Tangent Planes to Surfaces

Let $F$ be a differentiable function of three variables $x$, $y$, and $z$. For a constant $k$, the equation $F(x, y, z) = k$ represents a surface $S$ in space. For example, the equation $x^2 + y^2 + z^2 = 9$ represents the sphere with radius 3 and center at the origin.

Let $P(x_0, y_0, z_0)$ be a point on $S$. We wish to find the tangent plane to the surface $S$ at $P$. Let $C$ be a smooth curve that lies on the surface $S$ and passes through the point $P$. The curve $C$ is described by a differentiable vector function

$$r(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad a < t < b.$$ 

There exists a parameter $t_0 \in (a, b)$ such that

$$r(t_0) = (x_0, y_0, z_0).$$
Since the curve $C$ lies on the surface $S$, any point $(x(t), y(t), z(t))$ must satisfy the equation of $S$, that is,

$$F(x(t), y(t), z(t)) = k, \quad a < t < b.$$  

We use the chain rule to differentiate both sides of the above equation as follows:

$$\frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} + \frac{\partial F}{\partial z} \frac{dz}{dt} = 0.$$

Recall that $\nabla F = \frac{\partial F}{\partial x} \mathbf{i} + \frac{\partial F}{\partial y} \mathbf{j} + \frac{\partial F}{\partial z} \mathbf{k}$. Moreover, $r'(t) = \frac{dx}{dt} \mathbf{i} + \frac{dy}{dt} \mathbf{j} + \frac{dz}{dt} \mathbf{k}$. Thus we obtain

$$\nabla F \cdot r'(t) = 0, \quad a < t < b.$$  

In particular, for $t = t_0$ we have

$$\nabla F(x_0, y_0, z_0) \cdot r'(t_0) = 0.$$  

Note that $r'(t_0)$ is a tangent vector to the curve $C$ at $P$.  

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Normal Vectors

The above equation shows that the gradient vector at $P$, $\nabla F(x_0, y_0, z_0)$, is perpendicular to the tangent vector $\mathbf{r}'(t_0)$ to any curve $C$ on $S$ that passes through $P$.

**Conclusion.** If the gradient $\nabla F(x_0, y_0, z_0) \neq 0$, then $\nabla F(x_0, y_0, z_0)$ is a normal vector to the tangent plane to the surface $F(x, y, z) = k$ at $(x_0, y_0, z_0)$. 
Equations of Tangent Planes

Let $S$ be the surface represented by the equation $F(x, y, z) = k$, where $k$ is a constant and $F$ is a differentiable function. For a point $P = (x_0, y_0, z_0)$ on $S$, let 

$$a = \frac{\partial F}{\partial x} \bigg|_P, \quad b = \frac{\partial F}{\partial y} \bigg|_P, \quad c = \frac{\partial F}{\partial z} \bigg|_P.$$ 

If $\nabla F(x_0, y_0, z_0) = (a, b, c) \neq 0$, then the tangent plane to the surface $S$ at the point $P(x_0, y_0, z_0)$ has an equation 

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$ 

If a surface $S$ is represented by the equation $z = f(x, y)$, then we may rewrite it as $f(x, y) - z = 0.$
Suppose \( f \) has continuous partial derivatives. The tangent plane to the surface \( z = f(x, y) \) at the point \( P(x_0, y_0, z_0) \) has an equation

\[
z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).
\]

**Example.** Find the tangent plane to the paraboloid \( z = 2x^2 + y^2 \) at the point \( (1, 1, 3) \).

**Solution.** Let \( f(x, y) = 2x^2 + y^2 \). Then

\[
f_x(x, y) = 4x, \quad f_y(x, y) = 2y,
\]

\[
f_x(1, 1) = 4, \quad f_y(1, 1) = 2.
\]

Hence, the tangent plane at \( (1, 1, 3) \) has an equation

\[
z - 3 = 4(x - 1) + 2(y - 1).
\]

This is simplified to

\[
z = 4x + 2y - 3.
\]
Example. Find an equation of the tangent plane and symmetric equations of the normal line to the surface $4x^2 + 9y^2 - z^2 = 16$ at the point $(2, 1, 3)$.

Solution. Let $F(x, y, z) = 4x^2 + 9y^2 - z^2$. We have

\[
\nabla F(x, y, z) = 8x \mathbf{i} + 18y \mathbf{j} - 2z \mathbf{k}.
\]

It follows that

\[
\nabla F(2, 1, 3) = 16 \mathbf{i} + 18 \mathbf{j} - 6 \mathbf{k}.
\]

Hence, the tangent plane at $(2, 1, 3)$ has an equation

\[
16(x - 2) + 18(y - 1) - 6(z - 3) = 0,
\]

which is simplified to $8x + 9y - 3z = 16$. Moreover, the normal line to the surface at $(2, 1, 3)$ has symmetric equations

\[
\frac{x - 2}{8} = \frac{y - 1}{9} = \frac{z - 3}{-3}.
\]