Chapter 5. Integration

§1. The Riemann Integral

Let $a$ and $b$ be two real numbers with $a < b$. Then $[a, b]$ is a closed and bounded interval in $\mathbb{R}$. By a partition $P$ of $[a, b]$ we mean a finite ordered set $\{t_0, t_1, \ldots, t_n\}$ such that

$$a = t_0 < t_1 < \cdots < t_n = b.$$ 

The norm of $P$ is defined by $\|P\| := \max\{t_i - t_{i-1} : i = 1, 2, \ldots, n\}$.

Suppose $f$ is a bounded real-valued function on $[a, b]$. Given a partition $\{t_0, t_1, \ldots, t_n\}$ of $[a, b]$, for each $i = 1, 2, \ldots, n$, let

$$m_i := \inf\{f(x) : t_{i-1} \leq x \leq t_i\} \quad \text{and} \quad M_i := \sup\{f(x) : t_{i-1} \leq x \leq t_i\}.$$ 

The upper sum $U(f, P)$ and the lower sum $L(f, P)$ for the function $f$ and the partition $P$ are defined by

$$U(f, P) := \sum_{i=1}^{n} M_i (t_i - t_{i-1}) \quad \text{and} \quad L(f, P) := \sum_{i=1}^{n} m_i (t_i - t_{i-1}).$$

The upper integral $U(f)$ of $f$ over $[a, b]$ is defined by

$$U(f) := \inf\{U(f, P) : P \text{ is a partition of } [a, b]\}$$

and the lower integral $L(f)$ of $f$ over $[a, b]$ is defined by

$$L(f) := \sup\{L(f, P) : P \text{ is a partition of } [a, b]\}.$$ 

A bounded function $f$ on $[a, b]$ is said to be (Riemann) integrable if $L(f) = U(f)$. In this case, we write

$$\int_{a}^{b} f(x) \, dx = L(f) = U(f).$$

By convention we define

$$\int_{b}^{a} f(x) \, dx := -\int_{a}^{b} f(x) \, dx \quad \text{and} \quad \int_{a}^{a} f(x) \, dx := 0.$$ 

A constant function on $[a, b]$ is integrable. Indeed, if $f(x) = c$ for all $x \in [a, b]$, then $L(f, P) = c(b - a)$ and $U(f, P) = c(b - a)$ for any partition $P$ of $[a, b]$. It follows that

$$\int_{a}^{b} c \, dx = c(b - a).$$
Let \( f \) be a bounded function from \([a, b]\) to \( \mathbb{R} \) such that \( |f(x)| \leq M \) for all \( x \in [a, b] \). Suppose that \( P = \{t_0, t_1, \ldots, t_n\} \) is a partition of \([a, b]\), and that \( P_1 \) is a partition obtained from \( P \) by adding one more point \( t^* \in (t_{i-1}, t_i) \) for some \( i \). The lower sums for \( P \) and \( P_1 \) are the same except for the terms involving \( t_{i-1} \) or \( t_i \). Let \( m_i := \inf \{f(x) : t_{i-1} \leq x \leq t_i\}, \quad m' := \inf \{f(x) : t_{i-1} \leq x \leq t^*\}, \quad \text{and} \quad m'' := \inf \{f(x) : t^* \leq x \leq t_i\} \). Then

\[
L(f, P_1) - L(f, P) = m'(t^* - t_{i-1}) + m''(t_i - t^*) - m_i(t_i - t_{i-1}).
\]

Since \( m' \geq m_i \) and \( m'' \geq m_i \), we have \( L(f, P) \leq L(f, P_1) \). Moreover, \( m' - m \leq 2M \) and \( m'' - m \leq 2M \). It follows that

\[
m'(t^* - t_{i-1}) + m''(t_i - t^*) - m_i(t_i - t_{i-1}) \leq 2M(t_i - t_{i-1}).
\]

Consequently,

\[
L(f, P_1) - 2M\|P\| \leq L(f, P) \leq L(f, P_1).
\]

Now suppose that \( P_N \) is a mesh obtained from \( P \) by adding \( N \) points. An induction argument shows that

\[
L(f, P_N) - 2MN\|P\| \leq L(f, P) \leq L(f, P_N).
\]

Similarly we have

\[
U(f, P_N) \leq U(f, P) \leq U(f, P_N) + 2MN\|P\|.
\]

By the definition of \( L(f) \) and \( U(f) \), for each \( n \in \mathbb{N} \) there exist partitions \( P \) and \( Q \) of \([a, b]\) such that

\[
L(f) - 1/n \leq L(f, P) \quad \text{and} \quad U(f) + 1/n \geq U(f, Q).
\]

Consider the partition \( P \cup Q \) of \([a, b]\). Since \( P \subseteq P \cup Q \) and \( Q \subseteq P \cup Q \), by (1) and (2) we get

\[
L(f, P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q).
\]

It follows that \( L(f) - 1/n \leq U(f) + 1/n \) for all \( n \in \mathbb{N} \). Letting \( n \to \infty \) in the last inequality, we obtain \( L(f) \leq U(f) \).

We are in a position to establish the following criterion for a bounded function to be integrable.
**Theorem 1.1.** A bounded function $f$ on $[a, b]$ is integrable if and only if for each $\varepsilon > 0$ there exists a partition $P$ of $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon.$$ 

**Proof.** Suppose that $f$ is integrable on $[a, b]$. For $\varepsilon > 0$, there exist partitions $P_1$ and $P_2$ such that

$$L(f, P_1) > L(f) - \frac{\varepsilon}{2} \quad \text{and} \quad U(f, P_2) < U(f) + \frac{\varepsilon}{2}.$$ 

For $P := P_1 \cup P_2$ we have

$$L(f) - \frac{\varepsilon}{2} < L(f, P_1) \leq U(f, P) \leq U(f, P_2) < U(f) + \frac{\varepsilon}{2}.$$ 

Since $L(f) = U(f)$, it follows that $U(f, P) - L(f, P) < \varepsilon$.

Conversely, suppose that for each $\varepsilon > 0$ there exists a partition $P$ of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$. Then $U(f, P) < L(f, P) + \varepsilon$. It follows that

$$U(f) \leq U(f, P) < L(f, P) + \varepsilon \leq L(f) + \varepsilon.$$ 

Since $\varepsilon > 0$ is arbitrary, we have $U(f) \leq L(f)$. But $L(f) \leq U(f)$. Therefore $U(f) = L(f)$; that is, $f$ is integrable. \qed

Let $f$ be a bounded real-valued function on $[a, b]$ and let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a, b]$. For each $i = 1, 2, \ldots, n$, choose $\xi_i \in [x_{i-1}, x_i]$. The sum

$$\sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1})$$

is called a **Riemann sum** of $f$ with respect to the partition $P$ and points $\{\xi_1, \ldots, \xi_n\}$.

**Theorem 1.2.** Let $f$ be a bounded real-valued function on $[a, b]$. Then $f$ is integrable on $[a, b]$ if and only if there exists a real number $I$ with the following property: For any $\varepsilon > 0$ there exists some $\delta > 0$ such that

$$\left| \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) - I \right| \leq \varepsilon \quad (3)$$

whenever $P = \{t_0, t_1, \ldots, t_n\}$ is a partition of $[a, b]$ with $\left\| P \right\| < \delta$ and $\xi_i \in [t_{i-1}, t_i]$ for $i = 1, 2, \ldots, n$. If this is the case, then

$$\int_{a}^{b} f(x) \, dx = I.$$
Proof. Let $\varepsilon$ be an arbitrary positive number. Suppose that (3) is true for some partition $P = \{t_0, t_1, \ldots, t_n\}$ of $[a, b]$ and points $\xi_i \in [t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$. Then

$$L(f, P) = \inf \left\{ \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) : \xi_i \in [x_{i-1}, x_i] \text{ for } i = 1, 2, \ldots, n \right\} \geq I - \varepsilon$$

and

$$U(f, P) = \sup \left\{ \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) : \xi_i \in [x_{i-1}, x_i] \text{ for } i = 1, 2, \ldots, n \right\} \leq I + \varepsilon.$$

It follows that $U(f, P) - L(f, P) \leq 2\varepsilon$. By Theorem 1.1 we conclude that $f$ is integrable on $[a, b]$. Moreover, $L(f) = U(f) = I$.

Conversely, suppose that $f$ is integrable on $[a, b]$. Let $M := \sup\{|f(x)| : x \in [a, b]\}$ and $I := L(f) = U(f)$. Given an arbitrary $\varepsilon > 0$, there exists a partition $Q$ of $[a, b]$ such that $L(f, Q) > I - \varepsilon/2$ and $U(f, Q) < I + \varepsilon/2$. Suppose that $Q$ has $N$ points. Let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a, b]$ with $\|P\| < \delta$. Consider the partition $P \cup Q$ of $[a, b]$. By (1) and (2) we have

$$L(f, P) \geq L(f, P \cup Q) - 2MN\delta \quad \text{and} \quad U(f, P) \leq U(f, P \cup Q) + 2MN\delta.$$

But $L(f, P \cup Q) \geq L(f, Q) > I - \varepsilon/2$ and $U(f, P \cup Q) \leq U(f, Q) < I + \varepsilon/2$. Choose $\delta := \varepsilon/(4MN)$. Since $\|P\| < \delta$, we deduce from the foregoing inequalities that

$$I - \varepsilon < L(f, P) \leq U(f, P) < I + \varepsilon.$$

Thus, with $\xi_i \in [t_{i-1}, t_i]$ for $i = 1, 2, \ldots, n$ we obtain

$$I - \varepsilon < L(f, P) \leq \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) \leq U(f, P) < I + \varepsilon.$$

This completes the proof. \qed

**Theorem 1.3.** Let $f$ be a bounded function from a bounded closed interval $[a, b]$ to $\mathbb{R}$. If the set of discontinuities of $f$ is finite, then $f$ is integrable on $[a, b]$.

**Proof.** Let $D$ be the set of discontinuities of $f$. By our assumption, $D$ is finite. So the set $D \cup \{a, b\}$ can be expressed as $\{d_0, d_1, \ldots, d_N\}$ with $a = d_0 < d_1 < \cdots < d_N = b$. Let $M := \sup\{|f(x)| : x \in [a, b]\}$. For an arbitrary positive number $\varepsilon$, we choose $\eta > 0$ such
that $\eta < \varepsilon/(8MN)$ and $\eta < (d_j - d_{j-1})/3$ for all $j = 1, \ldots, N$. For $j = 0, 1, \ldots, N$, let $x_j := d_j - \eta$ and $y_j := d_j + \eta$. Then we have

$$a = d_0 < y_0 < x_1 < d_1 < y_1 \cdots < x_N < d_N = b.$$ 

Let $E$ be the union of the intervals $[d_0, y_0]$, $[x_1, d_1]$, $[d_1, y_1]$, $\ldots$, $[x_{N-1}, d_{N-1}]$, $[d_{N-1}, y_{N-1}]$, and $[x_N, d_N]$. There are $2N$ intervals in total. For $j = 1, \ldots, N$, let $F_j := [y_{j-1}, x_j]$. Further, let $F := \bigcup_{j=1}^N F_j$. The function $f$ is continuous on $F$, which is a finite union of bounded closed intervals. Hence $f$ is uniformly continuous on $F$. There exists some $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon/(2(b-a))$ whenever $x, y \in F$ satisfying $|x - y| < \delta$. For each $j \in \{1, \ldots, N\}$, let $P_j$ be a partition of $F_j$ such that $\|P_j\| < \delta$. Let

$$P := \{a, b\} \cup D \cup \left(\bigcup_{j=1}^N P_j\right).$$

The set $P$ can be arranged as $\{t_0, t_1, \ldots, t_n\}$ with $a = t_0 < t_1 < \cdots < t_n = b$. Consider

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1}),$$

where $M_i := \sup\{f(x) : t_{i-1} \leq x \leq t_i\}$ and $m_i := \inf\{f(x) : t_{i-1} \leq x \leq t_i\}$. Each interval $[t_{i-1}, t_i]$ is either contained in $E$ or in $F$, but not in both. Hence

$$\sum_{i=1}^{n} (M_i - m_i)(t_i - t_{i-1}) = \sum_{[t_{i-1}, t_i] \subseteq E} (M_i - m_i)(t_i - t_{i-1}) + \sum_{[t_{i-1}, t_i] \subseteq F} (M_i - m_i)(t_i - t_{i-1}).$$

There are $2N$ intervals $[t_{i-1}, t_i]$ contained in $E$. Each interval has length $\eta < \varepsilon/(8MN)$. Noting that $M_i - m_i \leq 2M$, we obtain

$$\sum_{[t_{i-1}, t_i] \subseteq E} (M_i - m_i)(t_i - t_{i-1}) \leq 2N(2M)\eta < \frac{\varepsilon}{2}.$$ 

If $[t_{i-1}, t_i] \subseteq F$, then $t_i - t_{i-1} < \delta$; hence $M_i - m_i < \varepsilon/(2(b-a))$. Therefore,

$$\sum_{[t_{i-1}, t_i] \subseteq F} (M_i - m_i)(t_i - t_{i-1}) \leq \frac{\varepsilon}{2(b-a)} \sum_{[t_{i-1}, t_i] \subseteq F} (t_i - t_{i-1}) < \frac{\varepsilon}{2(b-a)}(b-a) = \frac{\varepsilon}{2}.$$ 

From the above estimates we conclude that $U(f, P) - L(f, P) < \varepsilon$. By Theorem 1.1, the function $f$ is integrable on $[a, b]$. 

**Example 1.** Let $[a, b]$ be a closed interval with $a < b$, and let $f$ be the function on $[a, b]$ given by $f(x) = x$. By Theorem 1.3, $f$ is integrable on $[a, b]$. Let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a, b]$ and choose $\xi_i := (t_{i-1} + t_i)/2 \in [t_{i-1}, t_i]$ for $i = 1, 2, \ldots, n$. Then

$$\sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) = \frac{1}{2} \sum_{i=1}^{n} (t_i + t_{i-1})(t_i - t_{i-1}) = \frac{1}{2} \sum_{i=1}^{n} (t_i^2 - t_{i-1}^2) = \frac{1}{2}(t_n^2 - t_0^2) = \frac{1}{2}(b^2 - a^2).$$
By Theorem 1.2 we have
\[ \int_a^b x \, dx = \frac{1}{2} (b^2 - a^2). \]
More generally, for a positive integer \( k \), let \( f_k \) be the function given by \( f_k(x) = x^k \) for \( x \in [a, b] \). Choose
\[ \xi_i := \left( \frac{t_{i-1}^k + k t_{i-1}^{k-1} t_i + \cdots + t_i^k}{k+1} \right)^{1/k}, \quad i = 1, 2, \ldots, n. \]
We have \( t_{i-1} \leq \xi_i \leq t_i \) for \( i = 1, 2, \ldots, n \). Moreover,
\[ \sum_{i=1}^n f_k(\xi_i)(t_i - t_{i-1}) = \frac{1}{k+1} \sum_{i=1}^n (t_{i+1}^k - t_{i-1}^k) = \frac{1}{k+1} (t_{n+1}^k - t_0^k) = \frac{1}{k+1} (b^{k+1} - a^{k+1}). \]
By Theorem 1.2 we conclude that
\[ \int_a^b x^k \, dx = \frac{1}{k+1} (b^{k+1} - a^{k+1}). \]

**Example 2.** Let \( g \) be the function on \([0, 1]\) defined by \( g(x) := \cos(1/x) \) for \( 0 < x \leq 1 \) and \( g(0) := 0 \). The only discontinuity point of \( g \) is 0. By Theorem 1.3, \( g \) is integrable on \([0, 1]\). Note that \( g \) is not uniformly continuous on \((0, 1)\). Indeed, let \( x_n := 1/(2n\pi) \) and \( y_n := 1/(2n\pi + \pi/2) \) for \( n \in \mathbb{N} \). Then \( \lim_{n \to \infty} (x_n - y_n) = 0 \). But
\[ |f(x_n) - f(y_n)| = |\cos(2n\pi) - \cos(2n\pi + \pi/2)| = 1 \quad \forall n \in \mathbb{N}. \]
Hence \( g \) is not uniformly continuous on \((0, 1)\). On the other hand, the function \( u \) given by \( u(x) := 1/x \) for \( 0 < x \leq 1 \) and \( u(0) := 0 \) is not integrable on \([0, 1]\), even though \( u \) is continuous on \((0, 1)\). Theorem 1.3 is not applicable to \( u \), because \( u \) is unbounded.

**Example 3.** Let \( h \) be the function on \([0, 1]\) defined by \( h(x) := 1 \) if \( x \) is a rational number in \([0, 1]\) and \( h(x) := 0 \) if \( x \) is an irrational number in \([0, 1]\). Let \( P = \{t_0, t_1, \ldots, t_n\} \) be a partition of \([0, 1]\). For \( i = 1, \ldots, n \) we have
\[ m_i := \inf\{h(x) : x \in [t_{i-1}, t_i]\} = 0 \quad \text{and} \quad M_i := \sup\{h(x) : x \in [t_{i-1}, t_i]\} = 1. \]
Hence \( L(h, P) = 0 \) and \( U(h, P) = 1 \) for every partition \( P \) of \([0, 1]\). Consequently, \( L(h) = 0 \) and \( U(h) = 1 \). This shows that \( h \) is not Riemann integrable on \([0, 1]\).
§2. Properties of the Riemann Integral

In this section we establish some basic properties of the Riemann integral.

Theorem 2.1. Let \( f \) and \( g \) be integrable functions from a bounded closed interval \([a, b]\) to \( \mathbb{R} \). Then

1. For any real number \( c \), \( cf \) is integrable on \([a, b]\) and \( \int_a^b (cf)(x) \, dx = c \int_a^b f(x) \, dx \);
2. \( f + g \) is integrable on \([a, b]\) and \( \int_a^b (f + g)(x) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \).

Proof. Suppose that \( f \) and \( g \) are integrable functions on \([a, b]\). Write \( I(f) := \int_a^b f(x) \, dx \) and \( I(g) := \int_a^b g(x) \, dx \). Let \( \varepsilon \) be an arbitrary positive number. By Theorem 1.2, there exists some \( \delta > 0 \) such that

\[
\left| \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) - I(f) \right| \leq \varepsilon \quad \text{and} \quad \left| \sum_{i=1}^{n} g(\xi_i)(t_i - t_{i-1}) - I(g) \right| \leq \varepsilon
\]

whenever \( P = \{t_0, t_1, \ldots, t_n\} \) is a partition of \([a, b]\) with \( \|P\| < \delta \) and \( \xi_i \in [t_{i-1}, t_i] \) for \( i = 1, 2, \ldots, n \). It follows that

\[
\left| \sum_{i=1}^{n} (cf)(\xi_i)(t_i - t_{i-1}) - cI(f) \right| = |c| \left| \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) - I(f) \right| \leq |c|\varepsilon.
\]

Hence \( cf \) is integrable on \([a, b]\) and \( \int_a^b (cf)(x) \, dx = c \int_a^b f(x) \, dx \). Moreover,

\[
\left| \sum_{i=1}^{n} (f + g)(\xi_i)(t_i - t_{i-1}) - [I(f) + I(g)] \right|
\leq \left| \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) - I(f) \right| + \left| \sum_{i=1}^{n} g(\xi_i)(t_i - t_{i-1}) - I(g) \right| \leq 2\varepsilon.
\]

Therefore \( f + g \) is integrable on \([a, b]\) and \( \int_a^b (f + g)(x) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx \).

Theorem 2.2. Let \( f \) and \( g \) be integrable functions on \([a, b]\). Then \( fg \) is an integrable function on \([a, b]\).

Proof. Let us first show that \( f^2 \) is integrable on \([a, b]\). Since \( f \) is bounded, there exists some \( M > 0 \) such that \( |f(x)| \leq M \) for all \( x \in [a, b] \). It follows that

\[
\left| [f(x)]^2 - [f(y)]^2 \right| = |f(x) + f(y)||f(x) - f(y)| \leq 2M|f(x) - f(y)| \quad \text{for all} \quad x, y \in [a, b].
\]

We deduce from the above inequality that \( U(f^2, P) - L(f^2, P) \leq 2M [U(f, P) - L(f, P)] \) for any partition \( P \) of \([a, b]\). Let \( \varepsilon > 0 \). Since \( f \) is integrable on \([a, b]\), by Theorem 1.1
there exists a partition $P$ of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon/(2M)$. Consequently, $U(f^2, P) - L(f^2, P) < \varepsilon$. By Theorem 1.1 we conclude that $f^2$ is integrable on $[a, b]$.

Note that $fg = [(f + g)^2 - (f - g)^2]/4$. By Theorem 2.1, $f + g$ and $f - g$ are integrable on $[a, b]$. By what has been proved, both $(f + g)^2$ and $(f - g)^2$ are integrable on $[a, b]$. Using Theorem 2.1 again, we conclude that $fg$ is integrable on $[a, b]$. \qed

**Theorem 2.3.** Let $a, b, c, d$ be real numbers such that $a \leq c < d \leq b$. If a real-valued function $f$ is integrable on $[a, b]$, then $f|_{[c,d]}$ is integrable on $[c, d]$.

**Proof.** Suppose that $f$ is integrable on $[a, b]$. Let $\varepsilon$ be an arbitrary positive number. By Theorem 1.1, there exists a partition $P$ of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$. It follows that $U(f, P \cup \{c, d\}) - L(f, P \cup \{c, d\}) < \varepsilon$. Let $Q := (P \cup \{c, d\}) \cap [c, d]$. Then $Q$ is a partition of $[c, d]$. We have

$$U(f|_{[c,d]}, Q) - L(f|_{[c,d]}, Q) \leq U(f, P \cup \{c, d\}) - L(f, P \cup \{c, d\}) < \varepsilon.$$ 

Hence $f|_{[c,d]}$ is integrable on $[c, d]$. \qed

**Theorem 2.4.** Let $f$ be a bounded real-valued function on $[a, b]$. If $a < c < b$, and if $f$ is integrable on $[a, c]$ and $[c, b]$, then $f$ is integrable on $[a, b]$ and

$$\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.$$

**Proof.** Suppose that $f$ is integrable on $[a, c]$ and $[c, b]$. We write $I_1 := \int_a^c f(x) \, dx$ and $I_2 := \int_c^b f(x) \, dx$. Let $\varepsilon > 0$. By Theorem 1.1, there exist a partition $P_1 = \{s_0, s_1, \ldots, s_m\}$ of $[a, c]$ and a partition $P_2 = \{t_0, t_1, \ldots, t_n\}$ of $[c, b]$ such that

$$U(f, P_1) - L(f, P_1) < \frac{\varepsilon}{2} \quad \text{and} \quad U(f, P_2) - L(f, P_2) < \frac{\varepsilon}{2}.$$ 

Let $P := P_1 \cup P_2 = \{s_0, \ldots, s_{m-1}, t_0, \ldots, t_n\}$. Then $P$ is a partition of $[a, b]$. We have

$$L(f) \geq L(f, P) = L(f, P_1) + L(f, P_2) > U(f, P_1) + U(f, P_2) - \varepsilon \geq I_1 + I_2 - \varepsilon$$

and

$$U(f) \leq U(f, P) = U(f, P_1) + U(f, P_2) < L(f, P_1) + L(f, P_2) + \varepsilon \leq I_1 + I_2 + \varepsilon.$$ 

It follows that

$$\int_a^c f(x) \, dx + \int_c^b f(x) \, dx - \varepsilon < L(f) \leq U(f) < \int_a^c f(x) \, dx + \int_c^b f(x) \, dx + \varepsilon.$$
Since the above inequalities are valid for all \( \varepsilon > 0 \), we conclude that \( f \) is integrable on \([a, b]\) and \( \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx. \)

Let \( a, b, c \) be real numbers in any order, and let \( J \) be a bounded closed interval containing \( a, b, \) and \( c \). If \( f \) is integrable on \( J \), then by Theorems 2.3 and 2.4 we have

\[
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.
\]

**Theorem 2.5.** Let \( f \) and \( g \) be integrable functions on \([a, b]\). If \( f(x) \leq g(x) \) for all \( x \in [a, b] \), then \( \int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx. \)

**Proof.** By Theorem 2.1, \( h := g - f \) is integrable on \([a, b]\). Since \( h(x) \geq 0 \) for all \( x \in [a, b] \), it is clear that \( L(h, P) \geq 0 \) for any partition \( P \) of \([a, b]\). Hence, \( \int_a^b h(x) \, dx = L(h) \geq 0. \) Applying Theorem 2.1 again, we see that

\[
\int_a^b g(x) \, dx - \int_a^b f(x) \, dx = \int_a^b h(x) \, dx \geq 0. \qed
\]

**Theorem 2.6.** If \( f \) is an integrable function on \([a, b]\), then \( \lvert f \rvert \) is integrable on \([a, b]\) and

\[
\int_a^b \lvert f(x) \rvert \, dx \leq \int_a^b \lvert f(x) \rvert \, dx.
\]

**Proof.** Let \( P = \{t_0, t_1, \ldots, t_n\} \) be a partition of \([a, b]\). For each \( i \in \{1, \ldots, n\} \), let \( M_i \) and \( m_i \) denote the supremum and infimum respectively of \( f \) on \([t_{i-1}, t_i]\), and let \( M_i^* \) and \( m_i^* \) denote the supremum and infimum respectively of \( \lvert f \rvert \) on \([t_{i-1}, t_i]\). Then

\[
M_i - m_i = \sup \{f(x) - f(y) : x, y \in [t_{i-1}, t_i]\}
\]

and

\[
M_i^* - m_i^* = \sup \{|f(x)| - |f(y)| : x, y \in [t_{i-1}, t_i]\}.
\]

By the triangle inequality, \( \lvert f(x) \rvert - |f(y)| \leq |f(x) - f(y)| \). Hence \( M_i^* - m_i^* \leq M_i - m_i \) and

\[
\sum_{i=1}^n (M_i^* - m_i^*)(t_i - t_{i-1}) \leq \sum_{i=1}^n (M_i - m_i)(t_i - t_{i-1}).
\]

It follows that \( U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P) \). Let \( \varepsilon \) be an arbitrary positive number. By our assumption, \( f \) is integrable on \([a, b]\). By Theorem 1.1, there exists a partition \( P \) such that \( U(f, P) - L(f, P) < \varepsilon \). Hence \( U(|f|, P) - L(|f|, P) < \varepsilon \). By using Theorem 1.1 again we conclude that \( \lvert f \rvert \) is integrable on \([a, b]\). Furthermore, since \( f(x) \leq |f(x)| \) and \( -f(x) \leq |f(x)| \) for all \( x \in [a, b] \), by Theorem 2.5 we have

\[
\int_a^b f(x) \, dx \leq \int_a^b |f(x)| \, dx \quad \text{and} \quad -\int_a^b f(x) \, dx \leq \int_a^b |f(x)| \, dx.
\]

Therefore \( \int_a^b f(x) \, dx \leq \int_a^b |f(x)| \, dx. \) \qed
§3. Fundamental Theorem of Calculus

In this section we give two versions of the Fundamental Theorem of Calculus and their applications.

Let $f$ be a real-valued function on an interval $I$. A function $F$ on $I$ is called an antiderivative of $f$ on $I$ if $F'(x) = f(x)$ for all $x \in I$. If $F$ is an antiderivative of $f$, then so is $F + C$ for any constant $C$. Conversely, if $F$ and $G$ are antiderivatives of $f$ on $I$, then $G'(x) - F'(x) = 0$ for all $x \in I$. Thus, there exists a constant $C$ such that $G(x) - F(x) = C$ for all $x \in I$. Consequently, $G = F + C$.

The following is the first version of the Fundamental Theorem of Calculus.

**Theorem 3.1.** Let $f$ be an integrable function on $[a, b]$. If $F$ is a continuous function on $[a, b]$ and if $F$ is an antiderivative of $f$ on $(a, b)$, then

$$
\int_a^b f(x) \, dx = F(x) \bigg|_a^b := F(b) - F(a).
$$

**Proof.** Let $\varepsilon > 0$. By Theorem 1.1, there exists a partition $P = \{t_0, t_1, \ldots, t_n\}$ of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$. Since $t_0 = a$ and $t_n = b$ we have

$$
F(b) - F(a) = \sum_{i=1}^n [F(t_i) - F(t_{i-1})].
$$

By the Mean Value Theorem, for each $i \in \{1, \ldots, n\}$ there exists $x_i \in (t_{i-1}, t_i)$ such that

$$
F(t_i) - F(t_{i-1}) = F'(x_i)(t_i - t_{i-1}) = f(x_i)(t_i - t_{i-1}).
$$

Consequently,

$$
L(f, P) \leq F(b) - F(a) = \sum_{i=1}^n f(x_i)(t_i - t_{i-1}) \leq U(f, P).
$$

On the other hand,

$$
L(f, P) \leq \int_a^b f(x) \, dx \leq U(f, P).
$$

Thus both $F(b) - F(a)$ and $\int_a^b f(x) \, dx$ lie in $[L(f, P), U(f, P)]$ with $U(f, P) - L(f, P) < \varepsilon$. Hence

$$
\left| [F(b) - F(a)] - \int_a^b f(x) \, dx \right| < \varepsilon.
$$

Since the above inequality is valid for all $\varepsilon > 0$, we obtain $\int_a^b f(x) \, dx = F(b) - F(a)$. \qed
Example 1. Let \( k \) be a positive integer. Find \( \int_a^b x^k \, dx \).

Solution. We know that the function \( g_k : x \mapsto x^{k+1}/(k + 1) \) is an antiderivative of the function \( f_k : x \mapsto x^k \). By the Fundamental Theorem of Calculus we obtain

\[
\int_a^b x^k \, dx = \frac{x^{k+1}}{k + 1} \bigg|_a^b = \frac{b^{k+1} - a^{k+1}}{k + 1}.
\]

Example 2. Find the integral \( \int_1^2 1/x \, dx \).

Solution. On the interval \((0, \infty)\), the function \( x \mapsto \ln x \) is an antiderivative the function \( x \mapsto 1/x \). By the Fundamental Theorem of Calculus we obtain

\[
\int_1^2 1/x \, dx = \ln x \bigg|_1^2 = \ln 2 - \ln 1 = \ln 2.
\]

This integral can be used to find the limit

\[
\lim_{n \to \infty} \left( \frac{1}{n + 1} + \frac{1}{n + 2} + \cdots + \frac{1}{2n} \right).
\]

Indeed, let \( f(x) := 1/x \) for \( x = [1, 2] \), and let \( t_i = 1 + i/n \) for \( i = 0, 1, \ldots, n \). Then \( P := \{t_0, t_1, \ldots, t_n\} \) is a partition of \([1, 2]\) and

\[
\frac{1}{n + 1} + \frac{1}{n + 2} + \cdots + \frac{1}{2n} = \sum_{i=1}^{n} f(t_i)(t_i - t_{i-1})
\]

is a Riemann sum of \( f \) with respect to \( P \) and points \( \{t_1, \ldots, t_n\} \). By Theorem 1.2 we get

\[
\lim_{n \to \infty} \left( \frac{1}{n + 1} + \frac{1}{n + 2} + \cdots + \frac{1}{2n} \right) = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_i)(t_i - t_{i-1}) = \int_1^2 1/x \, dx = \ln 2.
\]

Example 3. A curve in plane is represented by a continuous mapping \( u = (u_1, u_2) \) from \([a, b]\) to \( \mathbb{R}^2 \). We use \( L(u) \) to denote the length of the curve. Suppose that \( u_1' \) and \( u_2' \) are continuous on \([a, b]\). Then \( u \) is rectifiable. For \( t \in [a, b] \), let \( s(t) \) denote the length of the curve \( u|_{[a, t]} \). It was proved in Theorem 7.1 of Chapter 4 that

\[
s'(t) = \sqrt{[u'_1(t)]^2 + [u'_2(t)]^2}, \quad t \in [a, b].
\]

By Theorem 3.1 (the Fundamental Theorem of Calculus), we obtain

\[
L(u) = s(b) - s(a) = \int_a^b s'(t) \, dt = \int_a^b \sqrt{[u'_1(t)]^2 + [u'_2(t)]^2} \, dt.
\]

The following is the second version of the Fundamental Theorem of Calculus.
Theorem 3.2. Let $f$ be an integrable function on $[a, b]$. Define

$$F(x) := \int_a^x f(t) \, dt, \quad x \in [a, b].$$

Then $F$ is a continuous function on $[a, b]$. Furthermore, if $f$ is continuous at a point $c \in [a, b]$, then $F$ is differentiable at $c$ and

$$F'(c) = f(c).$$

**Proof.** Since $f$ is bounded on $[a, b]$, there exists a constant $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a, b]$. If $x, y \in [a, b]$ and $x < y$, then

$$F(y) - F(x) = \int_a^y f(t) \, dt - \int_a^x f(t) \, dt = \int_x^y f(t) \, dt.$$

Since $-M \leq f(t) \leq M$ for $x \leq t \leq y$, by Theorem 2.5 we have

$$-M(y - x) \leq \int_x^y f(t) \, dt \leq M(y - x).$$

It follows that $|F(y) - F(x)| \leq M|y - x|$. For given $\varepsilon > 0$, choose $\delta = \varepsilon/M$. Then $|y - x| < \delta$ implies $|F(y) - F(x)| \leq M|y - x| < \varepsilon$. This shows that $F$ is continuous on $[a, b]$.

Now suppose that $f$ is continuous at $c \in [a, b)$. Let $h > 0$. By Theorem 2.4 we have

$$\frac{F(c + h) - F(c)}{h} - f(c) = \frac{1}{h} \int_c^{c+h} f(t) \, dt - f(c) = \frac{1}{h} \int_c^{c+h} [f(t) - f(c)] \, dt.$$

Let $\varepsilon > 0$ be given. Since $f$ is continuous at $c$, there exists some $\delta > 0$ such that $|f(t) - f(c)| \leq \varepsilon$ whenever $c \leq t \leq c + \delta$. Therefore, if $0 < h < \delta$, then

$$\left| \frac{F(c + h) - F(c)}{h} - f(c) \right| = \left| \frac{1}{h} \int_c^{c+h} [f(t) - f(c)] \, dt \right| \leq \frac{1}{h} \int_c^{c+h} |f(t) - f(c)| \, dt \leq \varepsilon.$$

Consequently,

$$\lim_{h \to 0^+} \frac{F(c + h) - F(c)}{h} = f(c).$$

Similarly, if $f$ is continuous at $c \in (a, b]$, then

$$\lim_{h \to 0^-} \frac{F(c + h) - F(c)}{h} = f(c).$$

This completes the proof of the theorem.
**Example 4.** Let $f$ be a continuous function on $[a, b]$, and let $F(x) := \int_x^b f(t) \, dt$ for each $x \in [a, b]$. Then we have

$$F(x) = \int_x^b f(t) \, dt = -\int_b^x f(t) \, dt.$$  

By Theorem 3.2, $F$ is differentiable on $[a, b]$ and $F'(x) = -f(x)$ for $a \leq x \leq b$.

**Example 5.** Let $F(x) := \int_{-x}^x \sqrt{4 + t^2} \, dt$, $x \in \mathbb{R}$. Find $F'(x)$ for $x \in \mathbb{R}$.

*Solution.* We have

$$F(x) = \int_{-x}^0 \sqrt{4 + t^2} \, dt + \int_0^x \sqrt{4 + t^2} \, dt = -\int_0^{-x} \sqrt{4 + t^2} \, dt + \int_0^x \sqrt{4 + t^2} \, dt.$$  

By using the chain rule and Theorem 3.2 we obtain

$$F'(x) = \sqrt{4 + x^2} + 2x\sqrt{4 + x^4}.$$  

**Example 6.** Let $G(x) := \int_2^x x \cos(t^3) \, dt$, $x \in \mathbb{R}$. Find $G''(x)$ for $x \in \mathbb{R}$.

*Solution.* We have $G(x) = x \int_2^x \cos(t^3) \, dt$. By Theorem 3.2 and the product rule for differentiation, we obtain

$$G'(x) = \int_2^x \cos(t^3) \, dt + x \cos(x^3).$$  

Taking derivative once more, we get

$$G''(x) = \cos(x^3) + \cos(x^3) + x[-\sin(x^3)](3x^2) = 2 \cos(x^3) - 3x^3 \sin(x^3).$$

§4. Indefinite Integrals

An antiderivative of a function $f$ is also called an **indefinite integral** of $f$. It is customary to denote an indefinite integral of $f$ by

$$\int f(x) \, dx.$$  

For example, for $\mu \in \mathbb{R} \setminus \{-1\}$ we have

$$\int x^{\mu} \, dx = \frac{x^{\mu+1}}{\mu + 1} + C, \quad x \in (0, \infty).$$
If $\mu \in \mathbb{N}_0$, then the above formula is valid for all $x \in \mathbb{R}$. If $\mu \in \mathbb{Z}$ and $\mu \leq -2$, then the formula holds for $x \in (-\infty, 0) \cup (0, \infty)$. For $\mu = -1$ we have

$$\int \frac{1}{x} \, dx = \ln |x| + C, \quad x \in (-\infty, 0) \cup (0, \infty).$$

The following formulas for integration are easily derived from the corresponding formulas for differentiation:

$$\int e^x \, dx = e^x + C, \quad x \in (-\infty, \infty).$$

$$\int \cos x \, dx = \sin x + C, \quad x \in (-\infty, \infty),$$

$$\int \sin x \, dx = -\cos x + C, \quad x \in (-\infty, \infty),$$

$$\int \frac{1}{1 + x^2} \, dx = \arctan x + C \quad x \in (-\infty, \infty),$$

$$\int \frac{1}{\sqrt{1 - x^2}} \, dx = \arcsin x + C \quad x \in (-1, 1).$$

If $F_1$ and $F_2$ are differentiable functions on an interval, and if $F'_1 = f_1$ and $F'_2 = f_2$, then for $c_1, c_2 \in \mathbb{R}$ we have

$$[c_1 F_1 + c_2 F_2]' = c_1 F'_1 + c_2 F'_2 = c_1 f_1 + c_2 f_2.$$

It follows that

$$\int [c_1 f_1(x) + c_2 f_2(x)] \, dx = c_1 \int f_1(x) \, dx + c_2 \int f_2(x) \, dx.$$

Now let $u$ and $v$ be differentiable functions on an interval. By the product rule for differentiation we have

$$(uv)' = u'v + uv'.$$

From this we deduce the following formula for integration by parts:

$$\int u(x)v'(x) \, dx = u(x)v(x) - \int u'(x)v(x) \, dx.$$

It can also be written as

$$\int u \, dv = uv - \int v \, du.$$
Example 1. Find $\int x^2 e^x \, dx$.

Solution. By integration by parts we have

$$\int x^2 e^x \, dx = \int x^2 d(e^x) = x^2 e^x - \int e^x d(x^2) = x^2 e^x - 2 \int xe^x \, dx.$$ 

By using integration by parts again we obtain

$$\int xe^x \, dx = \int xd(e^x) = xe^x - \int e^x dx = xe^x - e^x + C.$$ 

Therefore

$$\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C.$$ 

In general, if $p$ is a polynomial, then

$$\int p(x)e^x \, dx = \int p(x)d(e^x) = p(x)e^x - \int p'(x)e^x \, dx,$$

where the degree of $p'$ is one less than that of $p$. Thus the integral $\int p(x)e^x$ can be computed by using integration by parts repeatedly. This method also applies to the integrals $\int p(x)\sin x \, dx$ and $\int p(x)\cos x \, dx$.

Example 2. Find $\int x \ln x \, dx$.

Solution. Integration by parts gives

$$\int x \ln x \, dx = \int \ln x \, d(x^2/2) = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \, d(ln x)$$

$$= \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \frac{1}{x} \, dx = \frac{x^2}{2} \ln x - \frac{1}{4}x^2 + C.$$ 

In general, if $p$ is a polynomial given by $p(x) = \sum_{k=0}^{n} a_kx^k$, then

$$\int p(x) \, dx = \sum_{k=0}^{n} a_k \frac{x^{k+1}}{k+1} + C.$$

Let $s(x) := \sum_{k=0}^{n} a_kx^{k+1}/(k+1)$. By using integration by parts we get

$$\int p(x) \ln x \, dx = \int \ln x d(s(x)) = s(x) \ln x - \int s(x) \, d(ln x) = s(x) \ln x - \int \frac{s(x)}{x} \, dx.$$ 

This method also applies to the integral $\int p(x) \arctan x \, dx$. 


Let $u$ be a differentiable function from an interval $I$ to an interval $J$, and let $F$ be a differentiable function from $J$ to $\mathbb{R}$. Suppose $F' = f$. By the chain rule the composition $F \circ u$ is differentiable on $I$ and

$$(F \circ u)'(x) = F'(u(x))u'(x) = f(u(x))u'(x), \quad x \in I.$$ 

Thus we have the following formula for change of variables in an integral:

$$\int f(u(x))u'(x) \, dx = F(u(x)) + C.$$ 

**Example 3.** Find $\int \sin^2 x \cos x \, dx$.

*Solution.* Let $u := \sin x$. Then $du = \cos x \, dx$. Hence

$$\int \sin^2 x \cos x \, dx = \int u^2 \, du = \frac{1}{3} u^3 + C = \frac{1}{3} \sin^3 x + C.$$ 

We can use this integral together with the identity $\sin^2 x + \cos^2 x = 1$ to find the integral $\int \cos^3 x \, dx$:

$$\int \cos^3 x \, dx = \int \cos^2 x \cos x \, dx = \int (1 - \sin^2 x) \cos x \, dx$$

$$= \int \cos x \, dx - \int \sin^2 x \cos x \, dx = \sin x - \frac{1}{3} \sin^3 x + C.$$ 

For integrals involving sine and cosine, the following double angle formulas will be useful:

$$\sin(2x) = 2 \sin x \cos x,$$

$$\cos(2x) = \cos^2 x - \sin^2 x.$$ 

The second formula together with the identity $\sin^2 x + \cos^2 x = 1$ gives

$$\sin^2 x = \frac{1 - \cos(2x)}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos(2x)}{2}.$$ 

Thus we have

$$\int \sin^2 x \, dx = \int \frac{1}{2} \, dx - \frac{1}{2} \int \cos(2x) \, dx = \frac{x}{2} - \frac{1}{4} \sin(2x) + C.$$ 

In general, for nonnegative integers $m$ and $n$, the integral

$$\int \sin^m x \cos^n x \, dx$$
can be calculated as follows: (1) If \( m \) is odd, use the substitution \( u = \cos x \) and the identity \( \sin^2 x = 1 - \cos^2 x \). (2) If \( n \) is odd, use the substitution \( u = \sin x \) and the identity \( \cos^2 x = 1 - \sin^2 x \). (3) If both \( m \) and \( n \) are even, use \( \sin^2 x = \frac{1 - \cos(2x)}{2} \) and \( \cos^2 x = \frac{1 + \cos(2x)}{2} \) to reduce the exponents of sine and cosine.

**Example 4.** Find the following integrals:

\[
\int \tan x \, dx, \quad \int \cot x \, dx, \quad \int \sec x \, dx, \quad \int \csc x \, dx.
\]

**Solution.** For the first integral we use the substitution \( u = \cos x \) and get

\[
\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = - \int \frac{1}{u} \, du = - \ln |u| + C = - \ln |\cos x| + C = \ln |\sec x| + C.
\]

Similarly,

\[
\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx = \int \frac{d(\sin x)}{\sin x} = \ln |\sin x| + C.
\]

In order to find \( \int \sec x \, dx \), we observe that

\[
\frac{d}{dx}(\sec x + \tan x) = \sec x \tan x + \sec^2 x = \sec x(\tan x + \sec x).
\]

It follows that

\[
\int \sec x \, dx = \int \frac{d(\sec x + \tan x)}{\sec x + \tan x} = \ln |\sec x + \tan x| + C.
\]

Similarly,

\[
\int \csc x \, dx = - \int \frac{d(\csc x + \cot x)}{\csc x + \cot x} = - \ln |\csc x + \cot x| + C.
\]

**Example 5.** For \( a > 0 \), calculate the following integrals:

\[
\int \frac{1}{\sqrt{x^2 + a^2}} \, dx \quad \text{and} \quad \int \frac{1}{\sqrt{x^2 - a^2}} \, dx.
\]

**Solution.** For the first integral we let \( x = a \tan t \) for \(-\pi/2 < t < \pi/2\). Then \( \sec t > 0 \) and \( x^2 + a^2 = a^2(\tan^2 t + 1) = a^2 \sec^2 t \). Hence

\[
\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \int \frac{a \sec^2 t}{a \sec t} \, dt = \int \sec t \, dt = \ln(\tan t + \sec t).
\]

But \( \sec t = \sqrt{\tan^2 t + 1} \). Consequently,

\[
\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln \left(\frac{x}{a} + \sqrt{\frac{x^2}{a^2} + 1}\right) + C = \ln \left(x + \sqrt{x^2 + a^2}\right) + C_1,
\]

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where $C_1 = C - \ln a$. Similarly,

$$
\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln|x + \sqrt{x^2 - a^2}| + C, \quad |x| > a.
$$

Let us consider $\int \sqrt{\alpha x^2 + \beta} \, dx$, where $\alpha, \beta \in \mathbb{R}$. Integrating by parts, we obtain

$$
\int \sqrt{\alpha x^2 + \beta} \, dx = x\sqrt{\alpha x^2 + \beta} - \int \frac{\alpha x^2}{\sqrt{\alpha x^2 + \beta}} \, dx.
$$

Note that

$$
\frac{\alpha x^2}{\sqrt{\alpha x^2 + \beta}} = \frac{\alpha x^2 + \beta - \beta}{\sqrt{\alpha x^2 + \beta}} = \sqrt{\alpha x^2 + \beta} - \frac{\beta}{\sqrt{\alpha x^2 + \beta}}.
$$

Hence

$$
\int \sqrt{\alpha x^2 + \beta} \, dx = x\sqrt{\alpha x^2 + \beta} - \int \sqrt{\alpha x^2 + \beta} \, dx + \int \frac{\beta}{\sqrt{\alpha x^2 + \beta}} \, dx.
$$

It follows that

$$
\int \sqrt{\alpha x^2 + \beta} \, dx = \frac{1}{2} x\sqrt{\alpha x^2 + \beta} + \frac{\beta}{2} \int \frac{1}{\sqrt{\alpha x^2 + \beta}} \, dx.
$$

In particular, we get

$$
\int \sqrt{x^2 + a^2} \, dx = \frac{1}{2} x\sqrt{x^2 + a^2} + \frac{a^2}{2} \ln(x + \sqrt{x^2 + a^2}) + C
$$

and

$$
\int \sqrt{x^2 - a^2} \, dx = \frac{1}{2} x\sqrt{x^2 - a^2} - \frac{a^2}{2} \ln|x + \sqrt{x^2 - a^2}| + C
$$

For $a > 0$, a simple substitution gives

$$
\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C, \quad -a < x < a.
$$

Therefore,

$$
\int \sqrt{a^2 - x^2} \, dx = \frac{1}{2} x\sqrt{a^2 - x^2} + \frac{a^2}{2} \arcsin \frac{x}{a} + C, \quad -a < x < a.
$$

A rational function has the form $p(x)/s(x)$, where $p$ and $s$ are polynomials. There exist unique polynomials $q$ and $r$ such that $p(x) = q(x)s(x) + r(x)$, where the degree of $r$ is less than the degree of $s$. It follows that

$$
\frac{p(x)}{s(x)} = q(x) + \frac{r(x)}{s(x)}.
$$
In order to find \( \int \frac{r(x)}{s(x)} \, dx \), we decompose \( \frac{r(x)}{s(x)} \) as the sum of terms of the following type:

\[
\frac{c_1}{x - \alpha} + \cdots + \frac{c_n}{(x - \alpha)^n} + \frac{d_1 + e_1 x}{(x - \beta)^2 + \gamma^2} + \cdots + \frac{d_m + e_m x}{[(x - \beta)^2 + \gamma^2]^m}.
\]

**Example 6.** For \( b, c, \lambda, \mu \in \mathbb{R} \), find the integral

\[
\int \frac{\lambda x + \mu}{x^2 + bx + c} \, dx.
\]

**Solution.** We may write

\[
\int \frac{\lambda x + \mu}{x^2 + bx + c} \, dx = \int \frac{\lambda}{2} \frac{2x + b}{x^2 + bx + c} \, dx + \int \frac{\mu - b\lambda/2}{x^2 + bx + c} \, dx.
\]

Clearly,

\[
\int \frac{\lambda}{2} \frac{2x + b}{x^2 + bx + c} \, dx = \frac{\lambda}{2} \ln |x^2 + bx + c| + C.
\]

So it remains to find the integral \( \int dx/(x^2 + bx + c) \). There are three possible cases:

- \( b^2 - 4c > 0 \), \( b^2 - 4c = 0 \), and \( b^2 - 4c < 0 \). If \( b^2 - 4c > 0 \), then \( x^2 + bx + c = (x - \alpha)(x - \beta) \), where \( \alpha \) and \( \beta \) are distinct real numbers. In this case,

\[
\int \frac{1}{(x - \alpha)(x - \beta)} \, dx = \int \frac{1}{\alpha - \beta} \left( \frac{1}{x - \alpha} - \frac{1}{x - \beta} \right) \, dx = \frac{1}{\alpha - \beta} \left[ \ln |x - \alpha| - \ln |x - \beta| \right] + C.
\]

If \( b^2 - 4c = 0 \), then \( x^2 + bx + c = (x - \alpha)^2 \), where \( \alpha = -b/2 \). In this case,

\[
\int \frac{1}{(x - \alpha)^2} \, dx = -\frac{1}{x - \alpha} + C.
\]

Finally, if \( b^2 - 4c < 0 \), we have \( x^2 + bx + c = (x + b/2)^2 + \gamma^2 \), where \( \gamma = \sqrt{c - b^2}/4 \). Thus

\[
\int \frac{1}{x^2 + bx + c} \, dx = \int \frac{1}{(x + b/2)^2 + \gamma^2} = \frac{1}{\gamma} \arctan \frac{x + b/2}{\gamma} + C.
\]

### §5. Definite Integrals

As an application of the Fundamental Theorem of Calculus, we establish the following formula of integration by parts.

**Theorem 5.1.** If \( u \) and \( v \) are continuous functions on \([a, b]\) that are differentiable on \((a, b)\), and if \( u' \) and \( v' \) are integrable on \([a, b]\), then

\[
\int_a^b u(x)v'(x) \, dx + \int_a^b u'(x)v(x) \, dx = u(b)v(b) - u(a)v(a).
\]
Proof. Let $F := uv$. Then $F'(x) = u'(x)v(x) + u(x)v'(x)$ for $x \in (a, b)$. By Theorem 3.1 we have

$$\int_a^b F'(x) \, dx = F(b) - F(a) = u(b)v(b) - u(a)v(a).$$

\[\square\]

Example 1. Find $\int_0^1 x \ln x \, dx$.

Solution. For $k = 1, 2, \ldots$, let $f_k(x) := x^k \ln x$, $x > 0$. Then $f_k$ is continuous on $(0, \infty)$. Moreover,

$$\lim_{x \to 0^+} x^k \ln x = \lim_{x \to 0^+} \frac{\ln x}{(1/x)^k} = \lim_{y \to +\infty} \frac{\ln(1/y)}{y^k} = \lim_{y \to +\infty} -\frac{\ln y}{y^k} = 0.$$

Thus, by defining $f_k(0) := 0$, $f_k$ is extended to a continuous function on $[0, \infty)$. Integration by parts gives

$$\int_0^1 x \ln x \, dx = \frac{x^2}{2} \ln x \bigg|_0^1 - \int_0^1 \frac{x^2}{2} \, dx = -\frac{1}{4} x^2 \bigg|_0^1 = -\frac{1}{4}.$$

Now let us consider the integral $\int_0^1 \ln x \, dx$. The function $f_0 : x \mapsto \ln x$ is unbounded on $(0, 1)$. So this is an improper integral. We define

$$\int_0^1 \ln x \, dx := \lim_{a \to 0^+} \int_a^1 \ln x \, dx.$$

Integration by parts gives

$$\int_a^1 \ln x \, dx = x \ln x \bigg|_a^1 - \int_a^1 dx = -a \ln a - (1 - a).$$

Consequently,

$$\int_0^1 \ln x \, dx = \lim_{a \to 0^+} [-a \ln a - (1 - a)] = -1.$$

Example 2. For $n = 0, 1, 2, \ldots$, let

$$I_n := \int_0^1 (1 - x^2)^n \, dx.$$

Find $I_n$.

Solution. We have $I_0 = 1$. For $n \geq 1$, integrating by parts, we get

$$I_n = \int_0^1 (1 - x^2)^n \, dx = x(1 - x^2)^n \bigg|_0^1 - \int_0^1 x d((1 - x^2)^n) = 2n \int_0^1 x^2 (1 - x^2)^{n-1} \, dx.$$
We may write \[ x^2(1 - x^2)^{n-1} = [1 - (1 - x^2)](1 - x^2)^{n-1} = (1 - x^2)^n - (1 - x^2)^n. \] Hence
\[
I_n = 2n \int_0^1 (1 - x^2)^{n-1} \, dx - 2n \int_0^1 (1 - x^2)^n \, dx = 2nI_{n-1} - 2nI_n.
\]
It follows that \((2n + 1)I_n = 2nI_{n-1}\). Thus \(I_1 = 2/3\). In general,
\[
I_n = \frac{2n}{2n+1} I_{n-1} = \frac{2n}{2n+1} \frac{2n-2}{2n-1} \ldots \frac{2}{3} = \prod_{k=1}^{n} \frac{2k}{2k+1}.
\]

As another application of the Fundamental Theorem of Calculus, we give the following formula for change of variables in a definite integral.

**Theorem 5.2.** Let \(u\) be a differentiable function on \([a, b]\) such that \(u'\) is integrable on \([a, b]\). If \(f\) is continuous on \(I := u([a, b])\), then
\[
\int_a^b f(u(t))u'(t) \, dt = \int_{u(a)}^{u(b)} f(x) \, dx.
\]

**Proof.** Since \(u\) is continuous, \(I = u([a, b])\) is a closed and bounded interval. Also, since \(f \circ u\) is continuous and \(u'\) is integrable on \([a, b]\), the function \((f \circ u)u'\) is integrable on \([a, b]\). If \(I = u([a, b])\) is a single point, then \(u\) is constant on \([a, b]\). In this case \(u'(t) = 0\) for all \(t \in [a, b]\) and both integrals above are zero. Otherwise, for \(x \in I\) define
\[
F(x) := \int_{u(a)}^{x} f(s) \, ds.
\]
Since \(f\) is continuous on \(I\), \(F'(x) = f(x)\) for all \(x \in I\), by Theorem 3.2. By the chain rule we have
\[
(F \circ u)'(t) = F'(u(t))u'(t) = f(u(t))u'(t), \quad t \in [a, b].
\]
Therefore by Theorem 3.1 we obtain
\[
\int_a^b f(u(t))u'(t) \, dt = (F \circ u)(b) - (F \circ u)(a) = F(u(b)) - F(u(a)) = \int_{u(a)}^{u(b)} f(x) \, dx.
\]
Example 3. For $a > 0$, find $\int_0^a \sqrt{a^2 - x^2} \, dx$.

Solution. Let $x = a \sin t$. When $t = 0$, $x = 0$. When $t = \pi/2$, $x = a$. By Theorem 5.2 we get

$$\int_0^a \sqrt{a^2 - x^2} \, dx = \int_0^{\pi/2} a^2 (1 - \sin^2 t) \cos t \, dt = \int_0^a a^2 \sqrt{\cos^2 t} \cos t \, dt.$$ 

Since $\cos t \geq 0$ for $0 \leq t \leq \pi/2$, we have $\sqrt{\cos^2 t} = \cos t$. Thus

$$\int_0^a \sqrt{a^2 - x^2} \, dx = a^2 \int_0^{\pi/2} \cos^2 t \, dt = a^2 \int_0^{\pi/2} \frac{1 + \cos(2t)}{2} \, dt = \frac{\pi a^2}{4}.$$

Example 4. Let $a > 0$. Suppose that $f$ is a continuous function on $[-a, a]$. Prove the following statements.

(1) If $f$ is an even function, i.e., $f(-x) = f(x)$ for all $x \in [0, a]$, then

$$\int_{-a}^a f(x) \, dx = 2 \int_0^a f(x) \, dx.$$

(2) If $f$ is an odd function, i.e., $f(-x) = -f(x)$ for all $x \in [0, a]$, then $\int_{-a}^a f(x) \, dx = 0$.

Proof. We have

$$\int_{-a}^a f(x) \, dx = \int_{-a}^0 f(x) \, dx + \int_0^a f(x) \, dx.$$

In the integral $\int_{-a}^0 f(x) \, dx$ we make the change of variables: $x = -t$. When $t = a$, $x = -a$; when $t = 0$, $x = 0$. By Theorem 5.2 we get

$$\int_{-a}^0 f(x) \, dx = \int_a^0 f(-t) \, dt = -\int_a^0 f(-t) \, dt = \int_0^a f(-t) \, dt.$$

It follows that

$$\int_{-a}^a f(x) \, dx = \int_0^a f(-t) \, dt + \int_0^a f(t) \, dt = \int_0^a [f(-t) + f(t)] \, dt.$$

If $f$ is an even function, then $f(-t) = f(t)$ for all $t \in [0, a]$; hence

$$\int_{-a}^a f(x) \, dx = 2 \int_0^a f(t) \, dt = 2 \int_0^a f(x) \, dx.$$

If $f$ is an odd function, then $f(-t) = -f(t)$ for all $t \in [0, a]$; hence

$$\int_{-a}^a f(x) \, dx = 0.$$