Chapter 3. Metric Spaces

§1. Metric Spaces

A **metric space** is a set X endowed with a metric $\rho : X \times X \to [0, \infty)$ that satisfies the following properties for all x, y, and z in X:

1. $\rho(x, y) = 0$ if and only if $x = y$,
2. $\rho(x, y) = \rho(y, x)$, and
3. $\rho(x, z) \le \rho(x, y) + \rho(y, z)$.

The third property is called the **triangle inequality**.

We will write (X, ρ) to denote the metric space X endowed with a metric ρ. If Y is a subset of X, then the metric space $(Y, \rho|_{Y\times Y})$ is called a **subspace** of (X, ρ).

Example 1. Let $\rho(x, y) := |x - y|$ for $x, y \in \mathbb{R}$. Then $(\mathbb{R}, \rho)$ is a metric space. The set $\mathbb{R}$ equipped with this metric is called the **real line**.

Example 2. Let $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$. For $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$, define

$$\rho(x, y) := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

Then ρ is a metric on $\mathbb{R}^2$. The set $\mathbb{R}^2$ equipped with this metric is called the **Euclidean plane**. More generally, for $k \in \mathbb{N}$, the Euclidean k space $\mathbb{R}^k$ is the Cartesian product of k copies of $\mathbb{R}$ equipped with the metric ρ given by

$$\rho(x, y) := \left(\sum_{j=1}^{k} (x_j - y_j)^2 \right)^{1/2}, \quad x = (x_1, \ldots, x_k) \text{ and } y = (y_1, \ldots, y_k) \in \mathbb{R}^k.$$

Example 3. Let X be a nonempty set. For $x, y \in X$, define

$$\rho(x, y) := \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

In this case, ρ is called the **discrete metric** on X.

Let (X, ρ) be a metric space. For $x \in X$ and $r > 0$, the **open ball** centered at $x \in X$ with radius r is defined as

$$B_r(x) := \{y \in X : \rho(x, y) < r\}.$$

A subset A of X is called an **open set** if for every $x \in A$, there exists some $r > 0$ such that $B_r(x) \subseteq A$.

Theorem 1.1. *For a metric space (X, ρ) the following statements are true.*

1. *X and $\emptyset$ are open sets.*
2. *Arbitrary unions of open sets are open sets.*
3. *Finite intersections of open sets are open sets.*

Proof. The first statement is obviously true.

For the second statement, we let $(A_i)_{i \in I}$ be a family of open subsets of X and wish to prove that $\cup_{i \in I} A_i$ is an open set. Suppose $x \in \cup_{i \in I} A_i$. Then $x \in A_{i_0}$ for some $i_0 \in I$. Since A_{i_0} is an open set, there exists some $r > 0$ such that $B_r(x) \subseteq A_{i_0}$. Consequently, $B_r(x) \subseteq \cup_{i \in I} A_i$. This shows that $\cup_{i \in I} A_i$ is an open set.

For the second statement, we let $\{A_1, \ldots, A_n\}$ be a finite collection of open subsets of X and wish to prove that $\cap_{i=1}^n A_i$ is an open set. Suppose $x \in \cap_{i=1}^n A_i$. Then $x \in A_i$ for every $i \in \{1, \ldots, n\}$. For each $i \in \{1, \ldots, n\}$, there exists $r_i > 0$ such that $B_{r_i}(x) \subseteq A_i$. Set $r := \min\{r_1, \ldots, r_n\}$. Then $r > 0$ and $B_r(x) \subseteq \cap_{i=1}^n A_i$. This shows that $\cap_{i=1}^n A_i$ is an open set. □

Let (X, ρ) be a metric space. A subset B of X is called an **closed set** if its complement $B^c := X \setminus B$ is an open set.

The following theorem is an immediate consequence of Theorem 1.1.

Theorem 1.2. *For a metric space (X, ρ) the following statements are true.*

1. *X and $\emptyset$ are closed sets.*
2. *Arbitrary intersections of closed sets are closed sets.*
3. *Finite unions of closed sets are closed sets.*

Let (X, ρ) be a metric space. Given a subset A of X and a point x in X, there are three possibilities:

1. There exists some $r > 0$ such that $B_r(x) \subseteq A$. In this case, x is called an **interior point** of A.
2. For any $r > 0$, $B_r(x)$ intersects both A and A^c. In this case, x is called a **boundary point** of A.
3. There exists some $r > 0$ such that $B_r(x) \subseteq A^c$. In this case, x is called an **exterior point** of A.

For example, if A is a subset of the real line $\mathbb{R}$ bounded above, then $\sup A$ is a boundary point of A. Also, if A is bounded below, then $\inf A$ is a boundary point of A.

A point x is called a **closure point** of A if x is either an interior point or a boundary point of A. We denote by $\overline{A}$ the set of closure points of A. Then $A \subseteq \overline{A}$. The set $\overline{A}$ is called the **closure** of A.

Theorem 1.3. *If A is a subset of a metric space, then $\overline{A}$ is the smallest closed set that includes A.*

Proof. Let A be a subset of a metric space. We first show that $\overline{A}$ is closed. Suppose $x \notin \overline{A}$. Then x is an exterior point of A; hence there exists some $r > 0$ such that $B_r(x) \subseteq A^c$. If $y \in B_r(x)$, then there exists $\delta > 0$ such that $B_\delta(y) \subseteq B_r(x) \subseteq A^c$. This shows $y \notin \overline{A}$. Consequently, $B_r(x) \subseteq \overline{A}^c$. Therefore, $\overline{A}^c$ is open. In other words, $\overline{A}$ is closed.

Now assume that B is a closed subset of X such that $A \subseteq B$. Let $x \in B^c$. Then there exists $r > 0$ such that $B_r(x) \subseteq B^c \subseteq A^c$. This shows $x \in \overline{A}^c$. Hence, $B^c \subseteq \overline{A}^c$. It follows that $\overline{A} \subseteq B$. Therefore, $\overline{A}$ is the smallest closed set that includes A. □

A subset A of a metric space (X, ρ) is said to be **dense** in X if $\overline{A} = X$. A metric space (X, ρ) is called **separable** if it has a countable dense subset.

§2. Completeness

Let $(x_n)_{n=1,2,\ldots}$ be a sequence of elements in a metric space (X, ρ). We say that $(x_n)_{n=1,2,\ldots}$ converges to x in X and write $\lim_{n\to\infty} x_n = x$, if

$$\lim_{n\to\infty} \rho(x_n, x) = 0.$$

From the triangle inequality it follows that a sequence in a metric space has at most one limit.

Theorem 2.1. *Let A be a subset of a metric space (X, ρ). Then a point $x \in X$ belongs to $\overline{A}$ if and only if there exists a sequence $(x_n)_{n=1,2,\ldots}$ in A such that $\lim_{n\to\infty} x_n = x$.*

Proof. If $x \in \overline{A}$, then $B_{1/n}(x) \cap A \neq \emptyset$ for every $n \in \mathbb{N}$. Choose $x_n \in B_{1/n}(x) \cap A$ for each $n \in \mathbb{N}$. Then $\rho(x_n, x) < 1/n$, and hence $\lim_{n\to\infty} x_n = x$.

Suppose $x \notin \overline{A}$. Then there exists some $r > 0$ such that $B_r(x) \cap A = \emptyset$. Consequently, for any sequence $(x_n)_{n=1,2,\ldots}$ in A, we have $\rho(x_n, x) \geq r$ for all $n \in \mathbb{N}$. Thus, there is no sequence of elements in A that converges to x. □

A sequence $(x_n)_{n=1,2,\ldots}$ in a metric space (X, ρ) is said to be a **Cauchy sequence** if for every $\varepsilon > 0$ there exists a positive integer N such that

$$\rho(x_m, x_n) < \varepsilon \quad \text{whenever} \quad m, n > N.$$

Clearly, every convergent sequence is a Cauchy sequence.

If a metric space has the property that every Cauchy sequence converges, then the metric space is said to be **complete**. For example, the real line is a complete metric space.

The **diameter** of a set A is defined by

$$d(A) := \sup\{\rho(x, y) : x, y \in A\}.$$

If $d(A) < \infty$, then A is called a **bounded set**.

Theorem 2.2. *Let (X, ρ) be a complete metric space. Suppose that $(A_n)_{n=1,2,\ldots}$ is a sequence of closed and nonempty subsets of X such that $A_{n+1} \subseteq A_n$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} d(A_n) = 0$. Then $\cap_{n=1}^{\infty} A_n$ consists of precisely one element.*

Proof. If $x, y \in \cap_{n=1}^{\infty} A_n$, then $x, y \in A_n$ for every $n \in \mathbb{N}$. Hence, $\rho(x, y) \le \rho(A_n)$ for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \rho(A_n) = 0$, it follows that $\rho(x, y) = 0$, *i.e.*, $x = y$.

To show $\cap_{n=1}^{\infty} A_n \ne \emptyset$, we proceed as follows. Choose $x_n \in A_n$ for each $n \in \mathbb{N}$. Since $A_m \subseteq A_n$ for $m \ge n$, we have $\rho(x_m, x_n) \le d(A_n)$ for $m \ge n$. This in connection with the assumption $\lim_{n\to\infty} d(A_n) = 0$ shows that $(x_n)_{n=1,2,\ldots}$ is a Cauchy sequence. Since (X, ρ) is complete, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. We have $x_m \in A_n$ for all $m \ge n$. Hence, $x \in \overline{A}_n = A_n$. This is true for all $n \in \mathbb{N}$. Therefore, $x \in \cap_{n=1}^{\infty} A_n$. □

§3. Compactness

Let (X, ρ) be a metric space. A subset A of X is said to be **sequentially compact** if every sequence in A has a subsequence that converges to a point in A.

For example, a finite subset of a metric space is sequentially compact. The real line $\mathbb{R}$ is *not* sequentially compact.

A subset A of a metric space is called **totally bounded** if, for every $r > 0$, A can be covered by finitely many open balls of radius r.

For example, a bounded subset of the real line is totally bounded. On the other hand, if ρ is the discrete metric on an infinite set X, then X is bounded but not totally bounded.

Theorem 3.1. *Let A be a subset of a metric space (X, ρ). Then A is sequentially compact if and only if A is complete and totally bounded.*

Proof. Suppose that A is sequentially compact. We first show that A is complete. Let $(x_n)_{n=1,2,\ldots}$ be a Cauchy sequence in A. Since A is sequentially compact, there exists a subsequence $(x_{n_k})_{k=1,2,\ldots}$ that converges to a point x in A. For any $\varepsilon > 0$, there exists a positive integer N such that $\rho(x_m, x_n) < \varepsilon/2$ whenever $m, n > N$. Moreover, there exists some $k \in \mathbb{N}$ such that $n_k > N$ and $\rho(x_{n_k}, x) < \varepsilon/2$. Thus, for $n > N$ we have $\rho(x_n, x) \le \rho(x_n, x_{n_k}) + \rho(x_{n_k}, x) < \varepsilon$. Hence, $\lim_{n\to\infty} x_n = x$. This shows that A is complete.

Next, if A is not totally bounded, then there exists some $r > 0$ such that A cannot be covered by finitely many open balls of radius r. Choose $x_1 \in A$. Suppose $x_1, \ldots, x_n \in A$ have been chosen. Let x_{n+1} be a point in the nonempty set $A \setminus \cup_{i=1}^n B_r(x_i)$. If $m, n \in \mathbb{N}$ and $m \neq n$, then $\rho(x_m, x_n) \geq r$. Therefore, the sequence $(x_n)_{n=1,2,\ldots}$ has no convergent subsequence. Thus, if A is sequentially compact, then A is totally bounded.

Conversely, suppose that A is complete and totally bounded. Let $(x_n)_{n=1,2,\ldots}$ be a sequence of points in A. We shall construct a subsequence of $(x_n)_{n=1,2,\ldots}$ that is a Cauchy sequence, so that the subsequence converges to a point in A, by the completeness of A. For this purpose, we construct open balls B_k of radius $1/k$ and corresponding infinite subsets I_k of $\mathbb{N}$ for $k \in \mathbb{N}$ recursively. Since A is totally bounded, A can be covered by finitely many balls of radius 1. Hence, we can choose a ball B_1 of radius 1 such that the set $I_1 := \{n \in \mathbb{N} : x_n \in B_1\}$ is infinite. Suppose that a ball B_k of radius $1/k$ and an infinite subset I_k of $\mathbb{N}$ have been constructed. Since A is totally bounded, A can be covered by finitely many balls of radius $1/(k+1)$. Hence, we can choose a ball B_{k+1} of radius $1/(k+1)$ such that the set $I_{k+1} := \{n \in I_k : x_n \in B_{k+1}\}$ is infinite.

Choose $n_1 \in I_1$. Given n_k, choose $n_{k+1} \in I_{k+1}$ such that $n_{k+1} > n_k$. By our construction, $I_{k+1} \subseteq I_k$ for all $k \in \mathbb{N}$. Therefore, for all $i, j \geq k$, the points x_{n_i} and x_{n_j} are contained in the ball B_k of radius $1/k$. It follows that $(x_{n_k})_{k=1,2,\ldots}$ is a Cauchy sequence, as desired. □

Theorem 3.2. *A subset of a Euclidean space is sequentially compact if and only if it is closed and bounded.*

Proof. Let A be a subset of $\mathbb{R}^k$. If A is sequentially compact, then A is totally bounded and complete. In particular, A is bounded. Moreover, as a complete subset of $\mathbb{R}^k$, A is closed.

Conversely, suppose A is bounded and closed in $\mathbb{R}^k$. Since $\mathbb{R}^k$ is complete and A is closed, A is complete. It is easily seen that a bounded subset of $\mathbb{R}^k$ is totally bounded. □

Let $(A_i)_{i \in I}$ be a family of subsets of X. We say that $(A_i)_{i \in I}$ is a **cover** of a subset A of X, if $A \subseteq \cup_{i \in I} A_i$. If a subfamily of $(A_i)_{i \in I}$ also covers A, then it is called a **subcover**. If, in addition, (X, ρ) is a metric space and each A_i is an open set, then $(A_i)_{i \in I}$ is said to be an **open cover**.

Let $(G_i)_{i \in I}$ be an open cover of A. A real number $\delta > 0$ is called a **Lebesgue number** for the cover $(G_i)_{i \in I}$ if, for each subset E of A having diameter less than δ, $E \subseteq G_i$ for some $i \in I$.

Theorem 3.3. *Let A be a subset of a metric space (X, ρ). If A is sequentially compact, then there exists a Lebesgue number $\delta > 0$ for any open cover of A.*

Proof. Let $(G_i)_{i \in I}$ be an open cover of A. Suppose that there is no Lebesgue number for the cover $(G_i)_{i \in I}$. Then for each $n \in \mathbb{N}$ there exists a subset E_n of A having diameter less than $1/n$ such that $E_n \cap G_i^c \neq \emptyset$ for all $i \in I$. Choose $x_n \in E_n$ for $n \in \mathbb{N}$. Since A is sequentially compact, there exists a subsequence $(x_{n_k})_{k=1,2,\ldots}$ which converges to a point x in A. Since $(G_i)_{i \in I}$ is a cover of A, $x \in G_i$ for some $i \in I$. But G_i is an open set. Hence, there exists some $r > 0$ such that $B_r(x) \subseteq G_i$. We can find a positive integer k such that $1/n_k < r/2$ and $\rho(x_{n_k}, x) < r/2$. Let y be a point in E_{n_k}. Since x_{n_k} also lies in the set E_{n_k} with diameter less than $1/n_k$, we have $\rho(x_{n_k}, y) < 1/n_k$. Consequently,

$$\rho(x, y) \leq \rho(x, x_{n_k}) + \rho(x_{n_k}, y) < \frac{r}{2} + \frac{1}{n_k} < r.$$

This shows $E_{n_k} \subseteq B_r(x) \subseteq G_i$. However, E_{n_k} was so chosen that $E_{n_k} \cap G_i^c \neq \emptyset$. This contradiction proves the existence of a Lebesgue number for the open cover $(G_i)_{i \in I}$. □

A subset A of (X, ρ) is said to be **compact** if each open cover of A possesses a finite subcover of A. If X itself is compact, then (X, ρ) is called a **compact metric space**.

Theorem 3.4. *Let A be a subset of a metric space (X, ρ). Then A is compact if and only if it is sequentially compact.*

Proof. If A is not sequentially compact, then A is an infinite set. Moreover, there exists a sequence $(x_n)_{n=1,2,\ldots}$ in A having no convergent subsequence. Consequently, for each $x \in A$, there exists an open ball B_x centered at x such that $\{n \in \mathbb{N} : x_n \in B_x\}$ is a finite set. Then $(B_x)_{x \in A}$ is an open cover of A which does not possess a finite subcover of A. Thus, A is not compact.

Now suppose A is sequentially compact. Let $(G_i)_{i \in I}$ be an open cover of A. By Theorem 3.3, there exists a Lebesgue number $\delta > 0$ for the open cover $(G_i)_{i \in I}$. By Theorem 3.1, A is totally bounded. Hence, A is covered by a finite collection $\{B_1, \ldots, B_m\}$ of open balls with radius less than $\delta/2$. For each $k \in \{1, \ldots, m\}$, the diameter of B_k is less than δ. Hence, $B_k \subseteq G_{i_k}$ for some $i_k \in I$. Thus, $\{G_{i_k} : k = 1, \ldots, m\}$ is a finite subcover of A. This shows that A is compact. □

§4. Continuous Functions

Let (X, ρ) and (Y, τ) be two metric spaces. A function f from X to Y is said to be **continuous** at a point $a \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ (depending on ε) such that $\tau(f(x), f(a)) < \varepsilon$ whenever $\rho(x, a) < \delta$.

The function f is said to be **continuous** on X if f is continuous at every point of X.

Theorem 4.1. *For a function f from a metric space (X, ρ) to a metric space (Y, τ), the following statements are equivalent:*

1. *f is continuous on X.*
2. *$f^{-1}(G)$ is an open subset of X whenever G is an open subset of Y.*
3. *If $\lim_{n\to\infty} x_n = x$ holds in X, then $\lim_{n\to\infty} f(x_n) = f(x)$ holds in Y.*
4. *$f(\overline{A}) \subseteq \overline{f(A)}$ holds for every subset A of X.*
5. *$f^{-1}(F)$ is a closed subset of X whenever F is a closed subset of Y.*

Proof. $1 \Rightarrow 2$: Let G be an open subset of Y and $a \in f^{-1}(G)$. Since $f(a) \in G$ and G is open, there exists some $\varepsilon > 0$ such that $B_\varepsilon(f(a)) \subseteq G$. By the continuity of f, there exists some $\delta > 0$ such that $\tau(f(x), f(a)) < \varepsilon$ whenever $\rho(x, a) < \delta$. This shows $B_\delta(a) \subseteq f^{-1}(G)$. Therefore, $f^{-1}(G)$ is an open set.

$2 \Rightarrow 3$: Assume $\lim_{n\to\infty} x_n = x$ in X. For $\varepsilon > 0$, let $V := B_\varepsilon(f(x))$. In light of statement 2, $f^{-1}(V)$ is an open subset of X. Since $x \in f^{-1}(V)$, there exists some $\delta > 0$ such that $B_\delta(x) \subseteq f^{-1}(V)$. Then there exists a positive integer N such that $x_n \in B_\delta(x)$ for all $n > N$. It follows that $f(x_n) \in V = B_\varepsilon(f(x))$ for all $n > N$. Consequently, $\lim_{n\to\infty} f(x_n) = f(x)$.

$3 \Rightarrow 4$: Let A be a subset of X. If $y \in f(\overline{A})$, then there exists $x \in \overline{A}$ such that $y = f(x)$. Since $x \in \overline{A}$, there exists a sequence $(x_n)_{n=1,2,\ldots}$ of A such that $\lim_{n\to\infty} x_n = x$. By statement 3 we have $\lim_{n\to\infty} f(x_n) = f(x)$. It follows that $y = f(x) \in \overline{f(A)}$. This shows $f(\overline{A}) \subseteq \overline{f(A)}$.

$4 \Rightarrow 5$: Let F be a closed subset of Y, and let $A := f^{-1}(F)$. By statement 4 we have $f(\overline{A}) \subseteq \overline{f(A)} = \overline{F} = F$. It follows that $\overline{A} \subseteq f^{-1}(F) = A$. Hence, A is a closed subset of X.

$5 \Rightarrow 1$: Let $a \in X$ and $\varepsilon > 0$. Consider the closed set $F := Y \setminus B_\varepsilon(f(a))$. By statement 5, $f^{-1}(F)$ is a closed subset of X. Since $a \notin f^{-1}(F)$, there exists some $\delta > 0$ such that $B_\delta(a) \subseteq X \setminus f^{-1}(F)$. Consequently, $\rho(x, a) < \delta$ implies $\tau(f(x), f(a)) < \varepsilon$. So f is continuous at a. This is true for every point a in X. Hence, f is continuous on X. □

As an application of Theorem 4.1, we prove the Intermediate Value Theorem for continuous functions.

Theorem 4.2. *Suppose that $a, b \in \mathbb{R}$ and $a < b$. If f is a continuous function from $[a, b]$ to $\mathbb{R}$, then f has the intermediate value property, that is, for any real number d between $f(a)$ and $f(b)$, there exists $c \in [a, b]$ such that $f(c) = d$.*

Proof. Without loss of any generality, we may assume that $f(a) < d < f(b)$. Since the interval $(-\infty, d]$ is a closed set, the set $F := f^{-1}((-\infty, d]) = \{x \in [a,b] : f(x) \le d\}$ is closed, by Theorem 4.1. Let $c := \sup F$. Then c lies in F and hence $f(c) \le d$. It follows that $a \le c < b$. We claim $f(c) = d$. Indeed, if $f(c) < d$, then by the continuity of f we could find $r > 0$ such that $c < c + r < b$ and $f(c+r) < d$. Thus, we would have $c + r \in F$ and $c + r > \sup F$. This contradiction shows $f(c) = d$. □

The following theorem shows that a continuous function maps compact sets to compact sets.

Theorem 4.3. *Let f be a continuous function from a metric space (X, ρ) to a metric space (Y, τ). If A is a compact subset of X, then $f(A)$ is compact.*

Proof. Suppose that $(G_i)_{i \in I}$ is an open cover of $f(A)$. Since f is continuous, $f^{-1}(G_i)$ is open for every $i \in I$, by Theorem 4.1. Hence, $(f^{-1}(G_i))_{i \in I}$ is an open cover of A. By the compactness of A, there exists a finite subset $\{i_1, \ldots, i_m\}$ of I such that $A \subseteq \cup_{k=1}^m f^{-1}(G_{i_k})$. Consequently, $f(A) \subseteq \cup_{k=1}^m G_{i_k}$. This shows that $f(A)$ is compact. □

Theorem 4.4. *Let A be a nonempty compact subset of a metric space (X, ρ). If f is a continuous function from A to the real line $\mathbb{R}$, then f is bounded and assumes its maximum and minimum.*

Proof. By Theorem 4.3, $f(A)$ is a compact set, and so it is bounded and closed. Let $t := \inf f(A)$. Then $t \in \overline{f(A)} = f(A)$. Hence, $t = \min f(A)$ and $t = f(a)$ for some $a \in A$. Similarly, Let $s := \sup f(A)$. Then $s \in \overline{f(A)} = f(A)$. Hence, $s = \max f(A)$ and $s = f(b)$ for some $b \in A$. □

A function f from a metric space (X, ρ) to a metric space (Y, τ) is said to be **uniformly continuous** on X if for every $\varepsilon > 0$ there exists $\delta > 0$ (depending on ε) such that $\tau(f(x), f(y)) < \varepsilon$ whenever $\rho(x, y) < \delta$. Clearly, a uniformly continuous function is continuous.

A function from (X, ρ) to (Y, τ) is said to be a **Lipschitz function** if there exists a constant C_f such that $\tau(f(x), f(y)) \le C_f \rho(x, y)$ for all $x, y \in X$. Clearly, a Lipschitz function is uniformly continuous.

Example. Let f and g be the functions from the interval $(0, 1]$ to the real line $\mathbb{R}$ given by $f(x) = x^2$ and $g(x) = 1/x$, $x \in (0, 1]$, respectively. Then f is uniformly continuous, while g is continuous but not uniformly continuous.

Theorem 4.5. *Let f be a continuous function from a metric space (X, ρ) to a metric space (Y, τ). If X is compact, then f is uniformly continuous on X.*

Proof. Let $\varepsilon > 0$ be given. Since f is continuous, for each $x \in X$ there exists $r_x > 0$ such that $\tau(f(x), f(y)) < \varepsilon/2$ for all $y \in B_{r_x}(x)$. Then $(B_{r_x}(x))_{x \in X}$ is an open cover of X. Since X is compact, Theorem 3.3 tells us that there exists a Lebesgue number $\delta > 0$ for this open cover. Suppose $y, z \in X$ and $\rho(y, z) < \delta$. Then $\{y, z\} \subseteq B_{r_x}(x)$ for some $x \in X$. Consequently,

$$\tau(f(y), f(z)) \leq \tau(f(y), f(x)) + \tau(f(x), f(z)) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This shows that f is uniformly continuous on X. $\square$