{-1}([-\infty,a))$$

are measurable, so g_1 and g_2 are measurable.

Furthermore, for $k \in \mathbb{N}$ and $x \in X$, let

$$u_k(x) := \sup\{f_n(x) : n \ge k\} \quad \text{and} \quad v_k(x) := \inf\{f_n(x) : n \ge k\}.$$

Then $g_3(x) = \inf\{u_k(x) : k \in \mathbb{N}\}$ and $g_4(x) = \sup\{u_k(x) : k \in \mathbb{N}\}$ for $x \in X$. By what has been proved, u_k and v_k are measurable for every $k \in \mathbb{N}$. Consequently, both g_3 and g_4 are measurable.

Finally, if $f(x) = \lim_{n\to\infty} f_n(x)$ exists for every $x \in X$, then $f = g_3 = g_4$, so f is measurable. □

Let $(f_n)_{n=1,2,\ldots}$ be a sequence of functions from a nonempty set X to $\mathbb{R}$. We say that the sequence **converges uniformly** to a function $f : X \to \mathbb{R}$ if, for any $\varepsilon > 0$, there exists a positive integer N such that

$$|f_n(x) - f(x)| < \varepsilon \quad \text{whenever } n \ge N \text{ and } x \in X.$$

Let X be a nonempty set. The **characteristic function** of a subset E of X is the function given by

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \in E^c. \end{cases}$$

A function f from X to $\overline{\mathbb{R}}$ is said to be **simple** if its range $f(X)$ is a finite set. Suppose $f(X) = \{a_1, \ldots, a_k\}$. Then f can be represented as

$$f = \sum_{j=1}^{k} a_j \chi_{A_j},$$

where $A_j := f^{-1}(\{a_j\})$, $j = 1, \ldots, k$.

Theorem 2.2. *Let f be a function from a nonempty set X to $\overline{\mathbb{R}}$. Then the following statements are true.*

(1) There exists a sequence $(f_n)_{n=1,2,\ldots}$ of simple functions from the set X to $\mathbb{R}$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in X$.

(2) If f is nonnegative, the sequence can be so chosen that $0 \le f_n \le f_{n+1} \le f$ for all $n \in \mathbb{N}$.

(3) If f is bounded, the sequence can be so chosen that it converges uniformly to f on X.

(4) If (X, S) is a measurable space and f is measurable on (X, S), then each f_n $(n \in \mathbb{N})$ can be chosen to be measurable.

Proof. For $n \in \mathbb{N}$, let $I_n := [n, \infty]$ and

$$I_{n,k} := \Big[\frac{k-1}{2^n}, \frac{k}{2^n}\Big), \quad k = 1, 2, \ldots, n2^n.$$

Then $[0, \infty]$ is the union $(\cup_{k=1}^{n2^n} I_{n,k}) \cup I_n$ of mutually disjoint sets.

Assume first that $f \ge 0$. For $n \in \mathbb{N}$, let $A_n := f^{-1}(I_n)$ and $A_{n,k} := f^{-1}(I_{n,k})$ for $1 \le k \le n2^n$. Then X is the union $(\cup_{k=1}^{n2^n} A_{n,k}) \cup A_n$ of mutually disjoint sets. We define

$$f_n := \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \chi_{A_{n,k}} + n\chi_{A_n}.$$

Then f_n is a simple function from X to $\mathbb{R}$. If f is measurable, then the sets A_n and $A_{n,k}$ are measurable. Hence, f_n is measurable for each $n \in \mathbb{N}$.

If $x \in X$ and $f(x) = \infty$, then $x \in A_n$ for all $n \in \mathbb{N}$. Hence, $f_n(x) = n$ for all $n \in \mathbb{N}$. It follows that $\lim_{n\to\infty} f_n(x) = \infty = f(x)$. Suppose $f(x) < \infty$. If $n \in \mathbb{N}$ and $f(x) \ge n$, then $x \in A_n$ and $f_n(x) = n \le f(x)$. If $f(x) < n$, then $x \in A_n^c$ and hence $x \in A_{n,k}$ for some

$k \in \{1, \ldots, n2^n\}$. It is easily seen that $f_n(x) = (k-1)/2^n$ and $(k-1)/2^n \le f(x) < k/2^n$. Therefore,

$$0 \le f(x) - f_n(x) < \frac{1}{2^n}.$$

The above inequalities are valid for all $n > f(x)$. Consequently, $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in X$. Moreover, if f is bounded, then there exists some $n_0 \in \mathbb{N}$ such that $f(x) < n_0$ for all $x \in X$. Thus, the above argument shows that the sequence $(f_n)_{n=1,2,\ldots}$ converges uniformly to f on X.

For $f \ge 0$, we have proved $0 \le f_n \le f$ for all $n \in \mathbb{N}$. Let us show $f_n \le f_{n+1}$ for all $n \in \mathbb{N}$. Suppose $x \in X$. If $f(x) \ge n$, then $f_{n+1}(x) \ge n \ge f_n(x)$. If $f(x) < n$, then $x \in A_{n,k}$ for some $k \in \{1, \ldots, n2^n\}$. Since $I_{n,k} = I_{n+1,2k-1} \cup I_{n+1,2k}$, we have $A_{n,k} = A_{n+1,2k-1} \cup A_{n+1,2k}$. Thus, $x \in A_{n+1,2k-1} \cup A_{n+1,2k}$, and hence

$$f_{n+1}(x) \ge \frac{2k-1-1}{2^{n+1}} = \frac{k-1}{2^n} = f_n(x).$$

This completes the proof for the case $f \ge 0$.

For the general case, we write $f = f^+ + f^-$, where $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$. By what has been proved, we can find two sequences $(g_n)_{n=1,2,\ldots}$ and $(h_n)_{n=1,2,\ldots}$ of simple functions from X to $\mathbb{R}$ such that $\lim_{n\to\infty} g_n(x) = f^+(x)$ and $\lim_{n\to\infty} h_n(x) = f^-(x)$ for each $x \in X$. Let $f_n(x) := g_n(x) - h_n(x)$ for $x \in X$. Since $f^+(x) - f^-(x)$ is well defined for each $x \in X$, we have $\lim_{n\to\infty} f_n(x) = f^+(x) - f^-(x) = f(x)$. If f is bounded, then f^+ and f^- are bounded. Hence, $(g_n)_{n=1,2,\ldots}$ can be so chosen that it converges to f^+ uniformly on X, and $(h_n)_{n=1,2,\ldots}$ can be so chosen that it converges to f^- uniformly on X. Consequently, $(f_n)_{n=1,2,\ldots}$ converges to f uniformly on X. Finally, if f is measurable, then f^+ and f^- are measurable. Hence, for $n \in \mathbb{N}$, g_n and h_n can be chosen to be measurable. Thus, f_n is measurable for every $n \in \mathbb{N}$. This completes the proof of the theorem. □

§3. Modes of Convergence

Let (X, S, μ) be a measure space. A measurable set N is called a μ-null set if $\mu(N) = 0$. A property is said to hold μ-**almost everywhere** if there is a null set N such that it holds for all $x \in X \setminus N$. If the measure μ is clear from the context, the reference to μ will be omitted.

A measure space (X, S, μ) is called **complete** if every subset of a μ-null set is measurable. If this is the case, then the measure μ is said to be complete. For example, the Lebesgue measure on the real line is complete.

Theorem 3.1. *Let (X, S, μ) be a complete measure space. Then the following statements are true.*

(1) For two functions f and g from X to $\overline{\mathbb{R}}$, if f is measurable and $g = f$ almost everywhere, then g is measurable.

(2) If $(f_n)_{n=1,2,\ldots}$ is a sequence of measurable functions from X to $\overline{\mathbb{R}}$, and if f is a function from X to $\overline{\mathbb{R}}$ such that $f(x) = \lim_{n\to\infty} f_n(x)$ holds for almost every $x \in X$, then f is measurable.

Proof. (1) Let $E := \{x \in X : f(x) = g(x)\}$. Then E^c is a null set. For $a \in \mathbb{R}$, we have

$$\{x \in X : g(x) > a\} = \{x \in E : g(x) > a\} \cup \{x \in E^c : g(x) > a\}.$$

Moreover,

$$\{x \in E : g(x) > a\} = \{x \in E : f(x) > a\} = \{x \in X : f(x) > a\} \setminus \{x \in E^c : f(x) > a\}.$$

Since μ is a complete measure and E^c is a null set, both sets $\{x \in E^c : g(x) > a\}$ and $\{x \in E^c : f(x) > a\}$ are measurable. Furthermore, the set $\{x \in X : f(x) > a\}$ is measurable, because f is measurable. This shows that $\{x \in X : g(x) > a\}$ is measurable for every $a \in \mathbb{R}$. Therefore, g is a measurable function.

(2) There exists a null set N such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in X \setminus N$. Let $E := X \setminus N$, $g := f\chi_E$, and $g_n := f_n\chi_E$ for each $n \in \mathbb{N}$. Since each f_n ($n \in \mathbb{N}$) is a measurable function and E is a measurable set, each g_n ($n \in \mathbb{N}$) is measurable. But $g(x) = \lim_{n\to\infty} g_n(x)$ for all $x \in X$. Hence, by Theorem 2.1, g is measurable. Furthermore, $f = g$ for $x \in E = X \setminus N$. By part (1), f is a measurable function. □

Let $(f_n)_{n=1,2,\ldots}$ be a sequence of functions from a nonempty set X to $\mathbb{R}$, and let f be a function from X to $\mathbb{R}$. We say that $(f_n)_{n=1,2,\ldots}$ **converges to f pointwise** if $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in X$. We say that $(f_n)_{n=1,2,\ldots}$ **converges to f uniformly** if, for any $\varepsilon > 0$, there exists a positive integer N such that

$$|f_n(x) - f(x)| < \varepsilon \quad \text{whenever } n \geq N \text{ and } x \in X.$$

Let (X, S, μ) be a measure space. Suppose that $(f_n)_{n=1,2,\ldots}$ is a sequence of functions from X to $\mathbb{R}$ and f is a function from X to $\mathbb{R}$. We say that $(f_n)_{n=1,2,\ldots}$ **converges to f almost everywhere** if there is a null set N such that $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in X \setminus N$. Suppose in addition that f_n ($n \in \mathbb{N}$) and f are measurable. We say that $(f_n)_{n=1,2,\ldots}$ **converges to f in measure** if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu\big(\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}\big) = 0.$$

Clearly, pointwise convergence implies almost everywhere convergence. Moreover, uniform convergence implies both pointwise convergence and convergence in measure.

In the following examples, all the functions are from $\mathbb{R}$ to $\mathbb{R}$. The measure on $\mathbb{R}$ is understood to be the Lebesgue measure.

Example 1. For $n \in \mathbb{N}$, let $f_n := \chi_{(0,n)}/n$. The sequence $(f_n)_{n=1,2,\ldots}$ converges to 0 uniformly.

Example 2. For $n \in \mathbb{N}$, let $g_n := \chi_{(n,n+1)}$. The sequence $(g_n)_{n=1,2,\ldots}$ converges to 0 pointwise, but not uniformly. Also, the sequence does not converge to 0 in measure.

Example 3. For $n \in \mathbb{N}$, let $u_n := n\chi_{[0,1/n]}$. The sequence $(u_n)_{n=1,2,\ldots}$ converges to 0 almost everywhere, but not pointwise. Also, the sequence converges to 0 in measure.

Example 4. Each $n \in \mathbb{N}$ can be uniquely written as $u = 2^k + j$, where $k \in \mathbb{N}_0$ and $0 \le j < 2^k$. Let v_n be the characteristic function of the interval $[j/2^k, (j+1)/2^k]$. Then $(v_n)_{n=1,2,\ldots}$ converges to 0 in measure. But, for any $x \in [0,1]$, the sequence $(v_n(x))_{n=1,2,\ldots}$ does not converges to 0.

Theorem 3.2. *Let (X, S, μ) be a measure space with $\mu(X) < \infty$. Suppose that f and f_n ($n \in \mathbb{N}$) are measurable functions from X to $\mathbb{R}$. If the sequence $(f_n)_{n=1,2\ldots}$ converges to f almost everywhere, then for every $\varepsilon > 0$ there exists a subset E of X such that $\mu(E) < \varepsilon$ and $(f_n)_{n=1,2,\ldots}$ converge to f uniformly on E^c.*

Proof. Without loss of any generality, we may assume that $(f_n)_{n=1,2\ldots}$ converges to f pointwise. For $k, n \in \mathbb{N}$, let

$$E_n(k) := \cup_{m=n}^{\infty}\{x \in X : |f_m(x) - f(x)| \ge 1/k\}.$$

For fixed k, we have $\lim_{n\to\infty} \mu(E_n(k)) = 0$, since $\mu(X) < \infty$. Given $\varepsilon > 0$ and $k \in \mathbb{N}$, choose n_k so large that $\mu(E_{n_k}(k)) < \varepsilon/2^k$. Let $E := \cup_{k=1}^{\infty} E_{n_k}(k)$. Then $\mu(E) < \varepsilon$. Moreover, $|f_n(x) - f(x)| < 1/k$ whenever $n > n_k$ and $x \in E^c$. This shows that $(f_n)_{n=1,2,\ldots}$ converges to f uniformly on E^c. □

The type of convergence involved in the conclusion of the above theorem is called **almost uniform convergence**.