Chapter 1. Metric Spaces

§1. Metric Spaces

A metric space is a set $X$ endowed with a metric $\rho : X \times X \to [0, \infty)$ that satisfies the following properties for all $x$, $y$, and $z$ in $X$:

1. $\rho(x, y) = 0$ if and only if $x = y$,
2. $\rho(x, y) = \rho(y, x)$, and
3. $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

The third property is called the triangle inequality.

We will write $(X, \rho)$ to denote the metric space $X$ endowed with a metric $\rho$. If $Y$ is a subset of $X$, then the metric space $(Y, \rho|_{Y \times Y})$ is called a subspace of $(X, \rho)$.

Example 1. Let $\rho(x, y) := |x - y|$ for $x, y \in \mathbb{R}$. Then $(\mathbb{R}, \rho)$ is a metric space. The set $\mathbb{R}$ equipped with this metric is called the real line.

Example 2. Let $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$. For $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$, define

$$\rho(x, y) := \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$ 

Then $\rho$ is a metric on $\mathbb{R}^2$. The set $\mathbb{R}^2$ equipped with this metric is called the Euclidean plane. More generally, for $k \in \mathbb{N}$, the Euclidean $k$ space $\mathbb{R}^k$ is the Cartesian product of $k$ copies of $\mathbb{R}$ equipped with the metric $\rho$ given by

$$\rho(x, y) := \left(\sum_{j=1}^{k} (x_j - y_j)^2\right)^{1/2}, \quad x = (x_1, \ldots, x_k) \text{ and } y = (y_1, \ldots, y_k) \in \mathbb{R}^k.$$ 

Example 3. Let $X$ be a nonempty set. For $x, y \in X$, define

$$\rho(x, y) := \begin{cases} 
1 & \text{if } x \neq y, \\
0 & \text{if } x = y. 
\end{cases}$$ 

In this case, $\rho$ is called the discrete metric on $X$.

Let $(X, \rho)$ be a metric space. For $x \in X$ and $r > 0$, the open ball centered at $x \in X$ with radius $r$ is defined as

$$B_r(x) := \{y \in X : \rho(x, y) < r\}.$$ 

A subset $A$ of $X$ is called an open set if for every $x \in A$, there exists some $r > 0$ such that $B_r(x) \subseteq A$. 

1
Theorem 1.1. For a metric space \((X, \rho)\) the following statements are true.
1. \(X\) and \(\emptyset\) are open sets.
2. Arbitrary unions of open sets are open sets.
3. Finite intersections of open sets are open sets.

Proof. The first statement is obviously true.

For the second statement, we let \((A_i)_{i \in I}\) be a family of open subsets of \(X\) and wish to prove that \(\bigcup_{i \in I} A_i\) is an open set. Suppose \(x \in \bigcup_{i \in I} A_i\). Then \(x \in A_{i_0}\) for some \(i_0 \in I\). Since \(A_{i_0}\) is an open set, there exists some \(r > 0\) such that \(B_r(x) \subseteq A_{i_0}\). Consequently, \(B_r(x) \subseteq \bigcup_{i \in I} A_i\). This shows that \(\bigcup_{i \in I} A_i\) is an open set.

For the third statement, we let \(\{A_1, \ldots, A_n\}\) be a finite collection of open subsets of \(X\) and wish to prove that \(\bigcap_{i=1}^n A_i\) is an open set. Suppose \(x \in \bigcap_{i=1}^n A_i\). Then \(x \in A_i\) for every \(i \in \{1, \ldots, n\}\). For each \(i \in \{1, \ldots, n\}\), there exists \(r_i > 0\) such that \(B_{r_i}(x) \subseteq A_i\). Set \(r := \min\{r_1, \ldots, r_n\}\). Then \(r > 0\) and \(B_r(x) \subseteq \bigcap_{i=1}^n A_i\). This shows that \(\bigcap_{i=1}^n A_i\) is an open set.

Let \((X, \rho)\) be a metric space. A subset \(B\) of \(X\) is called an closed set if its complement \(B^c := X \setminus B\) is an open set.

The following theorem is an immediate consequence of Theorem 1.1.

Theorem 1.2. For a metric space \((X, \rho)\) the following statements are true.
1. \(X\) and \(\emptyset\) are closed sets.
2. Arbitrary intersections of closed sets are closed sets.
3. Finite unions of closed sets are closed sets.

Let \((X, \rho)\) be a metric space. Given a subset \(A\) of \(X\) and a point \(x\) in \(X\), there are three possibilities:
1. There exists some \(r > 0\) such that \(B_r(x) \subseteq A\). In this case, \(x\) is called an interior point of \(A\).
2. For any \(r > 0\), \(B_r(x)\) intersects both \(A\) and \(A^c\). In this case, \(x\) is called a boundary point of \(A\).
3. There exists some \(r > 0\) such that \(B_r(x) \subseteq A^c\). In this case, \(x\) is called an exterior point of \(A\).

For example, if \(A\) is a subset of the real line \(\mathbb{R}\) bounded above, then \(\sup A\) is a boundary point of \(A\). Also, if \(A\) is bounded below, then \(\inf A\) is a boundary point of \(A\).

A point \(x\) is called a closure point of \(A\) if \(x\) is either an interior point or a boundary point of \(A\). We denote by \(\overline{A}\) the set of closure points of \(A\). Then \(A \subseteq \overline{A}\). The set \(\overline{A}\) is called the closure of \(A\).
Theorem 1.3. If \( A \) is a subset of a metric space \((X, \rho)\), then \( \overline{A} \) is the smallest closed set that includes \( A \).

Proof. Let \( A \) be a subset of a metric space. We first show that \( \overline{A} \) is closed. Suppose \( x \notin \overline{A} \). Then \( x \) is an exterior point of \( A \); hence there exists some \( r > 0 \) such that \( B_r(x) \subseteq A^c \). If \( y \in B_r(x) \), then \( \rho(x, y) < r \). For \( \delta := r - \rho(x, y) > 0 \), by the triangle inequality we have \( B_\delta(y) \subseteq B_r(x) \). It follows that \( B_\delta(y) \subseteq A^c \). This shows \( y \notin \overline{A} \). Consequently, \( B_r(x) \subseteq \overline{A}^c \). Therefore, \( \overline{A}^c \) is open. In other words, \( \overline{A} \) is closed.

Now assume that \( B \) is a closed subset of \( X \) such that \( A \subseteq B \). Let \( x \in B^c \). Then there exists \( r > 0 \) such that \( B_r(x) \subseteq B^c \subseteq A^c \). This shows \( x \notin \overline{A} \). Hence, \( B^c \subseteq \overline{A}^c \). It follows that \( \overline{A} \subseteq B \). Therefore, \( \overline{A} \) is the smallest closed set that includes \( A \). \( \square \)

A subset \( A \) of a metric space \((X, \rho)\) is said to be **dense** in \( X \) if \( \overline{A} = X \).

§2. Completeness

Let \((x_n)_{n=1,2,...} \) be a sequence of elements in a metric space \((X, \rho)\). We say that \((x_n)_{n=1,2,...} \) converges to \( x \) in \( X \) and write \( \lim_{n \to \infty} x_n = x \), if

\[
\lim_{n \to \infty} \rho(x_n, x) = 0.
\]

From the triangle inequality it follows that a sequence in a metric space has at most one limit.

Theorem 2.1. Let \( A \) be a subset of a metric space \((X, \rho)\). Then a point \( x \in X \) belongs to \( \overline{A} \) if and only if there exists a sequence \((x_n)_{n=1,2,...} \) in \( A \) such that \( \lim_{n \to \infty} x_n = x \).

Proof. If \( x \in \overline{A} \), then \( B_{1/n}(x) \cap A \neq \emptyset \) for every \( n \in \mathbb{N} \). Choose \( x_n \in B_{1/n}(x) \cap A \) for each \( n \in \mathbb{N} \). Then \( \rho(x_n, x) < 1/n \), and hence \( \lim_{n \to \infty} x_n = x \).

Suppose \( x \notin \overline{A} \). Then there exists some \( r > 0 \) such that \( B_r(x) \cap A = \emptyset \). Consequently, for any sequence \((x_n)_{n=1,2,...} \) in \( A \), we have \( \rho(x_n, x) \geq r \) for all \( n \in \mathbb{N} \). Thus, there is no sequence of elements in \( A \) that converges to \( x \). \( \square \)

A sequence \((x_n)_{n=1,2,...} \) in a metric space \((X, \rho)\) is said to be a **Cauchy sequence** if for any given \( \varepsilon > 0 \) there exists a positive integer \( N \) such that

\[
m, n > N \implies \rho(x_m, x_n) < \varepsilon.
\]

Clearly, every convergent sequence is a Cauchy sequence.

If a metric space has the property that every Cauchy sequence converges, then the metric space is said to be **complete**. For example, the real line is a complete metric space.
The diameter of a set $A$ is defined by

$$d(A) := \sup\{\rho(x, y) : x, y \in A\}.$$ 

If $d(A) < \infty$, then $A$ is called a **bounded set**.

**Theorem 2.2.** Let $(X, \rho)$ be a complete metric space. Suppose that $(A_n)_{n=1}^{\infty}$ is a sequence of closed and nonempty subsets of $X$ such that $A_{n+1} \subseteq A_n$ for every $n \in \mathbb{N}$ and $\lim_{n \to \infty} d(A_n) = 0$. Then $\bigcap_{n=1}^{\infty} A_n$ consists of precisely one element.

**Proof.** If $x, y \in \bigcap_{n=1}^{\infty} A_n$, then $x, y \in A_n$ for every $n \in \mathbb{N}$. Hence, $\rho(x, y) \leq d(A_n)$ for all $n \in \mathbb{N}$. Since $\lim_{n \to \infty} \rho(A_n) = 0$, it follows that $\rho(x, y) = 0$, i.e., $x = y$.

To show $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$, we proceed as follows. Choose $x_n \in A_n$ for each $n \in \mathbb{N}$. Since $A_m \subseteq A_n$ for $m \geq n$, we have $\rho(x_m, x_n) \leq d(A_n)$ for $m \geq n$. This in connection with the assumption $\lim_{n \to \infty} d(A_n) = 0$ shows that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence. Since $(X, \rho)$ is complete, there exists $x \in X$ such that $\lim_{n \to \infty} x_n = x$. We have $x_m \in A_n$ for all $m \geq n$. Hence, $x \in \overline{A_n} = A_n$. This is true for all $n \in \mathbb{N}$. Therefore, $x \in \bigcap_{n=1}^{\infty} A_n$. \qed

§3. Compactness

Let $(X, \rho)$ be a metric space. A subset $A$ of $X$ is said to be **sequentially compact** if every sequence in $A$ has a subsequence that converges to a point in $A$.

For example, a finite subset of a metric space is sequentially compact. The real line $\mathbb{R}$ is *not* sequentially compact. But a bounded closed interval in the real line is sequentially compact.

A subset $A$ of a metric space is called **totally bounded** if, for every $r > 0$, $A$ can be covered by finitely many open balls of radius $r$.

For example, a bounded subset of the real line is totally bounded. On the other hand, if $\rho$ is the discrete metric on an infinite set $X$, then $X$ is bounded but not totally bounded.

**Theorem 3.1.** Let $A$ be a subset of a metric space $(X, \rho)$. Then $A$ is sequentially compact if and only if $A$ is complete and totally bounded.

**Proof.** Suppose that $A$ is sequentially compact. We first show that $A$ is complete. Let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in $A$. Since $A$ is sequentially compact, there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ that converges to a point $x$ in $A$. For any $\varepsilon > 0$, there exists a positive integer $N$ such that $\rho(x_m, x) < \varepsilon/2$ whenever $m, n > N$. Moreover, there exists some $k \in \mathbb{N}$ such that $n_k > N$ and $\rho(x_{n_k}, x) < \varepsilon/2$. Thus, for $n > N$ we have
\( \rho(x_n, x) \leq \rho(x_n, x_{n_k}) + \rho(x_{n_k}, x) < \varepsilon \). Hence, \( \lim_{n \to \infty} x_n = x \). This shows that \( A \) is complete.

Next, if \( A \) is not totally bounded, then there exists some \( r > 0 \) such that \( A \) cannot be covered by finitely many open balls of radius \( r \). Choose \( x_1 \in A \). Suppose \( x_1, \ldots, x_n \in A \) have been chosen. Let \( x_{n+1} \) be a point in the nonempty set \( A \setminus \bigcup_{i=1}^n B_r(x_i) \). If \( m, n \in \mathbb{N} \) and \( m \neq n \), then \( \rho(x_m, x_n) \geq r \). Therefore, the sequence \( (x_n)_{n=1,2,\ldots} \) has no convergent subsequence. Thus, if \( A \) is sequentially compact, then \( A \) is totally bounded.

Conversely, suppose that \( A \) is complete and totally bounded. Let \( (x_n)_{n=1,2,\ldots} \) be a sequence of points in \( A \). We shall construct a subsequence of \( (x_n)_{n=1,2,\ldots} \) that is a Cauchy sequence, so that the subsequence converges to a point in \( A \), by the completeness of \( A \). For this purpose, we construct open balls \( B_k \) of radius \( 1/k \) and corresponding infinite subsets \( I_k \) of \( \mathbb{N} \) for \( k \in \mathbb{N} \) recursively. Since \( A \) is totally bounded, \( A \) can be covered by finitely many balls of radius 1. Hence, we can choose a ball \( B_1 \) of radius 1 such that the set \( I_1 := \{ n \in \mathbb{N} : x_n \in B_1 \} \) is infinite. Suppose that a ball \( B_k \) of radius \( 1/k \) and an infinite subset \( I_k \) of \( \mathbb{N} \) have been constructed. Since \( A \) is totally bounded, \( A \) can be covered by finitely many balls of radius \( 1/(k+1) \). Hence, we can choose a ball \( B_{k+1} \) of radius \( 1/(k+1) \) such that the set \( I_{k+1} := \{ n \in I_k : x_n \in B_{k+1} \} \) is infinite.

Choose \( n_1 \in I_1 \). Given \( n_k \), choose \( n_{k+1} \in I_{k+1} \) such that \( n_{k+1} > n_k \). By our construction, \( I_{k+1} \subseteq I_k \) for all \( k \in \mathbb{N} \). Therefore, for all \( i, j \geq k \), the points \( x_{n_i} \) and \( x_{n_j} \) are contained in the ball \( B_k \) of radius \( 1/k \). It follows that \( (x_{n_k})_{k=1,2,\ldots} \) is a Cauchy sequence, as desired.

**Theorem 3.2.** A subset of a Euclidean space is sequentially compact if and only if it is closed and bounded.

**Proof.** Let \( A \) be a subset of \( \mathbb{R}^k \). If \( A \) is sequentially compact, then \( A \) is totally bounded and complete. In particular, \( A \) is bounded. Moreover, as a complete subset of \( \mathbb{R}^k \), \( A \) is closed.

Conversely, suppose \( A \) is bounded and closed in \( \mathbb{R}^k \). Since \( \mathbb{R}^k \) is complete and \( A \) is closed, \( A \) is complete. It is easily seen that a bounded subset of \( \mathbb{R}^k \) is totally bounded. \( \square \)

Let \( (A_i)_{i \in I} \) be a family of subsets of \( X \). We say that \( (A_i)_{i \in I} \) is a **cover** of a subset \( A \) of \( X \), if \( A \subseteq \bigcup_{i \in I} A_i \). If a subfamily of \( (A_i)_{i \in I} \) also covers \( A \), then it is called a **subcover**. If, in addition, \( (X, \rho) \) is a metric space and each \( A_i \) is an open set, then \( (A_i)_{i \in I} \) is said to be an **open cover**.

Let \( (G_i)_{i \in I} \) be an open cover of \( A \). A real number \( \delta > 0 \) is called a **Lebesgue number** for the cover \( (G_i)_{i \in I} \) if, for each subset \( E \) of \( A \) having diameter less than \( \delta \), \( E \subseteq G_i \) for
some \( i \in I \).

**Theorem 3.3.** Let \( A \) be a subset of a metric space \((X, \rho)\). If \( A \) is sequentially compact, then there exists a Lebesgue number \( \delta > 0 \) for any open cover of \( A \).

**Proof.** Let \((G_i)_{i \in I}\) be an open cover of \( A \). Suppose that there is no Lebesgue number for the cover \((G_i)_{i \in I}\). Then for each \( n \in \mathbb{N} \) there exists a subset \( E_n \) of \( A \) having diameter less than \( 1/n \) such that \( E_n \cap G_i^c \neq \emptyset \) for all \( i \in I \). Choose \( x_n \in E_n \) for \( n \in \mathbb{N} \). Since \( A \) is sequentially compact, there exists a subsequence \((x_{n_k})_{k=1,2,\ldots}\) which converges to a point \( x \) in \( A \). Since \((G_i)_{i \in I}\) is a cover of \( A \), \( x \in G_i \) for some \( i \in I \). But \( G_i \) is an open set. Hence, there exists some \( r > 0 \) such that \( B_r(x) \subseteq G_i \). We can find a positive integer \( k \) such that \( 1/n_k < r/2 \) and \( \rho(x_{n_k}, x) < r/2 \). Let \( y \) be a point in \( E_{n_k} \). Since \( x_{n_k} \) also lies in the set \( E_{n_k} \) with diameter less than \( 1/n_k \), we have \( \rho(x_{n_k}, y) < 1/n_k \). Consequently,

\[
\rho(x, y) \leq \rho(x, x_{n_k}) + \rho(x_{n_k}, y) < \frac{r}{2} + \frac{1}{n_k} < r.
\]

This shows \( E_{n_k} \subseteq B_r(x) \subseteq G_i \). However, \( E_{n_k} \) was so chosen that \( E_{n_k} \cap G_i^c \neq \emptyset \). This contradiction proves the existence of a Lebesgue number for the open cover \((G_i)_{i \in I}\). \( \square \)

A subset \( A \) of \((X, \rho)\) is said to be **compact** if each open cover of \( A \) possesses a finite subcover of \( A \). If \( X \) itself is compact, then \((X, \rho)\) is called a **compact metric space**.

**Theorem 3.4.** Let \( A \) be a subset of a metric space \((X, \rho)\). Then \( A \) is compact if and only if it is sequentially compact.

**Proof.** If \( A \) is not sequentially compact, then \( A \) is an infinite set. Moreover, there exists a sequence \((x_n)_{n=1,2,\ldots}\) in \( A \) having no convergent subsequence. Consequently, for each \( x \in A \), there exists an open ball \( B_x \) centered at \( x \) such that \( \{n \in \mathbb{N} : x_n \in B_x\} \) is a finite set. Then \((B_x)_{x \in A}\) is an open cover of \( A \) which does not possess a finite subcover of \( A \). Thus, \( A \) is not compact.

Now suppose \( A \) is sequentially compact. Let \((G_i)_{i \in I}\) be an open cover of \( A \). By Theorem 3.3, there exists a Lebesgue number \( \delta > 0 \) for the open cover \((G_i)_{i \in I}\). By Theorem 3.1, \( A \) is totally bounded. Hence, \( A \) is covered by a finite collection \( \{B_1, \ldots, B_m\} \) of open balls with radius less than \( \delta/2 \). For each \( k \in \{1, \ldots, m\} \), the diameter of \( B_k \) is less than \( \delta \). Hence, \( B_k \subseteq G_{i_k} \) for some \( i_k \in I \). Thus, \( \{G_{i_k} : k = 1, \ldots, m\} \) is a finite subcover of \( A \). This shows that \( A \) is compact. \( \square \)
§4. Continuous Functions

Let \((X, \rho)\) and \((Y, \tau)\) be two metric spaces. A function \(f\) from \(X\) to \(Y\) is said to be **continuous** at a point \(a \in X\) if for every \(\varepsilon > 0\) there exists \(\delta > 0\) (depending on \(\varepsilon\)) such that \(\tau(f(x), f(a)) < \varepsilon\) whenever \(\rho(x, a) < \delta\).

The function \(f\) is said to be **continuous** on \(X\) if \(f\) is continuous at every point of \(X\).

**Theorem 4.1.** For a function \(f\) from a metric space \((X, \rho)\) to a metric space \((Y, \tau)\), the following statements are equivalent:

1. \(f\) is continuous on \(X\).
2. \(f^{-1}(G)\) is an open subset of \(X\) whenever \(G\) is an open subset of \(Y\).
3. If \(\lim_{n \to \infty} x_n = x\) holds in \(X\), then \(\lim_{n \to \infty} f(x_n) = f(x)\) holds in \(Y\).
4. \(f(A) \subseteq \overline{f(A)}\) holds for every subset \(A\) of \(X\).
5. \(f^{-1}(F)\) is a closed subset of \(X\) whenever \(F\) is a closed subset of \(Y\).

**Proof.**

1 \(\Rightarrow\) 2: Let \(G\) be an open subset of \(Y\) and \(a \in f^{-1}(G)\). Since \(f(a) \in G\) and \(G\) is open, there exists some \(\varepsilon > 0\) such that \(B_\varepsilon(f(a)) \subseteq G\). By the continuity of \(f\), there exists some \(\delta > 0\) such that \(\tau(f(x), f(a)) < \varepsilon\) whenever \(\rho(x, a) < \delta\). This shows \(B_\delta(a) \subseteq f^{-1}(G)\). Therefore, \(f^{-1}(G)\) is an open set.

2 \(\Rightarrow\) 3: Assume \(\lim_{n \to \infty} x_n = x\) in \(X\). For \(\varepsilon > 0\), let \(V := B_\varepsilon(f(x))\). In light of statement 2, \(f^{-1}(V)\) is an open subset of \(X\). Since \(x \in f^{-1}(V)\), there exists some \(\delta > 0\) such that \(B_\delta(x) \subseteq f^{-1}(V)\). Then there exists a positive integer \(N\) such that \(x_n \in B_\delta(x)\) for all \(n > N\). It follows that \(f(x_n) \in V = B_\varepsilon(f(x))\) for all \(n > N\). Consequently, \(\lim_{n \to \infty} f(x_n) = f(x)\).

3 \(\Rightarrow\) 4: Let \(A\) be a subset of \(X\). If \(y \in f(\overline{A})\), then there exists \(x \in \overline{A}\) such that \(y = f(x)\). Since \(x \in \overline{A}\), there exists a sequence \((x_n)_{n=1,2,...}\) of \(A\) such that \(\lim_{n \to \infty} x_n = x\). By statement 3 we have \(\lim_{n \to \infty} f(x_n) = f(x)\). It follows that \(y = f(x) \in f(\overline{A})\). This shows \(f(\overline{A}) \subseteq \overline{f(A)}\).

4 \(\Rightarrow\) 5: Let \(F\) be a closed subset of \(Y\), and let \(A := f^{-1}(F)\). By statement 4 we have \(f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{F} = F\). It follows that \(\overline{A} \subseteq f^{-1}(F) = A\). Hence, \(A\) is a closed subset of \(X\).

5 \(\Rightarrow\) 1: Let \(a \in X\) and \(\varepsilon > 0\). Consider the closed set \(F := Y \setminus B_\varepsilon(f(a))\). By statement 5, \(f^{-1}(F)\) is a closed subset of \(X\). Since \(a \notin f^{-1}(F)\), there exists some \(\delta > 0\) such that \(B_\delta(a) \subseteq X \setminus f^{-1}(F)\). Consequently, \(\rho(x, a) < \delta\) implies \(\tau(f(x), f(a)) < \varepsilon\). So \(f\) is continuous at \(a\). This is true for every point \(a\) in \(X\). Hence, \(f\) is continuous on \(X\). \(\square\)

As an application of Theorem 4.1, we prove the Intermediate Value Theorem for continuous functions.
Theorem 4.2. Suppose that \( a, b \in \mathbb{R} \) and \( a < b \). If \( f \) is a continuous function from \([a, b]\) to \( \mathbb{R} \), then \( f \) has the intermediate value property, that is, for any real number \( d \) between \( f(a) \) and \( f(b) \), there exists \( c \in [a, b] \) such that \( f(c) = d \).

Proof. Without loss of any generality, we may assume that \( f(a) < d < f(b) \). Since the interval \((-\infty, d]\) is a closed set, the set \( F := f^{-1}((-\infty, d]) = \{ x \in [a, b] : f(x) \leq d \} \) is closed, by Theorem 4.1. Let \( c := \sup F \). Then \( c \) lies in \( F \) and hence \( f(c) \leq d \). It follows that \( a \leq c < b \). We claim \( f(c) = d \). Indeed, if \( f(c) < d \), then by the continuity of \( f \) we could find \( r > 0 \) such that \( c < c + r < b \) and \( f(c + r) < d \). Thus, we would have \( c + r \in F \) and \( c + r > \sup F \). This contradiction shows \( f(c) = d \). \( \square \)

The following theorem shows that a continuous function maps compact sets to compact sets.

Theorem 4.3. Let \( f \) be a continuous function from a metric space \((X, \rho)\) to a metric space \((Y, \tau)\). If \( A \) is a compact subset of \( X \), then \( f(A) \) is compact.

Proof. Suppose that \((G_i)_{i \in I}\) is an open cover of \( f(A) \). Since \( f \) is continuous, \( f^{-1}(G_i) \) is open for every \( i \in I \), by Theorem 4.1. Hence, \((f^{-1}(G_i))_{i \in I}\) is an open cover of \( A \). By the compactness of \( A \), there exists a finite subset \( \{i_1, \ldots, i_m\} \) of \( I \) such that \( A \subseteq \bigcup_{k=1}^{m} f^{-1}(G_{i_k}) \). Consequently, \( f(A) \subseteq \bigcup_{k=1}^{m} G_{i_k} \). This shows that \( f(A) \) is compact. \( \square \)

Theorem 4.4. Let \( A \) be a nonempty compact subset of a metric space \((X, \rho)\). If \( f \) is a continuous function from \( A \) to the real line \( \mathbb{R} \), then \( f \) is bounded and assumes its maximum and minimum.

Proof. By Theorem 4.3, \( f(A) \) is a compact set, and so it is bounded and closed. Let \( t := \inf f(A) \). Then \( t \in \overline{f(A)} = f(A) \). Hence, \( t = \min f(A) \) and \( t = f(a) \) for some \( a \in A \). Similarly, Let \( s := \sup f(A) \). Then \( s \in \overline{f(A)} = f(A) \). Hence, \( s = \max f(A) \) and \( s = f(b) \) for some \( b \in A \). \( \square \)

A function \( f \) from a metric space \((X, \rho)\) to a metric space \((Y, \tau)\) is said to be uniformly continuous on \( X \) if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) (depending on \( \varepsilon \)) such that \( \tau(f(x), f(y)) < \varepsilon \) whenever \( \rho(x, y) < \delta \). Clearly, a uniformly continuous function is continuous.

A function from \((X, \rho)\) to \((Y, \tau)\) is said to be a Lipschitz function if there exists a constant \( C_f \) such that \( \tau(f(x), f(y)) \leq C_f \rho(x, y) \) for all \( x, y \in X \). Clearly, a Lipschitz function is uniformly continuous.
Example. Let $f$ and $g$ be the functions from the interval $(0,1]$ to the real line $\mathbb{R}$ given by $f(x) = x^2$ and $g(x) = 1/x$, $x \in (0,1]$, respectively. Then $f$ is uniformly continuous, while $g$ is continuous but not uniformly continuous.

Theorem 4.5. Let $f$ be a continuous function from a metric space $(X, \rho)$ to a metric space $(Y, \tau)$. If $X$ is compact, then $f$ is uniformly continuous on $X$.

Proof. Let $\varepsilon > 0$ be given. Since $f$ is continuous, for each $x \in X$ there exists $r_x > 0$ such that $\tau(f(x), f(y)) < \varepsilon/2$ for all $y \in B_{r_x}(x)$. Then $(B_{r_x}(x))_{x \in X}$ is an open cover of $X$. Since $X$ is compact, Theorem 3.3 tells us that there exists a Lebesgue number $\delta > 0$ for this open cover. Suppose $y, z \in X$ and $\rho(y, z) < \delta$. Then $\{y, z\} \subseteq B_{r_x}(x)$ for some $x \in X$. Consequently,

$$\tau(f(y), f(z)) \leq \tau(f(y), f(x)) + \tau(f(x), f(z)) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This shows that $f$ is uniformly continuous on $X$. \qed

9