Chapter 4. Measure Theory

§1. Measure Spaces

Let X be a nonempty set. A collection S of subsets of X is said to be an **algebra** on X if S has the following properties:

1. $X \in S$;
2. if $A \in S$, then $A^c \in S$;
3. if $A_1, \ldots, A_n \in S$, then $\cup_{k=1}^n A_k \in S$.

An algebra S on X is called a **σ-algebra** provided it is closed under *countable* unions, that is, if $A_k \in S$ for $k \in \mathbb{N}$, then $\cup_{k=1}^\infty A_k \in S$.

It is easily seen that an algebra is closed under finite intersections and a σ-algebra is closed under countable intersections. Moreover, an algebra is a σ-algebra provided it is closed under countable *disjioint* unions. Indeed, suppose that $(A_n)_{n=1,2,\ldots}$ is a sequence of subsets of X. Let $B_1 := A_1$ and $B_n := A_n \setminus \cup_{i=1}^{n-1} A_i$ for $n > 1$. Then $B_m \cap B_n = \emptyset$ whenever $m \neq n$ and $\cup_{n=1}^\infty A_n = \cup_{n=1}^\infty B_n$. If A_n belongs to an algebra S for every $n \in \mathbb{N}$, then $B_n \in S$ for every $n \in \mathbb{N}$.

For example, the power set $\mathcal{P}(X)$ is a σ-algebra on X. It is the largest σ-algebra on X. The smallest σ-algebra on X is the collection $\{X, \emptyset\}$.

If X is a nonempty set and S is a σ-algebra on X, then (X, S) is called a **measurable space** and the sets in S are called **measurable sets**.

Let (X, S) be a measurable space. A function μ from S to $[0, \infty]$ is called a **measure** if $\mu(\emptyset) = 0$ and μ has the σ-additivity, that is, if $(A_k)_{k=1,2,\ldots}$ is a sequence of *disjoint* sets in S, then

$$\mu\big(\cup_{k=1}^\infty A_k\big) = \sum_{k=1}^\infty \mu(A_k).$$

These two properties imply finite additivity of μ, that is, if $A_1, \ldots, A_n$ are disjoint sets in S, then

$$\mu\big(\cup_{k=1}^n A_k\big) = \sum_{k=1}^n \mu(A_k),$$

because one can take $A_k = \emptyset$ for $k > n$.

The triple (X, S, μ) is called a **measure space**. If $\mu(X) = 1$, then μ is called a **probability measure** and (X, S, μ) is called a **probability space**. If $\mu(X) < \infty$, then μ is called a **finite measure**. A measure μ is said to be **σ-finite** if there exists a sequence $(X_k)_{k=1,2,\ldots}$ of sets in S such that $X = \cup_{k=1}^\infty X_k$ and $\mu(X_k) < \infty$ for every $k \in \mathbb{N}$.

Example 1. Let X be a nonempty set, and let $S := \mathcal{P}(X)$. Define $\mu : S \to [0, \infty]$ by

$$\mu(A) := \begin{cases} \infty & \text{if } A \text{ is an infinite subset of } X, \\ \#A & \text{if } A \text{ is a finite subset of } X. \end{cases}$$

Then μ is a measure on (X, S). It is called the **counting measure**.

Example 2. Let X be a nonempty set, and let $S := \mathcal{P}(X)$. For a fixed element a of X, define $\mu \to [0, \infty]$ by

$$\mu(A) := \begin{cases} 1 & \text{if } a \in A, \\ 0 & \text{if } a \notin A. \end{cases}$$

Then μ is a measure on (X, S). It is called the **Dirac measure**.

Example 3. Let X be a (nonempty) countable set. There is a bijection $i \mapsto x_i$ from a subset I of $\mathbb{N}$ to X. Thus, we write $X = \{x_i : i \in I\}$. Suppose that a real number $p_i \geq 0$ is associated to each $i \in I$. Moreover, $\sum_{i \in I} p_i = 1$. For $A \in S := \mathcal{P}(X)$, define

$$\mu(A) := \sum_{x_i \in A} p_i.$$

Then μ is a probability measure and (X, S, μ) is a discrete probability space.

The following are some elementary properties of measures.

Theorem 1.1. *Let (X, S, μ) be a measure space. Then the following statements are true:*

(1) (monotonicity) If $A, B \in S$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

(2) (σ-subadditivity) If $A_k \in S$ for all $k \in \mathbb{N}$, then

$$\mu\big(\cup_{k=1}^{\infty} A_k\big) \leq \sum_{k=1}^{\infty} \mu(A_k).$$

(3) (Continuity from below) If $A_k \in S$ and $A_k \subseteq A_{k+1}$ for all $k \in \mathbb{N}$, then

$$\mu\big(\cup_{k=1}^{\infty} A_k\big) = \lim_{n \to \infty} \mu(A_n).$$

(4) (Continuity from above) If $A_k \in S$ and $A_k \supseteq A_{k+1}$ for all $k \in \mathbb{N}$, and if $\mu(A_1) < \infty$, then

$$\mu\big(\cap_{k=1}^{\infty} A_k\big) = \lim_{n \to \infty} \mu(A_n).$$

Proof. (1) Since B is the union of two disjoint sets A and $B \setminus A$, we have

$$\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A).$$

(2) Let $B_1 := A_1$ and $B_k := A_k \setminus \cup_{i=1}^{k-1} A_i$ for $k > 1$. Then $(B_k)_{k=1,2,\ldots}$ is a sequence of mutually disjoint sets, $B_k \subseteq A_k$ for every $k \in \mathbb{N}$, and $\cup_{k=1}^{\infty} B_k = \cup_{k=1}^{\infty} A_k$. Hence,

$$\mu\big(\cup_{k=1}^{\infty} A_k\big) = \mu\big(\cup_{k=1}^{\infty} B_k\big) = \sum_{k=1}^{\infty} \mu(B_k) \le \sum_{k=1}^{\infty} \mu(A_k).$$

(3) Let $A_0 := \emptyset$ and $B_k := A_k \setminus A_{k-1}$ for $k \in \mathbb{N}$. Then we have

$$\mu\big(\cup_{k=1}^{\infty} A_k\big) = \mu\big(\cup_{k=1}^{\infty} B_k\big) = \sum_{k=1}^{\infty} \mu(B_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(A_k \setminus A_{k-1}) = \lim_{n\to\infty} \mu(A_n).$$

(4) Let $E_k := A_1 \setminus A_k$ for $k \in \mathbb{N}$, $E := \cup_{k=1}^{\infty} E_k$ and $A := \cap_{k=1}^{\infty} A_k$. Then $E = A_1 \setminus A$ and $E_k \subseteq E_{k+1}$ for every $k \in \mathbb{N}$. By (3) we have

$$\mu(A_1) - \mu(A) = \mu(A_1 \setminus A) = \mu(E) = \lim_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} \big[\mu(A_1) - \mu(A_n)\big].$$

It follows that $\lim_{n\to\infty} \mu(A_n) = \mu(A)$. ☐

The last statement in the above theorem is still true if $\mu(A_{k_0}) < \infty$ for some $k_0 \in \mathbb{N}$. This condition is necessary. For example, consider the measure space $(\mathbb{N}, S, \mu)$, where S is the power set of $\mathbb{N}$, and μ is the counting measure. For $k \in \mathbb{N}$, let $A_k := \{n \in \mathbb{N} : n \ge k\}$. Then $A := \cap_{k=1}^{\infty} A_k$ is the empty set. But $\mu(A_k) = \infty$ for every $k \in \mathbb{N}$.

§2. Outer Measure

Let X be a set, and let $\mathcal{P}(X)$ denote its power set. A function μ^* from $\mathcal{P}(X)$ to $[0, \infty]$ is called an **outer measure** on X if it has the following properties:

1. $\mu^*(\emptyset) = 0$;
2. μ^* is monotone, that is, $\mu^*(A) \le \mu^*(B)$ whenever $A \subseteq B \subseteq X$;
3. μ^* is σ-subadditive, that is, $\mu^*(\cup_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} \mu^*(A_n)$ holds for every sequence $(A_n)_{n=1,2,\ldots}$ of subsets of X.

A subset E of X is called **μ^*-measurable** if

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

holds for all $A \subseteq X$.

If $\mu^*(E) = 0$, then E is μ^*-measurable, by the above definition.

The inequality $\mu^*(A) \le \mu^*(A \cap E) + \mu^*(A \cap E^c)$ holds for any subsets A and E of X. Hence, in order to prove that E is μ^*-measurable, it suffices to prove the reverse inequality. But the reverse inequality is trivial if $\mu^*(A) = \infty$. Thus, E is μ^*-measurable if and only if

$$\mu^*(A) \ge \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

holds for every subset A of X with $\mu^*(A) < \infty$.

Suppose that $E_1, \ldots, E_n$ are *disjoint* μ^*-measurable sets. Then

$$\mu^*\big(A \cap (\cup_{i=1}^n E_i)\big) = \sum_{i=1}^n \mu^*(A \cap E_i)$$

holds for every $A \subseteq X$. The proof of this assertion proceeds by induction on n. It is obviously true for $n = 1$. Suppose that the statement is true for $n \in \mathbb{N}$. We wish to prove it is true for $n+1$. Since E_{n+1} is μ^*-measurable, we have

$$\mu^*\big(A \cap (\cup_{i=1}^{n+1} E_i)\big) = \mu^*\big(A \cap (\cup_{i=1}^{n+1} E_i) \cap E_{n+1}\big) + \mu^*\big(A \cap (\cup_{i=1}^{n+1} E_i) \cap E_{n+1}^c\big).$$

It follows that

$$\mu^*\big(A \cap (\cup_{i=1}^{n+1} E_i)\big) = \mu^*\big(A \cap E_{n+1}\big) + \mu^*\big(A \cap (\cup_{i=1}^n E_i)\big).$$

By the induction hypothesis, $\mu^*\big(A \cap (\cup_{i=1}^n E_i)\big) = \sum_{i=1}^n \mu^*(A \cap E_i)$. Hence,

$$\mu^*\big(A \cap (\cup_{i=1}^{n+1} E_i)\big) = \mu^*\big(A \cap E_{n+1}\big) + \sum_{i=1}^n \mu^*(A \cap E_i) = \sum_{i=1}^{n+1} \mu^*(A \cap E_i).$$

This completes the induction procedure.

Theorem 2.1. *The collection Λ of all μ^*-measurable sets is a σ-algebra on X.*

Proof. By the definition of μ^*-measurability, we see that $X \in \Lambda$ and $\emptyset \in \Lambda$. Moreover, $E \in \Lambda$ implies $E^c \in \Lambda$. Let us show that Λ is closed under finite unions. Suppose that $E_1, E_2 \in \Lambda$. We wish to show $E := E_1 \cup E_2 \in \Lambda$. Note that $E = E_1 \cup (E_1^c \cap E_2)$ and $E^c = E_1^c \cap E_2^c$. Hence, for any subset A of X, we have

$$\begin{aligned}
\mu^*(A) &\le \mu^*(A \cap E) + \mu^*(A \cap E^c) \\
&\le \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c \cap E_2) + \mu^*(A \cap E_1^c \cap E_2^c) \\
&\le \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c) = \mu^*(A).
\end{aligned}$$

This shows $E = E_1 \cup E_2 \in \Lambda$.

It remains to show that Λ is closed under countable unions. Suppose that $(E_n)_{n=1,2,\ldots}$ is a sequence of sets in Λ. We wish to show $E := \cup_{n=1}^\infty E_n \in \Lambda$. Without loss of any generality, we may assume that $E_m \cap E_n = \emptyset$ for $m \ne n$. For each n, let $F_n := \sum_{i=1}^n E_i$. Then $F_n \in \Lambda$ for every $n \in \mathbb{N}$. For every subset A of X and every $n \in \mathbb{N}$ we have

$$\mu^*(A) = \mu^*(A \cap F_n) + \mu^*(A \cap F_n^c) \ge \mu^*(A \cap F_n) + \mu^*(A \cap E^c) = \sum_{i=1}^n \mu^*(A \cap E_i) + \mu^*(A \cap E^c).$$

Letting $n \to \infty$, we obtain

$$\mu^*(A) \geq \sum_{i=1}^{\infty} \mu^*(A \cap E_i) + \mu^*(A \cap E^c) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c) \geq \mu^*(A).$$

This shows $E \in \Lambda$. Therefore, Λ is a σ-algebra. $\square$

Theorem 2.2. *Let μ^* be an outer measure on a set X, and let Λ be the collection of all μ^*-measurable subsets of X. Then Λ is a σ-algebra and $(X, \Lambda, \mu^*|_\Lambda)$ is a measure space.*

Proof. By Theorem 2.1, Λ is a σ-algebra. Moreover, $\mu^*(\emptyset) = 0$. Suppose that $(E_k)_{k=1,2,\ldots}$ is a sequence of disjoint sets in Λ. Let $E := \cup_{k=1}^{\infty} E_k$. By the σ-subadditivity of μ^* we have $\mu^*(E) \leq \sum_{k=1}^{\infty} \mu^*(E_k)$. On the other hand, for every $n \in \mathbb{N}$ we have

$$\sum_{k=1}^{n} \mu^*(E_k) = \mu^*(\cup_{k=1}^{n} E_k) \leq \mu^*(E).$$

Letting $n \to \infty$ in the above inequality, we obtain $\sum_{k=1}^{\infty} \mu^*(E_k) \leq \mu^*(E)$. Therefore, $\mu^*(E) = \sum_{k=1}^{\infty} \mu^*(E_k)$. This shows that $\mu^*|_\Lambda$ is a measure. $\square$

§3. Lebesgue Measure

Suppose that $a, b \in \mathbb{R}$ and $a < b$. Let I be one of the intervals (a,b), $[a,b]$, $[a,b)$, and $(a,b]$. The **length** of I is defined to be $\ell(I) := b - a$.

For a subset A of $\mathbb{R}$, consider all possible sequences $(I_k)_{k=1,2,\ldots}$ of open intervals such that $A \subseteq \cup_{k=1}^{\infty} I_k$. Define

$$\lambda^*(A) := \inf\left\{ \sum_{k=1}^{\infty} \ell(I_k) : A \subseteq \cup_{k=1}^{\infty} I_k \right\}.$$

It is easily seen that $\lambda^*(\emptyset) = 0$ and $\lambda^*(A) \leq \lambda^*(B)$ whenever $A \subseteq B \subseteq \mathbb{R}$. Moreover, if A is a countable subset of $\mathbb{R}$, then $\lambda^*(A) = 0$.

Let us show that $\lambda^*(I) = \ell(I)$ for every bounded interval I. For this purpose, it suffices to consider a closed interval I. Clearly, $\lambda^*(I) \leq \ell(I)$. In order to show $\ell(I) \leq \lambda^*(I)$, we shall first prove that if I is covered by the union of open intervals $I_1, \ldots, I_n$, then $\ell(I) \leq \sum_{k=1}^{n} \ell(I_k)$. This statement is obviously true for $n = 1$. Assuming that it is true for n, we wish to show that it is also valid for $n+1$. Let $I_k = (a_k, b_k)$ $(k = 1, \ldots, n, n+1)$ be open intervals such that $I \subseteq \cup_{k=1}^{n+1} I_k$. Without loss of any generality, we may assume that $b_k \leq b_{k+1}$ for $k = 1, \ldots, n$. Suppose $I = [a, b]$. Then $b \leq b_{n+1}$. There are three possibilities:

$a_{n+1} < a$, $a_{n+1} > b$, or $a \le a_{n+1} \le b$. In the first case, $I \subseteq I_{n+1}$ and $\ell(I) \le \ell(I_{n+1})$. In the second case, $I = [a, b] \subseteq \cup_{k=1}^{n} E_k$, and hence $\ell(I) \le \sum_{k=1}^{n} \ell(I_k)$, by the induction hypothesis. In the third case, $b - a_{n+1} \le \ell(I_{n+1})$. Moreover, $[a, a_{n+1}] \cap I_{n+1} = \emptyset$. It follows that $[a, a_{n+1}] \subseteq \cup_{k=1}^{n} I_k$. By the induction hypothesis, $a_{n+1} - a \le \sum_{k=1}^{n} \ell(I_k)$. Therefore,

$$\ell(I) = b - a = (a_{n+1} - a) + (b - a_{n+1}) \le \sum_{k=1}^{n} \ell(I_k) + \ell(I_{n+1}) = \sum_{k=1}^{n+1} \ell(I_k).$$

This completes the induction procedure.

Now suppose that $(I_k)_{k=1,2,\ldots}$ is a sequence of open intervals such that $I \subseteq \cup_{k=1}^{\infty} I_k$. Since I is compact, there exists some positive integer n such that $I \subseteq \cup_{k=1}^{n} I_k$. By what has been proved we have

$$\ell(I) \le \sum_{k=1}^{n} \ell(I_k) \le \sum_{k=1}^{\infty} \ell(I_k).$$

Since $\ell(I) \le \sum_{k=1}^{\infty} \ell(I_k)$ as long as $I \subseteq \cup_{k=1}^{\infty} I_k$, we conclude that $\ell(I) \le \lambda^*(I)$. The proof for $\lambda^*(I) = \ell(I)$ is complete.

For a subset E of $\mathbb{R}$ and $r, s \in \mathbb{R}$, define

$$E + r := \{x + r : x \in E\} \quad \text{and} \quad sE := \{sx : x \in E\}.$$

Theorem 3.1. *The set function λ^* from $\mathcal{P}(\mathbb{R})$ to $[0, \infty]$ is an outer measure on $\mathbb{R}$. It has the following properties:*

1. *$\lambda^*([a, b]) = b - a$ for all $a, b \in \mathbb{R}$ with $a < b$;*
2. *$\lambda^*(E + r) = \lambda^*(E)$ for every subset E of $\mathbb{R}$ and $r \in \mathbb{R}$.*
3. *$\lambda^*(sE) = |s|\lambda^*(E)$ for every subset E of $\mathbb{R}$ and $s \in \mathbb{R}$.*

Proof. In order to prove that λ^* is an outer measure, it remains to show that λ^* is σ-subadditive.

Let $(A_j)_{j=1,2,\ldots}$ be a sequence of subsets of $\mathbb{R}$. We wish to show

$$\lambda^*(\cup_{j=1}^{\infty} A_j) \le \sum_{j=1}^{\infty} \lambda^*(A_j).$$

If $\lambda^*(A_j) = \infty$ for some j, then the inequality holds trivially. Thus, we may assume that $\lambda^*(A_j) < \infty$ for all $j \in \mathbb{N}$. Let $\varepsilon > 0$ be given. For each $j \in \mathbb{N}$, we can find a sequence $(I_{jk})_{k=1,2,\ldots}$ of open intervals such that $A_j \subseteq \cup_{k=1}^{\infty} I_{jk}$ and

$$\sum_{k=1}^{\infty} \ell(I_{jk}) < \lambda^*(A_j) + \frac{\varepsilon}{2^j}.$$

Thus, $(I_{jk})_{j,k\in\mathbb{N}}$ is a countable family of open intervals and it covers $\cup_{j=1}^\infty A_j$. Hence,

$$\lambda^*\big(\cup_{j=1}^\infty A_j\big) \le \sum_{j=1}^\infty\sum_{k=1}^\infty \ell(A_{jk}) \le \sum_{j=1}^\infty\big(\lambda^*(A_j)+\varepsilon/2^j\big) = \sum_{j=1}^\infty \lambda^*(A_j)+\varepsilon.$$

Since the inequality $\lambda^*\big(\cup_{j=1}^\infty A_j\big) \le \sum_{j=1}^\infty \lambda^*(A_j)+\varepsilon$ holds for all $\varepsilon>0$, we obtain $\lambda^*\big(\cup_{j=1}^\infty A_j\big) \le \sum_{j=1}^\infty \lambda^*(A_j)$, as desired.

Let us show that λ^* has the three properties as stated in the theorem. Property 1 was established before. As to properties 2 and 3, we let E be a subset of $\mathbb{R}$. For any $\varepsilon>0$, there exists a sequence $(I_k)_{k=1,2,\ldots}$ of open intervals such that $E\subseteq\cup_{k=1}^\infty I_k$ and $\sum_{k=1}^\infty \ell(I_k)\le\lambda^*(E)+\varepsilon$. We have $E+r\subseteq\cup_{k=1}^\infty(I_k+r)$, where each I_k+r is also an open interval and $\ell(I_k+r)=\ell(I_k)$. It follows that

$$\lambda^*(E+r)\le\sum_{k=1}^\infty \ell(I_k+r)=\sum_{k=1}^\infty \ell(I_k)\le\lambda^*(E)+\varepsilon.$$

Thus, $\lambda^*(E+r)\le\lambda^*(E)+\varepsilon$ for every $\varepsilon>0$. Consequently, $\lambda^*(E+r)\le\lambda^*(E)$. For the same reason, $\lambda^*(E)=\lambda^*(E+r-r)\le\lambda^*(E+r)$. This shows $\lambda^*(E)=\lambda^*(E+r)$. Furthermore, for $s\ne 0$ we have $sE\subseteq\cup_{k=1}^\infty(sI_k)$, where each sI_k is also an open interval and $\ell(sI_k)=|s|\ell(I_k)$. It follows that

$$\lambda^*(sE)\le\sum_{k=1}^\infty \ell(sI_k)=\sum_{k=1}^\infty |s|\ell(I_k)\le|s|(\lambda^*(E)+\varepsilon).$$

Thus, $\lambda^*(sE)\le|s|\lambda^*(E)+|s|\varepsilon$ for every $\varepsilon>0$. Consequently, $\lambda^*(sE)\le|s|\lambda^*(E)$. For the same reason, $\lambda^*(E)=\lambda^*(s^{-1}(sE))\le|s|^{-1}\lambda^*(sE)$. This shows $\lambda^*(sE)=|s|\lambda^*(E)$. Finally, if $s=0$ and E is nonempty, then $sE=\{0\}$ and $\lambda^*(sE)=0$. ☐

The set function λ^* is called the **Lebesgue outer measure**. A subset E of $\mathbb{R}$ is said to be **Lebesgue measurable** if it is λ^*-measurable. Let Λ be the collection of all Lebesgue measurable sets. By Theorem 2.2, Λ is a σ-algebra on $\mathbb{R}$. Moreover, $\lambda:=\lambda^*|_\Lambda$ is a measure, called the **Lebesgue measure**. The following theorem shows that Λ contains all open sets and closed sets.

Theorem 3.2. *Open sets and closed sets in the real line* $\mathbb{R}$ *are Lebesgue measurable.*

Proof. First, we show that (a,∞) is Lebesgue measurable for each $a\in\mathbb{R}$. For this purpose, it suffices to prove that, for any subset A of $\mathbb{R}$ with $\lambda^*(A)<\infty$,

$$\lambda^*(A)\ge\lambda^*(A')+\lambda^*(A''),$$

where $A' := A \cap (a, \infty)$ and $A'' := A \cap (-\infty, a]$. For given $\varepsilon > 0$, there exists a sequence $(I_k)_{k=1,2,\ldots}$ of open intervals such that $A \subseteq \cup_{k=1}^{\infty} I_k$ and $\sum_{k=1}^{\infty} \ell(I_k) \le \lambda^*(A) + \varepsilon$. For $k \in \mathbb{N}$, let $I_k' := I_k \cap (a, \infty)$ and $I_k'' := I_k \cap (-\infty, a]$. Then I_k' and I_k'' are intervals and

$$\ell(I_k) = \ell(I_k') + \ell(I_k'') = \lambda^*(I_k') + \lambda^*(I_k'').$$

Moreover, $A' \subseteq \cup_{k=1}^{\infty} I_k'$ and $A'' \subseteq \cup_{k=1}^{\infty} I_k''$. It follows that

$$\lambda^*(A') + \lambda^*(A'') \le \sum_{k=1}^{\infty} \lambda^*(I_k') + \sum_{k=1}^{\infty} \lambda^*(I_k'') = \sum_{k=1}^{\infty} \lambda^*(I_k) \le \lambda^*(A) + \varepsilon.$$

Thus, $\lambda^*(A') + \lambda^*(A'') \le \lambda^*(A) + \varepsilon$ for all $\varepsilon > 0$. Hence, $\lambda^*(A') + \lambda^*(A'') \le \lambda^*(A)$, as desired.

Second, we show that (a, b) is Lebesgue measurable for $a, b \in \mathbb{R}$ with $a < b$. Indeed, the above proof also shows $(-\infty, b]$ is Lebesgue measurable. Then $(-\infty, b) = (-\infty, b] \setminus \{b\}$ is Lebesgue measurable. Consequently, $(a, b) = (-\infty, b) \cap (a, \infty)$ is Lebesgue measurable.

Third, we show that open sets are Lebesgue measurable. Let G be an open subset of $\mathbb{R}$. Consider all open intervals $(a, b) \subseteq G$ with $a, b \in \mathbb{Q}$. The collection of these intervals is a countable set. But $G = \cup\{(a, b) \subseteq G : a, b \in \mathbb{Q}\}$. As a countable union of Lebesgue measurable sets, G itself is Lebesgue measurable.

Finally, if F is a closed subset of $\mathbb{R}$, then the set $G := F^c$ is open, and hence G is Lebesgue measurable. Consequently, $F = G^c$ is Lebesgue measurable. □

Let $\mathcal{B}$ denote the smallest σ-algebra that contains all open subsets of $\mathbb{R}$. Then $\mathcal{B}$ is called the **Borel σ-algebra**. The members in $\mathcal{B}$ are called **Borel sets**. Theorem 3.2 tells us that every Borel set is Lebesgue measurable.

A countable intersection of open sets is called a G_δ set, and a countable union of closed sets is called a F_σ set. Clearly, G_δ sets and F_σ sets are Lebesgue measurable.

Theorem 3.3. *Let λ^* denote the Lebesgue outer measure on $\mathbb{R}$. The following conditions are equivalent for a subset E of $\mathbb{R}$:*

(1) E is λ^*-measurable;
(2) for every $\varepsilon > 0$, there exists an open set $G \supseteq E$ such that $\lambda^*(G \setminus E) < \varepsilon$;
(3) there is a G_δ set $U \supseteq E$ such that $\lambda^*(U \setminus E) = 0$;
(4) for every $\varepsilon > 0$, there exists a closed set $F \subseteq E$ such that $\lambda^*(E \setminus F) < \varepsilon$;
(5) there is a F_σ set $V \subseteq E$ such that $\lambda^*(E \setminus V) = 0$.

Proof. (1) $\Rightarrow$ (2): First, consider the case $\lambda^*(E) < \infty$. For any $\varepsilon > 0$, there exists a sequence $(I_k)_{k=1,2,\ldots}$ of open intervals such that $E \subseteq \cup_{k=1}^{\infty} I_k$ and $\sum_{k=1}^{\infty} \ell(I_k) < \lambda^*(E) + \varepsilon$.

Let $G := \cup_{k=1}^\infty I_k$. Then G is an open set, $G \supseteq E$, and $\lambda^*(G) < \infty$. Since both G and E are λ^*-measurable, we have $\lambda^*(G) = \lambda^*(E) + \lambda^*(G \setminus E)$. It follows that

$$\lambda^*(G \setminus E) = \lambda^*(G) - \lambda^*(E) \le \sum_{k=1}^\infty \ell(I_k) - \lambda^*(E) < \varepsilon.$$

Next, consider the case $\lambda^*(E) = \infty$. We express E as a disjoint union $\cup_{j=1}^\infty E_j$, where each E_j is Lebesgue measurable and $\lambda^*(E_j) < \infty$. Let $\varepsilon > 0$ be given. For each $j \in \mathbb{N}$, there exists an open set $G_j \supseteq E_j$ such that $\lambda^*(G_j \setminus E_j) < \varepsilon/2^j$. Let $G := \cup_{j=1}^\infty G_j$. Then G is an open set, $G \supseteq E$, and

$$\lambda^*(G \setminus E) \le \sum_{j=1}^\infty \lambda^*(G_j \setminus E_j) < \varepsilon.$$

(2) $\Rightarrow$ (3): For each $n \in \mathbb{N}$, there exists an open set G_n such that $G_n \supseteq E$ and $\lambda^*(G_n \setminus E) < 1/n$. Let $U := \cap_{n=1}^\infty G_n$. Then U is a G_δ set and $G \supseteq E$. Moreover, $U \setminus E \subseteq G_n \setminus E$ for all n. Hence, $\lambda^*(U \setminus E) < 1/n$ for all n. This shows $\lambda^*(U \setminus E) = 0$.

(3) $\Rightarrow$ (1): In this case, both U and $U \setminus E$ are λ^* measurable. Hence, $E = U \setminus (U \setminus E)$ is λ^* measurable.

(2) $\Leftrightarrow$ (4): For any $\varepsilon > 0$, there is an open set G such that $G \supseteq E^c$ and $\lambda^*(G \setminus E^c) < \varepsilon$. Let $F := G^c$. Then F is a closed set and $F \subseteq E$. Moreover, $E \setminus F = E \cap F^c = E \cap G = G \setminus E^c$. Hence, $\lambda^*(E \setminus F) = \lambda^*(G \setminus E^c) < \varepsilon$. Thus, (2) implies (4). The proof for (4) $\Rightarrow$ (2) is similar.

(3) $\Leftrightarrow$ (5): There is a G_δ set $U \supseteq E^c$ such that $\lambda^*(U \setminus E^c) = 0$. Let $V := U^c$. Then V is an F_σ set and $V \subseteq E$. Moreover, $U \setminus E^c = U \cap E = E \setminus U^c = E \setminus V$. Hence, $\lambda^*(E \setminus V) = 0$. Thus, (3) implies (5). The proof for (5) $\Rightarrow$ (3) is similar. $\square$

If E is Lebesgue measurable, then $E + r$ is Lebesgue measurable for $r \in \mathbb{R}$. Indeed, by Theorem 3.3, there exists a G_δ set U such that $U \supseteq E$ and $\lambda^*(U \setminus E) = 0$. Clearly, $U + r$ is a G_δ set and $U + r \supseteq E + r$. Moreover, by Theorem 3.1 we have

$$\lambda^*\big((U + r) \setminus (E + r)\big) = \lambda^*(U \setminus E) = 0.$$

Hence, $E + r$ is Lebesgue measurable. Similarly, for $s \in \mathbb{R}$, sE is Lebesgue measurable.