The Cross Product

If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, then the **cross product** of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix},$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the unit coordinate vectors. Expanding the above determinant in terms of the first row, we obtain

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}.$$

Notice the minus sign in the second term. Further,

$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1).$$

Example. Let $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$.

Find $\mathbf{u} \times \mathbf{v}$, $\mathbf{v} \times \mathbf{u}$, $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u}$, and $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v}$.

Solution. We have

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 2 \\ 3 & 4 & -1 \end{vmatrix} = -7\,\mathbf{i} + 8\,\mathbf{j} + 11\,\mathbf{k}$$

and

$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & -1 \\ 2 & -1 & 2 \end{vmatrix} = 7\,\mathbf{i} - 8\,\mathbf{j} - 11\,\mathbf{k}.$$

Moreover,

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = (-7)2 + 8(-1) + 11(2) = -14 - 8 + 22 = 0$$

and

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = (-7)3 + 8(4) + 11(-1) = -21 + 32 - 11 = 0.$$

Properties of the Cross Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be three vectors and let c be a real number. Then

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u},$$

$$(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w},$$

$$\mathbf{w} \times (\mathbf{u} + \mathbf{v}) = \mathbf{w} \times \mathbf{u} + \mathbf{w} \times \mathbf{v},$$

$$(c\mathbf{u}) \times \mathbf{v} = c(\mathbf{u} \times \mathbf{v}).$$

Further, $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, and

$$|\mathbf{u} \times \mathbf{v}|^2 = |\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2.$$

This is valid, because $|\mathbf{u} \times \mathbf{v}|^2$ is equal to

$$(u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2$$

and $|\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2$ is equal to

$$(u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2.$$

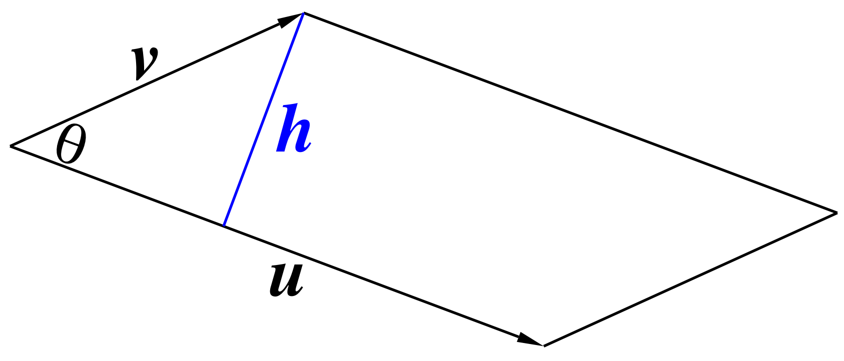
Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Then we have $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$. Hence,

$$\begin{aligned} |\mathbf{u} \times \mathbf{v}|^2 &= |\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u} \cdot \mathbf{v})^2 \\ &= |\mathbf{u}|^2|\mathbf{v}|^2 - (|\mathbf{u}||\mathbf{v}| \cos\theta)^2 \\ &= |\mathbf{u}|^2|\mathbf{v}|^2(1 - \cos^2\theta) \\ &= |\mathbf{u}|^2|\mathbf{v}|^2 \sin^2\theta. \end{aligned}$$

Since $\sin\theta \geq 0$, it follows that

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta.$$

In other words, the length of $\mathbf{u} \times \mathbf{v}$ is equal to the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$.

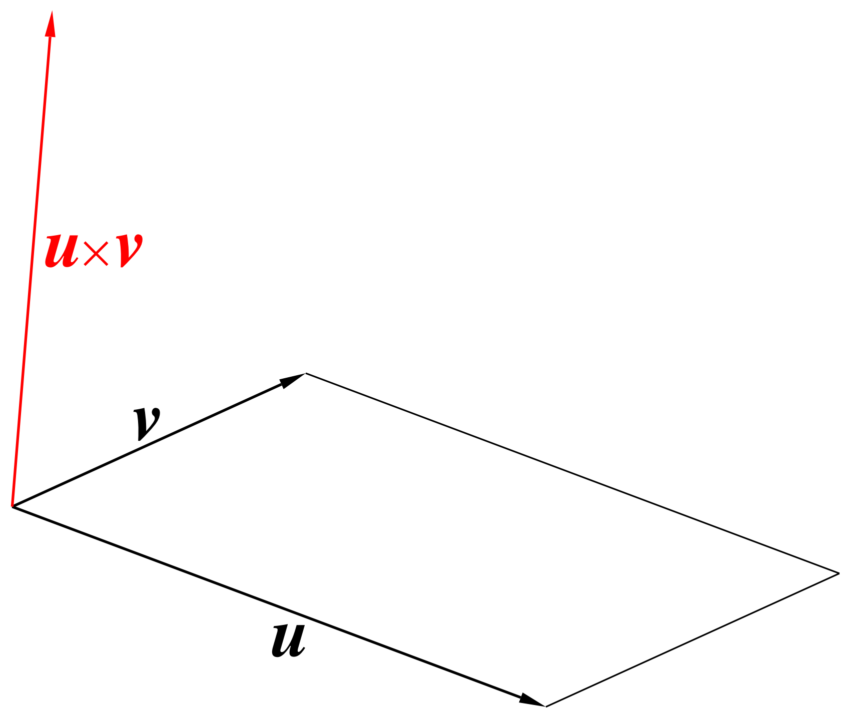


The Geometric Meaning of the Cross Product

1. $\mathbf{u} \times \mathbf{v}$ is a vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

2. Its length is equal to the area of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$.

3. Its direction is given by the right-hand rule: If the fingers of the *right* hand curl in the direction of a rotation (through an angle less than 180°) from $\mathbf{u}$ to $\mathbf{v}$, then the thumb points in the direction of $\mathbf{u} \times \mathbf{v}$.



Example. Find the area of the triangle with vertices $A(2,-1,3)$, $B(1,1,1)$, and $C(5,-2,1)$.

Solution. We have

$$\mathbf{u} = \overrightarrow{AB} = (-1,2,-2) \text{ and } \mathbf{v} = \overrightarrow{AC} = (3,-1,-2).$$

The cross product of **u** and **v** is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & -2 \\ 3 & -1 & -2 \end{vmatrix} = -6\,\mathbf{i} - 8\,\mathbf{j} - 5\,\mathbf{k}.$$

It follows that

$$|\mathbf{u} \times \mathbf{v}| = \sqrt{(-6)^2 + (-8)^2 + (-5)^2} = \sqrt{125} = 5\sqrt{5}.$$

Therefore, the area of $\triangle ABC$ is

$$\frac{1}{2}\,|\mathbf{u} \times \mathbf{v}| = \frac{5\sqrt{5}}{2}.$$

Example. Find two unit vectors orthogonal to both $\mathbf{u} = -4\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$ and $\mathbf{v} = 3\,\mathbf{i} - \mathbf{j} + \mathbf{k}$.

Solution. The cross product of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 2 & -1 \\ 3 & -1 & 1 \end{vmatrix} = \mathbf{i} + \mathbf{j} - 2\,\mathbf{k}.$$

It follows that

$$|\mathbf{u} \times \mathbf{v}| = \sqrt{1^2 + 1^2 + (-2)^2} = \sqrt{6}.$$

Thus,

$$\mathbf{w} = \frac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u} \times \mathbf{v}|} = \frac{1}{\sqrt{6}}\,\mathbf{i} + \frac{1}{\sqrt{6}}\,\mathbf{j} - \frac{2}{\sqrt{6}}\,\mathbf{k}$$

and

$$-\mathbf{w} = -\frac{1}{\sqrt{6}}\,\mathbf{i} - \frac{1}{\sqrt{6}}\,\mathbf{j} + \frac{2}{\sqrt{6}}\,\mathbf{k}$$

are two unit vectors orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.