Linear Approximations

Let $f$ be a function of two variables $x$ and $y$ defined in a neighborhood of $(a, b)$. The linear function

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linearization** of $f$ at $(a, b)$ and the approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linear approximation** of $f$ at $(a, b)$.

The function $f$ is said to be **differentiable** if

$$\lim_{x \to a, y \to b} \frac{|f(x, y) - L(x, y)|}{\sqrt{(x - a)^2 + (y - b)^2}} = 0.$$ 

**Theorem.** If the partial derivatives $f_x$ and $f_y$ exist in a neighborhood of $(a, b)$ and are continuous at $(a, b)$, then $f$ is differentiable at $(a, b)$. 

1
Example. Show that $f(x, y) = xe^{xy}$ is differentiable at $(1, 0)$ and find its linearization there. Then use it to approximate $f(1.1, -0.1)$.

Solution. The partial derivatives are

$$f_x(x, y) = e^{xy} + xye^{xy}, \quad f_y(x, y) = x^2 e^{xy},$$

$$f_x(1, 0) = 1, \quad f_y(1, 0) = 1.$$

Both $f_x$ and $f_y$ are continuous, so $f$ is differentiable everywhere. The linearization is

$$L(x, y) = f(1, 0) + f_x(1, 0)(x - 1) + f_y(1, 0)(y - 0)$$

$$= 1 + 1(x - 1) + 1 \cdot y = x + y.$$

The corresponding linear approximation is

$$xe^{xy} \approx x + y.$$

It follows that $f(1.1, -0.1) \approx 1.1 - 0.1 = 1$. In comparison, $f(1.1, -0.1) = 1.1e^{-0.11} \approx 0.98542$. 

2
Let $f$ be a continuous function on an open domain $G$. Suppose that $P(a, b)$ is a point in $G$. Let $h$ and $k$ be real numbers such that the line segment joining $P(a, b)$ and $Q(a + h, b + k)$ lies inside $G$. The line segment $PQ$ is represented by the parametric equations

$$x = a + th, \ y = b + tk, \ 0 \leq t \leq 1.$$ 

Let $F$ be the function defined by

$$F(t) = f(a + th, b + tk), \ 0 \leq t \leq 1.$$ 

Then $F$ is a continuous function on $[0, 1]$.

Suppose that $f$ has continuous partial derivatives up to order 2. Then $F'$ and $F''$ are continuous on $[0, 1]$. 

3
Taylor’s Formula for Functions of Two Variables

By Taylor’s theorem we have

\[ F(1) = F(0) + F'(0)(1 - 0) + \frac{F''(c)}{2!}(1 - 0)^2 \]

for some \( c \in (0, 1) \).

Recall that \( x = a + th \) and \( y = b + tk \). By the chain rule we obtain

\[ F'(t) = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = hf_x + kf_y. \]

Consequently,

\[ F''(t) = \frac{\partial}{\partial x} (hf_x + kf_y)h + \frac{\partial}{\partial y} (hf_x + kf_y)k \]
\[ = h^2 f_{xx} + hkf_{yx} + khf_{xy} + k^2 f_{yy} \]
\[ = h^2 f_{xx} + 2hk f_{yx} + k^2 f_{yy}. \]
Let \( x = a + h \) and \( y = b + k \). The first Taylor polynomial of \( f \) at \((a, b)\) is given by
\[
T_1(x, y) = F(0) + F'(0)(1 - 0)
= f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).
\]
Thus, \( T_1(x, y) \) is just the linearization of \( f \) at \((a, b)\).

Let \( R_1(x, y) = f(x, y) - T_1(x, y) \) be the remainder.

With \( h = x - a \) and \( k = y - b \) we have
\[
R_1(x, y) = \frac{F''(c)}{2!}(1 - 0)^2
= \frac{1}{2} [h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}] \big|_{(a+ch, b+ck)},
\]
where \( 0 < c < 1 \).

The second Taylor polynomial of \( f \) at \((a, b)\) is given by
\[
T_2(x, y) = f(a, b) + f_x(a, b)h + f_y(a, b)k
+ \frac{1}{2} [f_{xx}(a, b)h^2 + 2f_{xy}(a, b)hk + f_{yy}(a, b)k^2].
\]
Example. Let $f(x, y) = e^x \sin(x - y), (x, y) \in \mathbb{R}^2$.

(a) Find the linearization of $f$ at the point $(0, 0)$ and the corresponding remainder.

(b) Find the second Taylor polynomial of $f$ at $(0, 0)$.

Solution. We have

\[
\begin{align*}
    f_x &= e^x \sin(x - y) + e^x \cos(x - y), \\
    f_y &= -e^x \cos(x - y), \\
    f_{xx} &= 2e^x \cos(x - y), \\
    f_{xy} &= -e^x \cos(x - y) + e^x \sin(x - y), \\
    f_{yy} &= -e^x \sin(x - y).
\end{align*}
\]

Hence, $f(0, 0) = 0$, $f_x(0, 0) = 1$, $f_y(0, 0) = -1$. The linearization of $f$ at the point $(0, 0)$ is

\[
L(x, y) = f(0, 0) + f_x(0, 0)x + f_y(0, 0)y = x - y.
\]
The remainder is

\[ R(x, y) = \frac{1}{2} \left[ 2e^{cx} \cos(cx - cy)x^2 
\right.

\[ + 2(e^{cx} \sin(cx - cy) - e^{cx} \cos(cx - cy))xy 
\]

\[ - e^{cx} \sin(cx - cy)y^2 \], \]

where \(0 < c < 1\).

We have \(f_{xx}(0, 0) = 2\), \(f_{xy}(0, 0) = -1\), and \(f_{yy}(0, 0) = 0\). Consequently, the second Taylor polynomial of \(f\) at \((0, 0)\) is

\[ T_2(x, y) = x - y + \frac{1}{2}(2x^2 - 2xy) = x - y + x^2 - xy. \]