# Alternating Series

An **alternating series** is a series whose terms are alternatively positive and negative. Here are two examples:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}.$$

We see that the $n$th term of an alternating series can be written as

$$a_n = (-1)^{n-1} b_n \quad \text{or} \quad a_n = (-1)^n b_n,$$

where $b_n = |a_n| > 0$.

## The Alternating Series Test

**Theorem.** *The series*

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

*converges if all three of the following three conditions are satisfied:*

(1) $b_n > 0$ *for all* $n$;

(2) $b_{n+1} \leq b_n$ *for all* $n$;

(3) $\lim_{n\to\infty} b_n = 0$.

**Example.** The alternating harmonic series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges. Indeed, let $b_n := 1/n$ for $n = 1, 2, \ldots$. Then $b_n > 0$ and $b_n \geq b_{n+1}$ for all $n$. Moreover, $\lim_{n\to\infty} b_n = 0$. Thus, by the alternating series test, the series converges.

**Example.** The alternating series

$$\sum_{n=1}^{\infty}(-1)^{n}\frac{n}{n+1} = -\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \cdots$$

diverges. In this case, $b_n = \frac{n}{n+1}$. We have

$$\lim_{n\to\infty} b_n = 1 \neq 0.$$

By the test for divergence, the series diverges.

## Alternating Series Estimation

**Theorem.** *Let $s$ be the sum of the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ and let $s_n$ be its $n$th partial sum. Suppose that $0 < b_{n+1} \leq b_n$ for all $n$ and $\lim_{n\to\infty} b_n = 0$. Then*

$$|s - s_n| \leq b_{n+1}.$$

**Proof.** We have

$$\begin{aligned} s - s_n &= (-1)^n b_{n+1} + (-1)^{n+1} b_{n+2} \\ &\qquad + (-1)^{n+2} b_{n+3} + (-1)^{n+3} b_{n+4} + \cdots \\ &= (-1)^n (b_{n+1} - b_{n+2} + b_{n+3} - b_{n+4} + \cdots). \end{aligned}$$

Since $0 < b_{n+1} \leq b_n$ for all $n$, we deduce that

$$|s - s_n| = b_{n+1} - b_{n+2} + b_{n+3} - b_{n+4} + \cdots \leq b_{n+1}.$$

For a positive integer $n$, define the $n$ **factorial** as

$$n! = 1 \cdot 2 \cdots n.$$

In particular, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, and $5! = 120$. By convention, $0! = 1$ (not 0).

**Example.** Find the sum of the alternating series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places.

**Solution.** Let $b_n = 1/n!$ for $n = 0, 1, 2, \ldots$. Then $b_n \geq b_{n+1} > 0$ for all $n$. Since $0 < 1/n! \leq 1/n$, we have $\lim_{n\to\infty} 1/n! = 0$. Thus, the series converges.

We observe that $b_7 = 1/7! = 1/5040 < 0.0002$. By the Alternating Series Estimation Theorem we know that $|s - s_6| \leq b_7 < 0.0002$. But

$$s_6 = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \approx 0.368056.$$

Hence, $s \approx 0.368$ correct to three decimal places.