Taylor's Formula with Remainder

Let f be a function that is $(n+1)$ times differentiable in an interval I that contains a. Then for each x in I there is a number c strictly between x and a such that

$$f(x) = T_n(x) + R_n(x),$$

where

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

and

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Remarks

1. For the special case $n = 0$, we have

$$f(x) = f(a) + f'(c)(x - a).$$

This is just the Mean Value Theorem.

2. T_n is called the **Taylor polynomial** of order n or the nth Taylor polynomial of f at a. It is uniquely determined by the conditions

$$T_n(a) = f(a), T_n'(a) = f'(a), \ldots, T_n^{(n)}(a) = f^{(n)}(a).$$

3. R_n is called the **Remainder** of order n. This term is similar to the $(n+1)$th term in the Taylor series except that $f^{(n+1)}$ is evaluated at c with c between a and x.

Example. Write Taylor's formula for the case where

$f(x) = \ln x$, $a = 1$, $n = 3$.

Solution. We have

$$f(x) = \ln x, \qquad f(1) = 0;$$

$$f'(x) = \frac{1}{x} = x^{-1}, \qquad f'(1) = 1;$$

$$f''(x) = (-1)x^{-2}, \qquad f''(1) = -1;$$

$$f'''(x) = (-1)(-2)x^{-3}, \qquad f'''(1) = 2;$$

$$f^{(4)}(x) = (-1)(-2)(-3)x^{-4}, \quad f^{(4)}(c) = -\frac{6}{c^4}.$$

Therefore,

$$f(x) = T_3(x) + R_3(x),$$

where

$$T_3(x) = (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3$$

and, with c between 1 and x,

$$R_3(x) = \frac{1}{4!}\left(-\frac{6}{c^4}\right)(x-1)^4 = -\frac{1}{4c^4}(x-1)^4.$$

Example. Use the Taylor polynomial above to find an approximate value for $\ln 1.2$ and give an estimate for the error involved in approximation.

Solution. We have $f(x) = T_3(x) + R_3(x)$, where

$$T_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

and, with c between 1 and x, $R_3(x) = -\frac{1}{4c^4}(x-1)^4$.

It follows that

$$T_3(1.2) = 0.2 - \frac{1}{2} \times 0.2^2 + \frac{1}{3} \times 0.2^3 \approx 0.1826$$

and

$$R_3(1.2) = -\frac{1}{4c^4} \times 0.2^4, \quad 1 < c < 1.2.$$

Since $c > 1$, we have $c^4 > 1$ and hence $\frac{1}{c^4} < 1$.

Consequently,

$$|R_3(1.2)| = \frac{1}{4c^4} \times 0.0016 < 0.0004.$$

Taylor Polynomials as Partial Sums of Taylor Series

Theorem. If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x - a| < R$, then f is equal to the sum of its Taylor series on the interval $(a - R, a + R)$.

Example. Let $f(x) = e^x$ for $-\infty < x < \infty$. Prove that f is equal to the sum of its Maclaurin series.

Proof. By Taylor's formula, $f(x) = T_n(x) + R_n(x)$, where $T_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ is the nth-degree Taylor polynomial of f at 0 and

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} = \frac{e^c}{(n+1)!}x^{n+1}$$

with c between 0 and x. Suppose that M is a positive number and $|x| \leq M$. Then $|c| \leq M$. Consequently,

$$|R_n(x)| \leq e^M \frac{|x|^{n+1}}{(n+1)!}.$$

By the ratio test, the series $\sum_{n=0}^{\infty} \frac{|x|^{n+1}}{(n+1)!}$ converges. Thus, $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$, and hence $\lim_{n\to\infty} R_n(x) = 0$. Therefore,

$$e^x = \lim_{n\to\infty} T_n(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty.$$

Computation of e

Recall that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

for $-\infty < x < \infty$. In particular,

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots.$$

Let s_n be the nth partial sum of the above series:

$$s_n = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}.$$

Then $s_n = T_n(1)$ and $e - T_n(1) = R_n(1)$. By Taylor's formula

$$R_n(1) = \frac{e^c}{(n+1)!} \quad \text{with} \quad 0 < c < 1.$$

It follows that

$$|e - T_n(1)| = |R_n(1)| \leq \frac{e^c}{(n+1)!} < \frac{e}{(n+1)!} < \frac{3}{(n+1)!}.$$