

Kähler-Einstein metrics (following Berman)

Abstract

In this essay, Berman's proof of K-stability of Kähler-Einstein varieties is explained, assuming only basic knowledge of Kähler manifolds and algebraic geometry. After developing the necessary notions of connections and curvature on complex manifolds, a definition of a Kähler-Einstein form is first given for the smooth case, which is then generalised to Fano varieties. The Ding functional, which has Kähler-Einstein metrics as its minima, is also introduced. Test-configurations, the Donaldson-Futaki invariant and the K-stability condition are then introduced. The essay then proceed by giving an account of the Yau-Tian-Donaldson conjecture, which in general states that a polarised variety admits a Kähler-Einstein metric if and only if it is K-stable. Finally, an account of the proof of one direction in the Fano variety case, due to Berman, is given. This proof uses a special metric on the top Deligne pairing of the relative anticanonical line bundle on a test-configuration, whose weight over 0 is given by the Donaldson-Futaki invariant.

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1 Introduction

A Fano variety is a normal complex variety such that the anticanonical line bundle $-K_X$ is ample. A Kähler metric ω on such a variety is called Kähler-Einstein if $\text{Ric}(\omega) = \omega$. Equivalently, there exists a metric φ on $-K_X$ such that $\omega = dd^c\varphi$ and

$$\frac{(dd^c\varphi)^n}{\int_X (dd^c\varphi)^n} = \frac{e^{-\varphi}}{\int_X e^{-\varphi}} \quad (1.1)$$

This equation is also the Euler-Lagrange equation for a certain functional on the space of locally bounded metrics on $-K_X$, called the Ding metric.

However, not all Fano varieties admit a Kähler-Einstein metric. Since the 1970's, many have sought to find out exactly when X does admit such a metric. Yau first proposed that the obstruction could be an algebro-geometric stability condition, analogous to Hilbert-Mumford stability. Tian found such a condition, called K-stability, which was later generalised by Donaldson. Thus, the Yau-Tian-Donaldson conjecture states that a Fano variety admits a Kähler-Einstein metric if and only if it is K-stable [12].

The K-stability condition in its current form is described in terms of test-configurations, i.e. flat families $\mathcal{X} \rightarrow \mathbb{C}$ with a \mathbb{C}^* -action on $-K_{\mathcal{X}/\mathbb{C}}$ such that $\mathcal{X}_1 = X$. These test-configurations have a basic invariant, called the Donaldson-Futaki invariant, and K-stability of a variety is the condition that this invariant be non-positive for all test-configurations of the variety, being zero only for the trivial case.

In this paper, a proof of one direction of the Yau-Tian-Donaldson conjecture, i.e. the following theorem, will be proved.

Theorem 1.1 (Berman). *Any Fano variety admitting a Kähler-Einstein metric is K-stable.*

The proof of this theorem was first given by Berman [2], and we will largely follow his proof.

In the proof, the given Kähler-Einstein metric will be extended to a metric on the test-configuration, which will then induce a Deligne metric on the top Deligne pairing of $-K_{\mathcal{X}/\mathbb{C}}$. This Deligne metric will be modified by adding a certain function, so that the resulting Ding metric will be related to the Ding functional. This Ding metric will turn out to behave nicely around 0, which makes it useful in calculating the weight of the \mathbb{C}^* -action on the zero fibre of the Deligne pairing. As this weight is equal to the Donaldson-Futaki invariant by a result of Phong-Ross-Sturm[10], we here get a connection from the Kähler-Einstein metric, via the Ding functional and its Euler-Lagrange equation, to the Deligne pairing and its Ding metric, and ultimately to the Donaldson-Futaki invariant, whose sign behaviour then turns out to agree with K-stability.

Outline of the paper: Section 2 gives an outline of the general basic theory of Kähler-Einstein metrics and test-configurations. In subsection 2.1, the notions of connections and curvature of (smooth) Kähler manifolds and line bundles will be developed, after which the three cases of the Kähler-Einstein existence problem are

briefly discussed. Subsection 2.2 explains the Fano variety theory, focusing on the link between metrics on X and on $-K_X$. In subsection 2.3, test-configurations, the Donaldson-Futaki invariant and K-stability are defined. Section 3 gives a brief outline of the progress on the Yau-Tian-Donaldson conjecture, which is the third and hardest of the three cases considered in subsection 2.1. In section 4, the actual proof of theorem 1.1 is given. Subsection 4.1 extends the metric on $-K_X$ to a metric on $-K_{X/\mathbb{C}}$, which is then ‘pushed down’ to a metric on the Deligne pairing in subsection 4.2. After a slight modification, we obtain the Ding metric, whose properties are examined in subsection 4.3. Finally, in subsection 4.4, all parts of the proof are combined to give the final result.

I would like to thank Julius Ross, my essay setter, for his help and advise.

2 Definitions and notation

2.1 Background on metrics

This subsection is well-known background on connections and curvature on complex manifolds. A reference is e.g. Székelyhidi [12].

In this subsection, we will let X be a complex manifold and ω a Kähler form on X . Just as in the real case, one can define the Levi-Civita connection and curvature:

Definition 2.1. • The Levi-Civita connection, denoted by ∇ , is the unique connection on the tangent bundle TX (and hence also on the dual bundle T^*X and any of their products) satisfying $\nabla g = 0$ and $\nabla_{\bar{i}}X = 0$ for all holomorphic vector fields X .

The Riemannian curvature, $R = R_i^j{}_{k\bar{l}}$, is then defined by the equation

$$(\nabla_k \nabla_{\bar{l}} - \nabla_{\bar{l}} \nabla_k) \frac{\partial}{\partial z^i} = R_i^j{}_{k\bar{l}} \frac{\partial}{\partial z^j} \quad (2.1)$$

The Ricci curvature is $R_{i\bar{j}} := g^{k\bar{l}} R_{i\bar{j}k\bar{l}} = g^{k\bar{l}} g_{m\bar{j}} R_i^m{}_{k\bar{l}}$.

The Ricci form is given by $\text{Ric}(\omega) := \frac{i}{2\pi} \cdot R_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}$.

Finally, the scalar curvature is given by $S := g^{i\bar{j}} R_{i\bar{j}}$.

- Let $L \rightarrow X$ be a line bundle, with a hermitian metric h . The Chern connection, also denoted by ∇ , is the unique connection on L satisfying $\nabla h = 0$ and $\nabla_{\bar{i}}s = 0$ for all holomorphic sections s .

The curvature $F_{k\bar{l}}$ is then defined by the equation

$$F_{k\bar{l}} = \nabla_k \nabla_{\bar{l}} - \nabla_{\bar{l}} \nabla_k \quad (2.2)$$

The curvature form is given by $F(h) := \frac{i}{2\pi} \cdot F_{k\bar{l}} dz^k \wedge d\bar{z}^{\bar{l}}$.

Lemma 2.2. *The Ricci curvature can locally be expressed as*

$$R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log \det(g_{p\bar{q}}) \quad (2.3)$$

Hence, $\text{Ric}(\omega) = \frac{-i}{2\pi} \partial \bar{\partial} \log \det(g)$.

Similarly, the curvature $F(h)$ of $L \rightarrow X$ is locally given by

$$F_{k\bar{l}} = -\partial_k \partial_{\bar{l}} \log h(s) \quad (2.4)$$

where s is a local holomorphic section of L and $h(s) = |s|_h^2$.

Hence, $F(h) = \frac{-i}{2\pi} \partial \bar{\partial} \log h(s)$.

By recalling that $d = \partial + \bar{\partial}$ and defining $d^c := -\frac{i}{4\pi}(\partial - \bar{\partial})$, we can rewrite this as $\text{Ric}(\omega) = -dd^c \log \det(g)$ and $F(h) = -dd^c \log h(s)$.

Definition 2.3. The first Chern class of X is defined as $c_1(X) = [\text{Ric}(g)] \in H^2(M, \mathbb{R})$.

The first Chern class of a line bundle $L \rightarrow X$ is defined as $c_1(L) = [F(h)] \in H^2(M, \mathbb{R})$.

This is well-defined, as given two Kähler metrics g, h on X , $\text{Ric}(g) - \text{Ric}(h) = -dd^c \log \frac{g}{h}$ is an exact form, with a similar argument for $c_1(L)$. Note that $c_1(X) = c_1(-K_X)$, where $K_X \rightarrow X$ is the canonical line bundle.

An elemental lemma which will be instrumental throughout the essay is the following:

Lemma 2.4 ($\partial\bar{\partial}$ -lemma). *For any two real cohomologous $(1,1)$ -forms τ and η on a compact Kähler manifold X , there exists a smooth function $f : X \rightarrow \mathbb{R}$ such that*

$$\tau = \eta + \sqrt{-1} \partial \bar{\partial} f \quad (2.5)$$

Proof. [12] As τ and η are cohomologous, there exists a real 1-form α such that $\tau = \eta + d\alpha$. We can decompose α in its $(1,0)$ part $\alpha^{(1,0)}$ and its $(0,1)$ part $\alpha^{(0,1)}$, and as α is real, $\alpha^{(0,1)} = \overline{\alpha^{(1,0)}}$. As τ and η are $(1,1)$ -forms, we get that $\partial\alpha^{(1,0)} = \bar{\partial}\alpha^{(0,1)} = 0$ and

$$\tau = \eta + \bar{\partial}\alpha^{(1,0)} + \partial\alpha^{(0,1)}$$

Choosing a Kähler metric g on X , we get

$$\begin{aligned} \int_X (\partial^* \alpha^{(1,0)}) d\text{vol} &= - \int_X (*\partial * \alpha^{(1,0)}) \cdot d\text{vol} = \int_X (\partial * \alpha^{(1,0)}) \cdot (*d\text{vol}) \\ &= \int_X \partial * \alpha^{(1,0)} = 0 \end{aligned}$$

So by Hodge theory, there exists a function f' such that $\partial^* \alpha = \Delta_\partial f' = -\partial^* \partial f'$.

As $\partial\alpha^{(1,0)} = 0$, we get

$$\partial(\alpha^{(1,0)} + \partial f') = \partial^*(\alpha^{(1,0)} + \partial f') = 0$$

Therefore, $\alpha^{(1,0)} + \partial f'$ is ∂ -harmonic, and hence also $\bar{\partial}$ -harmonic, as g is Kähler. This gives $\bar{\partial}\alpha^{(1,0)} = -\bar{\partial}\partial f'$. Putting everything together,

$$\begin{aligned} \tau - \eta &= \bar{\partial}\alpha^{(1,0)} + \partial\alpha^{(0,1)} = \bar{\partial}\alpha^{(1,0)} + \overline{\partial\alpha^{(1,0)}} \\ &= \bar{\partial}\alpha^{(1,0)} + \overline{\bar{\partial}\partial f'} = -\bar{\partial}\partial f' - \bar{\partial}\partial f' \\ &= \partial\bar{\partial}f' - \partial\bar{\partial}f' = 2i\partial\bar{\partial}\Im(f') \end{aligned}$$

Choosing $f = 2\Im(f')$ gives the result. \square

The next theorem is a fundamental result about the first Chern class of Kähler manifolds:

Theorem 2.5 (Calabi-Yau theorem[15]). *Given a compact Kähler manifold (X, ω) , there exists for any real $(1,1)$ -form $\alpha \in c_1(X)$ a unique Kähler metric η , cohomologous to ω , such that $\text{Ric}(\eta) = 2\pi\alpha$.*

For a proof, see e.g. Tian [13].

Now we can go on to define one of the central concepts of this essay:

Definition 2.6. A Kähler metric ω is called Kähler-Einstein if for some $\lambda \in \mathbb{R}$,

$$\text{Ric}(\omega) = \lambda\omega \quad (2.6)$$

By rescaling the metric, we can and will always restrict ourselves to one of three cases: $\text{Ric}(\omega) = \omega$, $\text{Ric}(\omega) = 0$ or $\text{Ric}(\omega) = -\omega$. In these cases the first Chern class is, by definition, positive, zero, or negative respectively.

A main problem in this field has been to find about when a compact Kähler manifold admits a Kähler-Einstein metric. Trivially, the first Chern class should be definite, i.e. $c_1(X) = \lambda[\omega]$ for some Kähler form ω . The answer then turns out to depend on the sign of this λ :

- For the case $c_1(X) < 0$, the solution to the problem is given by the following theorem:

Theorem 2.7 (Aubin-Yau). *[1, 16] Suppose X is a compact Kähler manifold such that $c_1(X) < 0$. Then there exists a unique Kähler-Einstein metric ω on X . This metric must necessarily lie in the cohomology class $-2\pi c_1(X)$.*

This case is hence completely solved.

- 0 For the case $c_1(X) = 0$, there is no a priori confinement of Kähler-Einstein metrics to certain cohomology classes. It turns out there is no such restriction at all in this case.

Theorem 2.8 (Yau). *Let (X, ω) be a compact Kähler manifold with trivial first Chern class. Then there exists a unique Kähler metric ω' , with $[\omega] = [\omega']$, such that $\text{Ric}(\omega') = 0$.*

This is an easy corollary of the Calabi-Yau theorem.

So this case is also solved.

- + In the case $c_1(X) > 0$, things are more complicated. It turns out not all Kähler manifolds with positive first Chern class admit a Kähler-Einstein metric. According to the Yau-Tian-Donaldson conjecture, the obstruction is algebro-geometric in nature. This essay will deal with the question which manifolds with positive first Chern class do admit a Kähler-Einstein metric, by presenting a proof of one direction of a version of the Yau-Tian-Donaldson conjecture.

2.2 The Fano variety setting

From this section onwards we will work with Fano varieties, i.e. X is a normal compact projective complex variety and the anti-canonical line bundle $-K_X := \det(TX)$ on the regular locus X_{reg} extends to an ample \mathbb{Q} -line bundle on X , i.e. some power extends to an ample line bundle over X .

First, we fix some notation. Following Berman [2], line bundles and metrics on them will be written additively from now on, i.e. $2L := L \otimes L$ and a metric $|\cdot|$ on L is locally represented by an upper semicontinuous function φ_U such that for a certain generating section s , $|s|^2 = e^{-\varphi_U}$. Then we can write $dd^c\varphi$ for the curvature, cf. lemma 2.2. We write $\mathcal{H}_b(X, L)$ for the space of locally bounded metrics on L with positive curvature current.

A metric φ is called (locally) bounded if $\varphi - \varphi_0$ is a (locally) bounded function for some (hence, all) smooth metric φ_0 .

A function $v : X \rightarrow [-\infty, \infty)$ will be called plurisubharmonic, or psh for short, if it is upper semicontinuous and for any holomorphic $f : \Delta \rightarrow X$, $v \circ f$ is subharmonic. Here $\Delta \subset \mathbb{C}$ denotes the (closed) unit disc. If in addition $\omega + dd^c v \geq 0$ for some Kähler form ω , v is said to be ω -psh.

If $L = -K_X$, a metric $\varphi \in \mathcal{H}_b(X, -K_X)$ induces a measure μ_φ on X : on a coordinate chart, take local holomorphic coordinates z_1, \dots, z_n , and let φ_U be the representative of φ relative to the dual of $dz := dz_1 \wedge \dots \wedge dz_n$. Then define $\mu_\varphi|_U := e^{-\varphi_U} dz \wedge d\bar{z}$.

Lemma 2.9. *The measure $\mu_\varphi|_U$ defined above is independent of coordinates, and hence defines a measure μ_φ on all of X_{reg} , which can be extended by zero over all of X .*

Proof. Let (z_1, \dots, z_n) and (w_1, \dots, w_n) both be local coordinates on U . Then $(dz)^{-1}$ and $(dw)^{-1}$ are local sections of $-K_X$ over U and

$$(dz)^{-1} = \det\left(\frac{\partial w_i}{\partial z_j}\right)(dw)^{-1}$$

Therefore, we get that

$$e^{-\varphi_z} := |(dz)^{-1}|^2 = \left| \det\left(\frac{\partial w_i}{\partial z_j}\right) \right|^2 \cdot |(dw)^{-1}|^2 = \left| \det\left(\frac{\partial w_i}{\partial z_j}\right) \right|^2 \cdot e^{-\varphi_w}$$

Hence,

$$\begin{aligned} e^{-\varphi_z} dz \wedge d\bar{z} &= \left| \det\left(\frac{\partial w_i}{\partial z_j}\right) \right|^2 \cdot e^{-\varphi_w} \cdot \det\left(\frac{\partial z_i}{\partial w_j}\right) \cdot \det\left(\frac{\partial \bar{z}_i}{\partial \bar{w}_j}\right) dw \wedge d\bar{w} \\ &= e^{-\varphi_w} dw \wedge d\bar{w} \end{aligned}$$

□

The measure μ_φ will often just be written $e^{-\varphi}$.

Definition 2.10. A Fano variety X has Kawamata log terminal (klt) singularities if the total mass of $e^{-\varphi}$ is finite for some (equivalently any) $\varphi \in \mathcal{H}_b(X, -K_X)$. [2]

In the presence of singularities, the definition of a Kähler-Einstein metric must be adapted slightly:

Definition 2.11. Let X be a Fano variety. A metric ω on X is said to be a Kähler-Einstein metric if $\text{Ric}(\omega) = \omega$ and $\int_{X_{\text{reg}}}(\omega)^n = c_1(-K_X)^n =: V.[2]$

Note that $c_1(-K_X)^n$ means the top intersection number here.

As proved in [3], this implies that X actually has klt singularities.

There is a different characterisation of Kähler-Einstein metrics in terms of metrics on $-K_X$.

Lemma 2.12. *Let X be a Fano variety. A metric ω is Kähler-Einstein if and only if it is the curvature of a locally bounded metric φ on $-K_X$ satisfying the following equation:*

$$(dd^c\varphi)^n = e^{-\varphi} \frac{\int_X (dd^c\varphi)^n}{\int_X e^{-\varphi}} \quad (2.7)$$

Proof. That there exists a φ such that $\omega = dd^c\varphi$ is trivial: locally this is just the $\partial\bar{\partial}$ -lemma (note that φ need not be smooth, hence we can patch). The main point of the lemma is to show that 2.7 holds. This is a simple algebraic exercise.

$$\begin{aligned} -dd^c \log(dd^c\varphi)^n &= \text{Ric}(\omega) = \omega = dd^c\varphi \\ -\log(dd^c\varphi)^n &= \varphi + c \\ (dd^c\varphi)^n &= e^{-\varphi-c} \end{aligned}$$

where c is an arbitrary constant of integration. Hence we can choose c such that the lemma holds.

As the string of equations can also be read from the bottom upwards (the chain of implications is reversible), this is indeed an equivalent description. \square

Remark. The choice of the constant c in the proof might seem a bit arbitrary, but in lemma 2.15, it will turn out to yield the Euler-Lagrange equation for the Ding functional, which makes it a logical choice.

To define that functional, we first need the functional

$$\mathcal{E}(\varphi, \psi) := \frac{1}{n+1} \sum_{j=0}^n \int_X (\varphi - \psi) (dd^c\varphi)^{n-j} \wedge (dd^c\psi)^j \quad (2.8)$$

The variation of this functional with respect to the first variable is given as follows:

Lemma 2.13. *The variation of \mathcal{E} is given by:*

$$\left. \frac{d}{dt} \right|_{t=0} \mathcal{E}((1-t)\varphi + t\chi, \psi) = \int_X (\chi - \varphi) (dd^c\varphi)^n \quad (2.9)$$

Proof. This is just a computation:

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \mathcal{E}((1-t)\varphi + t\chi, \psi) &= \frac{1}{n+1} \sum_{j=0}^n \int_X \frac{d}{dt}\Big|_{t=0} \left[((1-t)\varphi + t\chi - \psi) \cdot \right. \\
&\quad \left. ((1-t)dd^c\varphi + tdd^c\chi)^{n-j} \wedge (dd^c\psi)^j \right] \\
&= \frac{1}{n+1} \sum_{j=0}^n \int_X \left[(\chi - \varphi)(dd^c\varphi)^{n-j} \wedge (dd^c\psi)^j \right. \\
&\quad \left. + (n-j)(\varphi - \psi)(dd^c\chi - dd^c\varphi) \wedge (dd^c\varphi)^{n-j-1} \wedge (dd^c\psi)^j \right] \\
&= \frac{1}{n+1} \sum_{j=0}^n \int_X \left[(\chi - \varphi)(dd^c\varphi)^{n-j} \wedge (dd^c\psi)^j \right. \\
&\quad \left. + (n-j)((\chi - \varphi)dd^c\varphi - (\chi - \varphi)dd^c\psi) \wedge (dd^c\varphi)^{n-j-1} \wedge (dd^c\psi)^j \right] \\
&= \frac{1}{n+1} \left[\sum_{j=0}^n \int_X (n-j+1)(\chi - \varphi)(dd^c\varphi)^{n-j} \wedge (dd^c\psi)^j \right. \\
&\quad \left. - \sum_{j=1}^n \int_X (n-j+1)(\chi - \varphi)(dd^c\varphi)^{n-j} \wedge (dd^c\psi)^j \right] \\
&= \int_X (\chi - \varphi)(dd^c\varphi)^n
\end{aligned}$$

where for the third equality sign, integration by parts has been used, and for the fourth, part of the summation has been re-indexed. \square

We will also need the following function on \mathbb{C} (for now knowing that $\mathcal{X}_1 = X$ and $\varphi_1 = \varphi$ is sufficient).

$$v_\varphi(\tau) := -\log \int_{\mathcal{X}_\tau} e^{-\varphi_\tau} \quad (2.10)$$

This function will play a very important role later on.

Definition 2.14. The Ding functional is defined as follows:

$$\mathcal{D}(\varphi, \psi) = \frac{-1}{V} \mathcal{E}(\varphi, \psi) + v_\varphi(1) - v_\psi(1) \quad (2.11)$$

Lemma 2.15. *The equation 2.7 is the Euler-Lagrange equation for the first variable of the Ding functional, keeping the second argument fixed.*

Proof. Using equation 2.9, this is a simple computation:

$$\begin{aligned}
0 &= \frac{d}{dt} \Big|_{t=0} \mathcal{D}((1-t)\varphi + t\chi, \psi) \\
&= \frac{-1}{V} \frac{d}{dt} \Big|_{t=0} \mathcal{E}((1-t)\varphi + t\chi, \psi) - \frac{d}{dt} \Big|_{t=0} \log \int_X e^{-((1-t)\varphi + t\chi)} - 0 \\
&= \frac{1}{\int_X (dd^c \varphi)^n} \int_X (\varphi - \chi) (dd^c \varphi)^n - \frac{1}{\int_X e^{-\varphi}} \cdot \frac{d}{dt} \Big|_{t=0} \int_X e^{-((1-t)\varphi + t\chi)} \\
&= \frac{1}{\int_X (dd^c \varphi)^n} \int_X (\varphi - \chi) (dd^c \varphi)^n - \frac{1}{\int_X e^{-\varphi}} \cdot \int_X (\varphi - \chi) e^{-\varphi} \\
&= \int_X (\varphi - \chi) \left(\frac{(dd^c \varphi)^n}{\int_X (dd^c \varphi)^n} - \frac{e^{-\varphi}}{\int_X e^{-\varphi}} \right)
\end{aligned}$$

As this must hold for all $\chi \in \mathcal{H}_b(X, -K_X)$, we conclude that

$$\frac{(dd^c \varphi)^n}{\int_X (dd^c \varphi)^n} = \frac{e^{-\varphi}}{\int_X e^{-\varphi}}$$

is the Euler-Lagrange equation for the Ding functional. As this is also equation 2.7, this finishes the proof. \square

2.3 Test-configurations and K-stability

As was mentioned before, the obstruction for the existence of a Kähler-Einstein metric on a Kähler manifold with positive first Chern class is algebro-geometric in nature. This section is devoted to developing this obstruction, called K-stability.

Definition 2.16. A test-configuration for a Fano variety X consists of a flat family $\pi : \mathcal{X} \rightarrow \mathbb{C}$, with a relatively ample \mathbb{Q} -line bundle $\mathcal{L} \rightarrow \mathcal{X}$, endowed with a \mathbb{C}^* -action ρ covering the standard action on \mathbb{C} such that

- $X = \mathcal{X}_1 := \pi^{-1}(1)$ and $(\mathcal{X}_1, \mathcal{L}_1) \cong (X, -K_X)$;
- The total space \mathcal{X} and the central fibre $\mathcal{X}_0 := \pi^{-1}(0)$ are normal \mathbb{Q} -Gorenstein varieties with klt singularities. The latter is reduced and irreducible.[2]

The condition that $\mathcal{L} \rightarrow \mathcal{X}$ be relatively ample means that, for all $\tau \in \mathbb{C}$, $\mathcal{L}_\tau \rightarrow \mathcal{X}_\tau$ is an ample line bundle. That \mathcal{X} and \mathcal{X}_0 are \mathbb{Q} -Gorenstein means that $K_{\mathcal{X}} - \pi^* K_{\mathbb{C}}$ is trivial near all singularities, and hence can be extended over them.

Remark. The definition of test configuration here is often called a special test configuration in the literature. However, we can restrict to using only these, according to Li-Xu [8] and lemma 2.1 of Berman [2].

Lemma 2.17. *If $(\mathcal{X}, \mathcal{L})$ is a test configuration for the Fano variety X , then \mathcal{L} is isomorphic to $-K := -K_{\mathcal{X}/\mathbb{C}} := -(K_{\mathcal{X}} - \pi^* K_{\mathbb{C}})$. As \mathcal{X} and \mathcal{X}_0 are \mathbb{Q} -Gorenstein, \mathcal{L} is in particular an actual line bundle, not only a \mathbb{Q} -line bundle.*

Proof. As $\mathcal{L}_1 \cong L = -K_X$, and $(\mathcal{X}_\tau, \mathcal{L}_\tau) \cong (\mathcal{X}_1, \mathcal{L}_1)$ for all $\tau \in \mathbb{C}^*$ via the action of ρ , the divisors \mathcal{L} and $-K_X$ are linearly equivalent on $\mathcal{X}^* := \mathcal{X} \setminus \mathcal{X}_0$. Hence $\mathcal{L} + K_X$ linearly equivalent to a Weil divisor supported in \mathcal{X}_0 . But as the central fibre is irreducible, $\mathcal{L} + K_X$ is equivalent to a multiple of \mathcal{X}_0 . As \mathcal{X}_0 is cut out by $\pi^*\tau$, where τ is the coordinate on \mathbb{C} , and this coordinate, inducing the action on \mathcal{L} , vanishes with order one there, \mathcal{L} is indeed isomorphic to $-K_X + \pi^*K_{\mathbb{C}} = -K_{X/\mathbb{C}}$. \square

Hence we will often just write \mathcal{X} for a test-configuration, in stead of $(\mathcal{X}, \mathcal{L})$.

Test-configurations have an associated numerical invariant, called the Futaki or Donaldson-Futaki invariant. To define it, consider for all $k \geq 1$ the vector space $H^0(\mathcal{X}_0, -kK_0)$, having a \mathbb{C}^* -action induced by that of K , whose dimension will be called d_k . The total weight of the action (or, equivalently, the weight of the action on the determinant bundle) will be denoted w_k . By Hilbert function theory, it can be shown that d_k and w_k , for large k , become polynomials of degree n and $n + 1$, respectively.[5] Hence, the following definition makes sense.

Definition 2.18. The Donaldson-Futaki invariant $\text{DF}(\mathcal{X})$ of a test-configuration \mathcal{X} is given by the following expansion:

$$\frac{w_k}{kd_k} = c_0 + \frac{1}{2} \text{DF}(\mathcal{X})k^{-1} + O(k^{-2}) \quad (2.12)$$

Note that different authors use different signs and normalisations. This definition follows Berman [2].

Definition 2.19. The Fano variety X is called K-semistable if $\text{DF}(\mathcal{X}) \leq 0$ for all test-configurations \mathcal{X} , and K-(poly)stable if furthermore equality implies that \mathcal{X} is isomorphic $X \times \mathbb{C}$. If X is not K-semistable, it is said to be K-unstable.

The condition of K-stability is the one that will turn out to be equivalent to admitting a Kähler-Einstein metric in the Fano variety case.

3 The Yau-Tian-Donaldson conjecture

The notion of Kähler-Einstein metrics has been generalised to that of constant scalar curvature Kähler (cscK) metrics, which have, as the name implies, constant scalar curvature, and even further to extremal metrics, which are extrema of the Calabi functional

$$\text{Ca}(\omega) := \int_X S(\omega)^2 \omega^n$$

There exist several theorems and conjectures giving necessary and/or sufficient conditions for the existence of a Kähler-Einstein metric or a cscK metric in $c_1(X)$ on different classes of manifolds or varieties with positive first Chern class.

It was noted the scalar curvature can be interpreted as a moment map, with the Calabi functional as the norm squared of this map. This gave a link between the existence of these special metrics and certain Mumford stability conditions, in the field of Geometric Invariance Theory.[12]

Hence, Yau proposed that the existence of Kähler-Einstein metrics could be equivalent to a certain such stability condition. This conjecture has been developed by looking for the right stability condition, and now exists in various degrees of generality. A general version is the following:

Conjecture 3.1 (Yau-Tian-Donaldson, v.1). *Let (X, L) be a polarised manifold (i.e. a manifold X with an ample line bundle $L \rightarrow X$). Then M admits a cscK metric in $c_1(L)$ if and only if (X, L) is K-stable (in a more general version than defined in this essay).[12]*

For smooth Kähler manifolds (M, ω) , the existence of a Kähler-Einstein metric was shown to be equivalent to the properness (a very specific property) of a certain functional on the automorphism-invariant functions φ such that $\omega + \partial\bar{\partial}\varphi > 0$, giving a way point for proving stability conditions.[13]

The version of the conjecture most relevant to us is this one:

Conjecture 3.2 (Yau-Tian-Donaldson, v.2). *A Fano variety admits a Kähler-Einstein metric if and only if it is K-stable.*

In fact, this second ‘conjecture’ has now been proven completely. Chen, Donaldson & Sun [5] proved that K-stable Fano varieties admit Kähler-Einstein metrics. Conversely, after partial results proved by Tian [14] (assuming X to be a smooth manifold) and Stoppa [11] (assuming X to be smooth and $\text{Aut}(X)$ to be discrete, put allowing for general polarised manifolds), Berman [2] proved that in fact all Fano varieties admitting Kähler-Einstein metrics are K-stable, i.e. this paper’s theorem 1.1. In fact, Berman gave two proofs in his paper. In the rest of this essay, we will explain one of Berman’s proofs.

4 Proof of theorem 1.1

The outline of the proof is as follows: given a Kähler-Einstein metric on X , it will be extended to a metric on a test configuration $(\mathcal{X}, -K)$, restricted to the closed unit disc Δ . Then, this metric will induce a metric called the Ding metric on a certain line bundle $\eta \rightarrow \Delta$ defined by Deligne pairings, related to the Ding functional. The Donaldson-Futaki invariant of $(\mathcal{X}, -K)$ will be related to this Ding metric, yielding the result.

As stated before, this theorem and its proof are due to Berman [2]. The proof, however, has been expanded upon from its very dense form found in [2], and some inaccuracies in notation have been resolved.

4.1 Extension of metrics to test configurations

Given a Fano manifold X with a locally bounded metric φ_1 on $-K_X \rightarrow X$ with positive curvature and a test configuration \mathcal{X} for X , we will construct a metric φ on $-K|_M \rightarrow M$, where $M := \mathcal{X}|_\Delta$, which is a variety with boundary. Then we will show this metric has certain desirable properties.

Denote by φ_1 also the \mathbb{C}^* -invariant metric on $-K^* \rightarrow \mathcal{X}^*$ over Δ^* , the punctured disc obtained from φ_1 on $X = \mathcal{X}_1$. Then define

$$\varphi := \sup \{ \psi \mid \psi \leq \varphi_1 \text{ on } \partial M \} \quad (4.1)$$

where ψ ranges over all (not necessarily continuous) locally bounded metrics on $-K|_M \rightarrow M$ with positive curvature. This metric φ is clearly S^1 -invariant.

In the following proposition, we will prove some properties of φ . As most relevant entities are S^1 -invariant, we will occasionally use $t = -\log |\tau|$ as a real coordinate. Also, to compare metrics, we will set $\varphi^t := \rho(\tau)^* \varphi_\tau$ as a metric on $(X, -K_X)$. By S^1 -invariance, this indeed only depends on t .

Proposition 4.1. *The metric φ , as defined above, is locally bounded, upper semi-continuous, and has positive curvature. Furthermore, $\varphi_\tau \rightarrow \varphi_1$ uniformly as $|\tau| \rightarrow 1$.*

Proof. First, we will construct a continuous metric $\bar{\varphi}$ on $-K$ that coincides with φ_1 on ∂M and converges to it as $|\tau| \rightarrow 1$. Note that as $-K$ is a relatively ample line bundle, for some r and N we get that the complete linear system $|-rK|$ embeds \mathcal{X} into $\mathbb{P}^N \times \mathbb{C}$. Hence we get a continuous metric $\frac{1}{r}\varphi_{FS}$ with positive curvature current on $-K$ by restricting the fibrewise Fubini-Study form. Now set $\bar{\varphi} := \max\{\varphi_1 + \log |\tau|, \frac{1}{r}\varphi_{FS} - C\}$, where C is chosen such that for small $|\tau|$, $\bar{\varphi} = \frac{1}{r}\varphi_{FS}$ and for $|\tau|$ close to 1, $\bar{\varphi} = \varphi_1 + \log |\tau|$. As $\bar{\varphi}$ is clearly an admissible ψ in equation 4.1, we get a lower bound

$$\varphi \geq \bar{\varphi} \geq \varphi_1 + \log |\tau|$$

For an upper bound, note that there exists a C such that $\psi \leq \frac{1}{r}\varphi_{FS} + C$ on ∂M for any ψ considered in the supremum, as $\psi = \varphi_1$ there and φ_1 is bounded. By general compactness arguments (see [2]), it follows that there exists a C' such that for all ψ , hence also for φ , $\psi \leq \frac{1}{r}\varphi_{FS} + C'$ on all of M . Hence φ is finite and it has positive curvature as all ψ do. By the upper bound, the upper semicontinuous regularisation of φ is a candidate for φ , hence φ is upper semicontinuous.

Finally, the uniform convergence statement. As φ is positively curved, φ^t is convex. Hence, it admits a right derivative $\dot{\varphi}$ at 0 and so for any positive t , we get $\dot{\varphi} \leq (\varphi^t - \varphi^0)/t$. Furthermore, as the upper bound is independent of t , and t is unbounded, convexity gives $\varphi^t \leq \varphi^0 = \varphi_1$. Therefore, $\dot{\varphi}t \leq \varphi^t - \varphi^0 \leq 0$ for all t , yielding the result. \square

4.2 Deligne pairings

Suppose $\pi : X \rightarrow B$ is a flat projective morphism of integral schemes of pure relative dimension n , and L_0, \dots, L_n are line bundles over X . These determine a line bundle $\langle L_0, \dots, L_n \rangle(X/B)$ over B , called the Deligne pairing, as follows. Let s_i be rational sections of L_i , whose divisors have empty intersection. Then $\langle L_0, \dots, L_n \rangle(X/B)$ is locally generated by symbols $\langle s_0, \dots, s_n \rangle$, satisfying the relations

$$\langle s_0, \dots, f s_i, \dots, s_n \rangle = f[Y] \cdot \langle s_0, \dots, s_n \rangle \quad (4.2)$$

where f is a rational function, $Y = \bigcap_{j \neq i} \text{div}(s_j) = \sum_i n_i Y_i$ is a flat divisor over B which has empty intersection with $\text{div}(f)$, and $f[Y](b) = \prod_i \prod_{x \in Y_i \cap \mathcal{X}_b} f(x)^{n_i}$.

Given smooth metrics φ_j on L_j , there exists an induced metric $\varphi_D = \langle \varphi_0, \dots, \varphi_n \rangle$ on $\langle L_0, \dots, L_n \rangle$, defined by induction on n :

$$\begin{aligned} \log \left\| \langle s_0, \dots, s_n \rangle \right\|_{\langle \varphi_0, \dots, \varphi_n \rangle} &= \log \left\| \langle s_0, \dots, s_{n-1} \rangle (\operatorname{div}(s_n)) \right\|_{\langle \varphi_0, \dots, \varphi_{n-1} \rangle} \\ &\quad + \pi_* \left(\log \|s_n\|_{\varphi_n} \bigwedge_{i=0}^{n-1} dd^c \log \|s_i\|_{\varphi_i} \right) \end{aligned} \quad (4.3)$$

where $\pi_*(f)(b) = \int_{\mathcal{X}_b} f \cdot [17]$

This pairing has some useful properties[10, 2]:

- it is multilinear in its arguments;
- its curvature is given by

$$dd^c \varphi_D = \pi_*(dd^c \varphi_0 \wedge \dots \wedge dd^c \varphi_n) \quad (4.4)$$

- if $\varphi, \psi \in \mathcal{H}_B(L)$ and φ_D, ψ_D are the Deligne pairings on $\langle L, \dots, L \rangle$ in case B is a point, then

$$\varphi_D - \psi_D = (n+1)\mathcal{E}(\varphi, \psi) \quad (4.5)$$

The importance of Deligne pairings in our current setting comes from the following corollary to the main result of Phong-Ross-Sturm.[10]

Proposition 4.2. *The Donaldson-Futaki invariant of a test configuration \mathcal{X} for a Fano variety X is minus the weight over 0 of the line bundle over \mathbb{C} given by*

$$\eta := \gamma \langle -K_{\mathcal{X}/\mathbb{C}}, \dots, -K_{\mathcal{X}/\mathbb{C}} \rangle \quad (4.6)$$

where $\gamma = \frac{-1}{(n+1)V}$.

Proof. This proof is due to Berman [2].

By proposition 6 of Li-Xu [8], we get the following formula for the Donaldson-Futaki invariant (remembering our different sign and normalisation):

$$\operatorname{DF}(\mathcal{X}, \mathcal{L}) = \frac{-1}{(n+1)(-K_X)^n} \left(n\bar{\mathcal{L}}^{n+1} + (n+1)K_{\bar{\mathcal{X}}/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^n \right)$$

where $\bar{\mathcal{X}}$ is the compactification of \mathcal{X} over \mathbb{P}^1 , and $\bar{\mathcal{L}}$ the \mathbb{C}^* -compatible extension of \mathcal{L} over $\bar{\mathcal{X}}$. The products are intersection numbers. Using that $\mathcal{L} = -K_{\mathcal{X}/\mathbb{C}}$, so $\bar{\mathcal{L}} = -K_{\bar{\mathcal{X}}/\mathbb{P}^1}$, and that intersection numbers can be computed as integrals of Chern classes, this gives

$$\begin{aligned} \operatorname{DF}(\mathcal{X}) &= \frac{1}{(n+1) \int_X c_1(X)^n} \int_{\bar{\mathcal{X}}} c_1(-K_{\bar{\mathcal{X}}/\mathbb{P}^1})^{n+1} \\ &= \frac{1}{(n+1)V} \int_{\mathbb{P}^1} \pi_*(c_1(-K_{\bar{\mathcal{X}}/\mathbb{P}^1})^{n+1}) \end{aligned}$$

Hence, by equation 4.4,

$$\operatorname{DF}(\mathcal{X}) = -\gamma \int_{\mathbb{P}^1} c_1 \left(\langle -K_{\bar{\mathcal{X}}/\mathbb{P}^1}, \dots, -K_{\bar{\mathcal{X}}/\mathbb{P}^1} \rangle \right) = \int_{\mathbb{P}^1} c_1(-\bar{\eta})$$

This is the degree of $-\bar{\eta}$ over \mathbb{P}^1 , which is exactly minus the weight over zero of the \mathbb{C}^* -action on η , as the extension to \mathbb{P}^1 is given by this action. \square

Now, we will define a metric on $\eta|_\Delta$. Given a metric φ_1 on $-K_X$, we already have the metric $\gamma\varphi_D$ on $\eta|_\Delta$, defined via the metric φ on $-K \rightarrow M$. However, it will be convenient to modify this metric by adding the function v_φ , defined just before definition 2.14.

Definition 4.3. The Ding metric Φ on η related to the metric φ on $-K_X$ is defined as

$$\Phi := \gamma\varphi_D + v_\varphi \quad (4.7)$$

This metric has been chosen so that in the case where \mathbb{C} is replaced by a point, and choosing two metrics φ, ψ , $\Phi - \Psi = \mathcal{D}(\varphi, \psi)$, as is obvious from the definition of \mathcal{D} .

Via the Ding functional and lemma 2.15, this metric is linked to the existence of Kähler-Einstein metrics on X . In order to link this Ding metric to K-stability as well, via proposition 4.2, the next lemma will be useful. However, we first need a definition.

Definition 4.4. Given a subharmonic function φ on \mathbb{C} , the Lelong number of φ at a point $x \in \mathbb{C}$ is defined to be

$$l_\varphi(x) = \liminf_{\tau \rightarrow x} \frac{\varphi(\tau)}{\log |\tau - x|^2} \quad (4.8)$$

Equivalently, $l_\varphi(x) = \sup \{l \mid \varphi(\tau) \leq l \log |\tau - x|^2 \text{ close to } x\}$

Lemma 4.5. Let $L \rightarrow \mathbb{C}$ be a line bundle with a \mathbb{C}^* -action ρ covering the standard action on \mathbb{C} and let φ be an S^1 -invariant metric on L with positive curvature current. Then

$$-\lim_{t \rightarrow \infty} \frac{d}{dt} \log \|\rho(\tau)s\|_\varphi^2 = 2w - 2l_\varphi(0) \quad (4.9)$$

where $t = -\log |\tau|$, s is a holomorphic section of L and w is the weight of ρ on L_0 .

Proof. We can assume that around $\tau = 0$, taking s to be our frame, $|\rho(\tau)s| = |\tau^w s| = e^{-tw}|s|$, by definition of w and t . Hence we get, using upper semicontinuity and S^1 -invariance of φ ,

$$\begin{aligned} -\lim_{t \rightarrow \infty} \frac{d}{dt} \log \|\rho(\tau)s\|_\varphi^2 &= -\lim_{t \rightarrow \infty} \frac{d}{dt} \log e^{-2tw - \varphi(\tau)} \\ &= \lim_{t \rightarrow \infty} \frac{d}{dt} (2tw + \varphi(\tau)) \\ &= 2w + \lim_{t \rightarrow \infty} \frac{d}{dt} l_\varphi(0) \log |\tau|^2 \\ &= 2w + \lim_{t \rightarrow \infty} \frac{d}{dt} (-2l_\varphi(0)t) \\ &= 2w - 2l_\varphi(0) \end{aligned}$$

□

An alternative characterisation of the Lelong number would be the following (where we assume for simplicity that $x = 0$, as that is the only relevant case for us):

Lemma 4.6. *The Lelong number of a subharmonic function φ at 0 can be calculated by*

$$l_\varphi(0) = \inf \left\{ l \geq 0 \mid \int_V e^{-(\varphi(\tau) + (1-l) \log |\tau|^2)} d\tau \wedge d\bar{\tau} < \infty \right\} \quad (4.10)$$

where $V \subset \mathbb{C}$ is a suitably small bounded neighbourhood of 0.

Proof. The main point here is that $\int_V |\tau|^{-\alpha} d\tau \wedge d\bar{\tau} < \infty$ if and only if $\alpha < 2$. So for the integral in the lemma to converge, we need that

$$\begin{aligned} 0 &= \lim_{\tau \rightarrow 0} e^{-(\varphi(\tau) + (1-l) \log |\tau|^2)} \cdot |\tau|^2 \\ &= \lim_{\tau \rightarrow 0} e^{-\varphi(\tau) + (l-1) \log |\tau|^2 + \log |\tau|^2} \\ &= \lim_{\tau \rightarrow 0} e^{-\varphi(\tau) + l \log |\tau|^2} \end{aligned}$$

This is equivalent to

$$\lim_{\tau \rightarrow 0} (\varphi(\tau) - l \log |\tau|^2) = \infty$$

Hence, using that the sets of l such that this limit is or is not equal to ∞ are both connected (such that the infimum of one is the supremum of the other),

$$\begin{aligned} l_\varphi(0) &= \liminf_{\tau \rightarrow 0} \frac{\varphi(\tau)}{\log |\tau|^2} \\ &= \sup \{ l \mid \varphi(\tau) \leq l \log |\tau|^2 \text{ close to } 0 \} \\ &= \sup \left\{ l \mid \lim_{\tau \rightarrow 0} \varphi(\tau) - l \log |\tau|^2 < \infty \right\} \\ &= \inf \left\{ l \mid \lim_{\tau \rightarrow 0} \varphi(\tau) - l \log |\tau|^2 = \infty \right\} \\ &= \inf \left\{ l \geq 0 \mid \int_V e^{-(\varphi(\tau) + (1-l) \log |\tau|^2)} d\tau \wedge d\bar{\tau} < \infty \right\} \end{aligned}$$

□

4.3 Properties of the Ding metric

In this section, we will first establish some properties of $v_\varphi(\tau)$, using them to obtain certain properties for the Ding metric that will be used in the proof via lemma 4.5.

Theorem 4.7. [4] *For a special test configuration \mathcal{X} and an S^1 -invariant locally bounded metric φ on $-K_{\mathcal{X}/\mathbb{C}}$ with positive curvature current, $v_\varphi(\tau)$ has the following properties on Δ^* :*

- $v_\varphi(\tau)$ is subharmonic in τ (convex in t);
- If $v_\varphi(\tau)$ is actually harmonic (affine in t), then \mathcal{X} is a trivial test configuration.

Proof. We will follow Berndtsson [4] in this proof in case X is smooth. It is extended to X with klt singularities in [3].

Consider the trivial line bundle $E \rightarrow \Delta$ with fibre $H^0(X, K_X + L)$ (as $L = -K_X$, this is indeed trivial). This bundle has a natural metric

$$\|u\|_\tau^2 := \int_{\mathcal{X}_\tau} |u|^2 e^{-\varphi_\tau} \quad (4.11)$$

Notice that for $u \equiv 1$, $v_\varphi(\tau) = -\log \|u\|_\tau^2$. Hence $v_\varphi(\tau)$ is subharmonic if and only if $dd^c v_\varphi(\tau) \geq 0$ if and only if E has non-negative curvature, and it is harmonic if and only if $dd^c v_\varphi(\tau) = 0$ if and only if E has zero curvature.

Define the operator

$$\partial^{\varphi_\tau} := e^{\varphi_\tau} \partial e^{-\varphi_\tau} = \partial - \partial \varphi_\tau \wedge$$

Then for any L -valued $(n, 0)$ -form η orthogonal to the space of holomorphic forms, the equation

$$\partial^{\varphi_\tau} w = \eta$$

has a solution. Moreover, if we assume $\bar{\partial} w \wedge \omega = 0$ for a certain fixed Kähler form ω , w is unique, as was shown in [4].

Now, if π_\perp is the projection on the orthogonal complement of the space of holomorphic forms, we solve for a holomorphic section u_τ of E the equation

$$\partial^{\varphi_\tau} w_\tau = \pi_\perp(\dot{\varphi}_\tau u_\tau) \quad (4.12)$$

where $\dot{\varphi}_\tau = \frac{\partial \varphi_\tau}{\partial \tau}$. Also set

$$\hat{u} = u_\tau - d\tau \wedge w_\tau$$

Then the following formula for the curvature form Θ on E , due to Berndtsson [4], can be used.

$$\langle \Theta u_\tau, u_\tau \rangle_\tau = \pi_*(2\pi dd^c \varphi \wedge \hat{u} \wedge \bar{\hat{u}} e^{-\varphi}) + \int_{\mathcal{X}_\tau} |\bar{\partial} w_\tau|^2 e^{-\varphi_\tau} i d\tau \wedge d\bar{\tau} \quad (4.13)$$

As both terms on the right hand side are non-negative, and taking $u_\tau \equiv 1$, we get $dd^c v_\varphi = \frac{1}{2\pi} \langle \Theta u_\tau, u_\tau \rangle \geq 0$, so the first point follows.

For the second point, assume $dd^c v_\varphi = 0$. Then both terms on the right hand side of equation 4.13 must vanish, so in particular w_τ is a holomorphic form. Taking $\bar{\partial}$ of equation 4.12, we get

$$\bar{\partial} \partial^{\varphi_\tau} w_\tau = \bar{\partial} \dot{\varphi}_\tau \wedge u_\tau$$

And since

$$\bar{\partial} \partial^{\varphi_\tau} + \partial^{\varphi_\tau} \bar{\partial} = \bar{\partial} \partial - \bar{\partial} \partial \varphi_\tau \wedge - \partial \varphi_\tau \wedge \bar{\partial} + \partial \bar{\partial} + \partial \varphi_\tau \wedge \bar{\partial} = \bar{\partial} \bar{\partial} \varphi_\tau \wedge$$

we get

$$\bar{\partial} \bar{\partial} \varphi_\tau \wedge w_\tau = \bar{\partial} \partial^{\varphi_\tau} w_\tau + \partial^{\varphi_\tau} \bar{\partial} w_\tau = \bar{\partial} \dot{\varphi}_\tau \wedge u_\tau$$

Now, defining the gradient vector field V_τ of $i\dot{\varphi}_\tau$ by

$$\iota_{V_\tau}(i\partial \bar{\partial} \varphi_\tau) = i\bar{\partial} \dot{\varphi}_\tau$$

where ι_{V_τ} denotes contraction, and noting $\partial\bar{\partial}\varphi_\tau \wedge u_\tau = 0$ as it should be an $(n+1, 1)$ -form,

$$\partial\bar{\partial}\varphi_\tau \wedge w_\tau = \bar{\partial}\dot{\varphi}_\tau \wedge u_\tau = \iota_{V_\tau}(\partial\bar{\partial}\varphi_\tau) \wedge u_\tau = -\partial\bar{\partial}\varphi_\tau \wedge \iota_{V_\tau}(u_\tau)$$

As we assumed $i\partial\bar{\partial}\varphi_\tau > 0$, we hence deduce

$$-w_\tau = \iota_{V_\tau}(u_\tau)$$

This proves V_τ is a holomorphic vector field for all τ .

From the vanishing of the first term of equation 4.13, it follows that $i\partial\bar{\partial}\varphi \wedge \hat{u} \wedge \bar{\hat{u}} \equiv 0$. As this is a semidefinite form in \hat{u} , we get $\partial\bar{\partial}\varphi \wedge \hat{u} = 0$. The $d\tau \wedge d\bar{\tau}$ term of this expression reads

$$\frac{\partial^2\varphi}{\partial\tau\partial\bar{\tau}} - \partial_X\left(\frac{\partial\varphi}{\partial\bar{\tau}}\right)(V_\tau) = 0$$

By equation 4.12, $\partial^{\varphi_\tau}w_t = \dot{\varphi} \wedge u + h_\tau$, where h_τ is some holomorphic form on X for all τ . From this it follows that

$$\begin{aligned} \partial^{\varphi_\tau}\frac{\partial w_\tau}{\partial\bar{\tau}} &= -\frac{\partial}{\partial\bar{\tau}}\partial^{\varphi_\tau}w_\tau - \partial\frac{\partial\varphi}{\partial\bar{\tau}} \wedge w_\tau \\ &= -\frac{\partial}{\partial\bar{\tau}}(\dot{\varphi} \wedge u + h_\tau) + \partial\frac{\partial\varphi}{\partial\bar{\tau}} \wedge \iota_{V_\tau}(u) \\ &= \left(\frac{\partial^2\varphi}{\partial\tau\partial\bar{\tau}} - \partial_X\left(\frac{\partial\varphi}{\partial\bar{\tau}}\right)(V_\tau)\right) \wedge u + \frac{\partial h_\tau}{\partial\bar{\tau}} \\ &= \frac{\partial h_\tau}{\partial\bar{\tau}} \end{aligned}$$

which is holomorphic. But as the left hand side is orthogonal to the holomorphic forms, $\partial^{\varphi_\tau}\frac{\partial w_\tau}{\partial\bar{\tau}} = 0$. By injectivity of ∂^{φ_τ} (see [4]), w_τ is holomorphic in τ , hence V_τ is holomorphic on $X \times \Delta^* \cong \mathcal{X}^*$.

Now define the holomorphic vector field $W := V_\tau - \frac{\partial}{\partial\tau}$ on \mathcal{X}^* . Then, as

$$L_{V_\tau}\partial\bar{\partial}\varphi_\tau = \partial\iota_{V_\tau}\partial\bar{\partial}\varphi_\tau = \partial\bar{\partial}\dot{\varphi}_\tau = \frac{\partial}{\partial\tau}\partial\bar{\partial}\varphi_\tau$$

we get $L_W\partial\bar{\partial}\varphi_\tau = 0$. Hence $\partial\bar{\partial}\varphi_\tau$ moves by the flow a holomorphic family of automorphisms $F_\tau : \mathcal{X}^* \rightarrow \mathcal{X}^*$.

The family $F_\tau|_{\mathcal{X}_1} : \mathcal{X}_1 \rightarrow \mathcal{X}_\tau$ can also be considered as a family of embeddings of X into \mathbb{P}^N , using the embedding in the proof of proposition 4.1. As $F_\tau^*\varphi_\tau = \varphi_1$ and φ is locally bounded, we get

$$\sup_X |F_\tau^*\varphi_{FS} - \varphi_1| \leq C$$

where C is independent of τ . Hence F_τ converges to a holomorphic map $F_0 : X_1 \rightarrow (X_0)_{\text{red}} = X_0$. As it pulls back an ample line bundle to an ample line bundle, F_0 is finite. Furthermore, it is generically one-to-one, so the map $X \times \mathbb{C} \rightarrow \mathcal{X} : (\tau, x) \rightarrow (\tau, F_\tau(x))$ is a biholomorphism away from a codimension two subvariety. By normality of \mathcal{X} , it then is a biholomorphism, proving the second point. \square

For the next proposition, we need the Ohsawa-Takegoshi theorem (formulation from [6]).

Theorem 4.8 (Ohsawa-Takegoshi). *Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain, and let L be an affine linear subspace of \mathbb{C}^n of codimension $p \geq 1$ given by an orthonormal system s of affine linear equations $s_1 = \dots = s_p = 0$. For every $\beta < p$, there exists a constant $C_{\beta,n,\Omega}$ depending only on β , n and the diameter of Ω , satisfying the following property. For every $\varphi \in PSH(\Omega)$ and $f \in \mathcal{O}(\omega \cap L)$ with $\int_{\Omega \cap L} |f|^2 e^{-\varphi} dV_L < +\infty$, there exists an extension $F \in \mathcal{O}(\Omega)$ of f such that*

$$\int_{\Omega} |F|^2 |s|^{-2\beta} e^{-\varphi} dV_{\mathbb{C}^n} \leq C_{\beta,n,\Omega} \int_{\Omega \cap L} |f|^2 e^{-\varphi} dV_L \quad (4.14)$$

where $dV_{\mathbb{C}^n}$ and dV_L are the Lebesgue volume elements in \mathbb{C}^n and L respectively.

Proposition 4.9. *The Lelong number of $v_{\varphi}(\tau)$ at 0 vanishes.*

Proof. Let $s \equiv 1$ be the standard trivialising section of the trivial line bundle $\mathcal{L} + K_{\mathcal{X}/\mathbb{C}} \rightarrow \mathcal{X}|_{\Delta}$. This corresponds to the trivial section over $\pi_*(\mathcal{L} + K_{\mathcal{X}/\mathbb{C}}) \rightarrow \Delta$. Then $v_{\varphi}(\tau) = -\log \|s\|^2$. Using lemma 4.6, write the Lelong number as

$$l_{v_{\varphi}}(0) = \inf \left\{ l \left| \int_{\Delta} e^{-(v_{\varphi}(\tau) + (1-l) \log |\tau|^2)} d\tau \wedge d\bar{\tau} < \infty \right. \right\}$$

By writing out the definition of $v_{\varphi}(\tau)$ in this integral, we get

$$\begin{aligned} \int_{\Delta} e^{-(v_{\varphi}(\tau) + (1-l) \log |\tau|^2)} d\tau \wedge d\bar{\tau} &= \int_{\Delta} e^{-(-\log \int_{\mathcal{X}_{\tau}} e^{-\varphi_{\tau}} + (1-l) \log |\tau|^2)} d\tau \wedge d\bar{\tau} \\ &= \int_{\Delta} \int_{\mathcal{X}_{\tau}} |s|^2 e^{-\varphi_{\tau}} e^{-(1-l) \log |\tau|^2} \otimes d\tau \wedge d\bar{\tau} \\ &= \int_{\mathcal{X}|_{\Delta}} |s|^2 e^{-\varphi} |\tau|^{-2(1-l)} \otimes d\tau \wedge d\bar{\tau} \end{aligned}$$

Now, as X is projective, it is compact (as a manifold). Hence $\mathcal{X}|_{\Delta}$ is as well. We may therefore cover $\mathcal{X}|_{\Delta}$ by a finite family of trivialising open neighbourhoods $\mathfrak{U} = \{U_i\}_{i=1}^n$. On such a U_i , complement the coordinate τ with a coordinate $z \in \mathbb{C}^n$ such that (τ, z) trivialise U_i (refine \mathfrak{U} if necessary for this). We may then write the measure $|s|^2 e^{-\varphi} \otimes d\tau \wedge d\bar{\tau} = |s|^2 e^{-\varphi} dz \wedge d\bar{z} \wedge d\tau \wedge d\bar{\tau}$, where φ becomes a psh function. By definition of τ , it is an affine function which vanishes exactly at \mathcal{X}_0 . Hence, by the Ohsawa-Takegoshi theorem, setting $\Omega = U_i$, $s = \tau$, $\beta = 1 - l < 1$, $f \equiv 1$, there exists an $F_i \in \mathcal{O}(U_i)$ extending f such that

$$\begin{aligned} \int_{U_i} |F_i s|^2 e^{-\varphi_{\tau}} |\tau|^{-2(1-l)} \otimes d\tau \wedge d\bar{\tau} &= \int_{U_i} |F_i|^2 e^{-\varphi_{\tau}} |\tau|^{-2(1-l)} dV_{\mathbb{C}^n} \\ &\leq C \int_{U_i \cap \{\tau=0\}} e^{-\varphi} dV_{\mathbb{C}^{n-1}} < \infty \end{aligned}$$

As F_i extends f , it is equal to 1 on $\{\tau = 0\}$. Hence, we can refine \mathfrak{U} such that F_i is bounded below by some $\varepsilon_i > 0$ for all i such that $U_i \cap \mathcal{X}_0 \neq \emptyset$. On such a U_i we then get

$$\int_{U_i} |s|^2 e^{-\varphi_{\tau}} |\tau|^{-2(1-l)} \otimes d\tau \wedge d\bar{\tau} \leq \frac{1}{\varepsilon_i} \int_{U_i} |F_i s|^2 e^{-\varphi_{\tau}} |\tau|^{-2(1-l)} \otimes d\tau \wedge d\bar{\tau} < \infty$$

As the finiteness of the integral is clear if $U_i \cap \mathcal{X}_0 = \emptyset$, and the cover is finite, it follows that

$$\int_{\mathcal{X}|\Delta} |s|^2 e^{-\varphi_\tau} |\tau|^{-2(1-l)} \otimes d\tau \wedge d\bar{\tau} < \infty$$

for all $l \in (0, \infty)$. Hence $l_{v_\varphi}(0) = 0$. \square

Corollary 4.10. *Let \mathcal{X} be a test configuration for a Fano manifold X . Suppose φ is a bounded S^1 -invariant metric on $-K$ with positive curvature current and let Φ be its Ding metric on η . Then*

- Φ has positive curvature current on Δ ;
- if the curvature on Δ^* vanishes, \mathcal{X} is a trivial test configuration;
- Φ is locally bounded and continuous on Δ^* (including the boundary) and we have $l_\Phi(0) = 0$.

Remark. There is some unclarity in Berman's sign conventions here. His proof of the first and second point is as follows:

By equation 4.4, we have that φ_D is positively curved. By theorem 4.7, v_φ is subharmonic, i.e. $dd^c v_\varphi \geq 0$. It follows that $\Phi = \gamma\varphi_D + v_\varphi$ also has positive curvature.

Similarly, if the curvature is zero, in particular $dd^c v_\varphi = 0$, so again by theorem 4.7, \mathcal{X} is trivial.

However, as $\gamma < 0$, this conclusion does not seem to follow.

Proof (of point three). By a result of Moriwaki [9], for any smooth metric ψ , ψ_D is continuous. As φ is bounded, $u := \varphi - \psi$ is a bounded function on \mathcal{X} . Hence, by the change of metrics formula (equation 4.5),

$$|\varphi_D - \psi_D| \leq (n+1) \left(\sup_{\mathcal{X}} |u| \right) c_1(X)^n$$

so

$$|\gamma(\varphi_D - \psi_D)| \leq \sup_{\mathcal{X}} |u|$$

As continuous metric are locally bounded (they can be approximated by smooth metrics), $\gamma\varphi_D$ is locally bounded.

As $\varphi_\tau \rightarrow \varphi_1$ uniformly as $|\tau| \rightarrow 1$ by proposition 4.1, again by the change of metric formula,

$$\left| \gamma((\rho(\tau)^* \varphi_\tau)_D - (\varphi_1)_D) \right| \leq C \sup_{\mathcal{X}_\tau} |\rho(\tau)^* \varphi_\tau - \varphi_1| \rightarrow 0$$

as $|\tau| \rightarrow 1$. For v_φ , it is clear that $v_\varphi(\tau) \rightarrow v_\varphi(1)$ as $|\tau| \rightarrow 1$, hence Φ is continuous on $\partial\Delta$. As it is also subharmonic, S^1 -invariant and bounded, it is convex in t , and hence continuous on Δ^* . As φ_D is bounded around 0, $l_\Phi(0) = l_{v_\varphi}(0) = 0$, by proposition 4.9, concluding the proof. \square

4.4 Putting the pieces together

Theorem 4.11. *Let X be a Fano variety admitting a Kähler-Einstein metric φ_1 . Then X is K-stable.*

Proof. Let $(\mathcal{X}, \mathcal{L})$ be a test configuration for X . By lemma 2.17, we have $\mathcal{L} = -K_{\mathcal{X}/\mathbb{C}}$. The metric φ_1 can be extended to a locally bounded, upper semicontinuous S^1 -invariant metric φ with positive curvature on $-K \rightarrow \mathcal{X}|_{\Delta}$ according to equation 4.1. This metric in turn induces a Ding metric Φ on $\eta \rightarrow \Delta$ according to definition 4.3. By corollary 4.10, Φ has positive curvature and is locally bounded on Δ^* . It also has vanishing Lelong number at 0.

By proposition 4.2, the Donaldson-Futaki invariant of X is minus the weight of the \mathbb{C}^* -action on η over 0. The weight of this action can be computed, using lemma 4.5 and the fact that for two metrics φ, ψ we have $\Phi - \Psi = \mathcal{D}(\varphi, \psi)$ on a point, by the formula

$$\begin{aligned} \text{DF}(\mathcal{X}) &= -w = \frac{1}{2} \lim_{t \rightarrow \infty} \frac{d}{dt} \log \|\rho(\tau)s\|_{\Phi}^2 - l_{\Phi}(0) \\ &= -\frac{1}{2} \lim_{t \rightarrow \infty} \frac{d}{dt} ((\rho(\tau)^*\Phi)|_{\mathcal{X}_1} - \Phi_1|_{\mathcal{X}_1}) \\ &= -\frac{1}{2} \lim_{t \rightarrow \infty} \frac{d}{dt} \mathcal{D}(\rho(\tau)^*\varphi_{\tau}, \varphi_1) =: -\frac{1}{2} \lim_{t \rightarrow \infty} \frac{d}{dt} \mathcal{D}(t) \end{aligned}$$

As φ_1 is a Kähler-Einstein metric, it is a minimum of the Donaldson-Futaki invariant, so at $t = 0$, $\frac{d}{dt} \mathcal{D}(t) = 0$. As $\mathcal{D}(t)$ is convex by corollary 4.10, its derivative is nondecreasing, so $\text{DF}(\mathcal{X}) = -\lim_{t \rightarrow \infty} \frac{d}{dt} \mathcal{D}(t) \leq 0$. Hence X is semi-stable.

Convexity also implies that if $\text{DF}(\mathcal{X}) = 0$, then Φ must have vanishing curvature, so by corollary 4.10 again, \mathcal{X} must be a trivial test configuration, i.e. X is actually K-stable. \square

References

- [1] T. Aubin, *Equations du type Monge-Ampère sur les variétés kählériennes compactes*, C. R. Acad. Sci. Paris Sér. A 283 (1976), 119.
- [2] R. Berman, *K-stability and \mathbb{Q} -Fano Varieties Admitting Kähler-Einstein metrics*, arXiv:1205.6214, 1982.
- [3] R. Berman, S. Boucksom, P. Eyssidieux, V Guedj, A Zeriahi. *Kähler-Einstein metrics and the Kähler-Ricci flow on log Fano varieties*, arXiv:1111.7158, 2012.
- [4] B. Berndtsson, *A Brunn-Minkowski type inequality for Fano manifolds and the Bando-Mabuchi uniqueness theorem*, arXiv:1103.0923, 2011
- [5] X. Chen, S. Donaldson, S. Song, *Kähler-Einstein metrics and stability*, arXiv:1210.7494, 2012.
- [6] J.P. Demailly, J. Kolla, *Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds*, Ann. Sci. École Norm. Sup. (4) 34 (2001), no. 4, 525–556.
- [7] S. Donaldson, *Kähler metrics with cone singularities along a divisor*, arXiv:1102.1196, 2011.
- [8] C. Li, C. Xu, *Special test configurations and K-stability of Fano varieties*, arXiv:1111.5398, 2013.
- [9] A. Moriawaki, *The continuity of Deligne’s pairing*, Internat. Math. Res. Notices 1999, no. 19, 1057-1066.
- [10] D. Phong, J. Ross, J. Sturm, *Deligne pairings and the Knudsen-Mumford expansion*, 2008.
- [11] J. Stoppa, *K-stability of constant scalar curvature Kähler manifolds*, Adv. Math. 221 (2009), no. 4, 1397-1408.
- [12] G. Székelyhidi, *Introduction to Extremal metrics*, <http://www3.nd.edu/~gszekely/>, 2011.
- [13] G. Tian, *Canonical Metrics in Kähler Geometry*, 1st edition, Birkhauser-Verlag, 2000.
- [14] G. Tian, *Kähler-Einstein metrics with positive scalar curvature*, Invent. Math. 130 (1997), no. 1, 137.
- [15] S.-T. Yau, *Calabi’s conjecture and some new results in algebraic geometry*, Proc. Natl. Acad. Sci. USA 74, no. 5, 1798-1799, 1977.
- [16] S.-T. Yau, *On The Ricci Curvature of a Compact Kähler Manifold and the Complex Monge-Ampère Equation, I*, Comm. Pure and App. Math. 31 (1978), 339-411.

- [17] S. Zhang, *Heights and reductions of semi-stable varieties*, Comp. Math. 104 (1996), 77-105.