MSc Mathematical Physics

Master Thesis

The Madsen-Weiss theorem

Author:
Reinier Kramer

Supervisor & master coordinator:
prof. dr. Sergey Shadrin

June 17, 2015

Universiteit van Amsterdam

Faculteit der Natuurwetenschappen, Wiskunde en Informatica

Korteweg-de Vriesinstituut

Abstract

This thesis aims to introduce master-level mathematics students to the statement and proof of the Madsen-Weiss theorem, calculating the stable homology type of the mapping class group of Riemann surfaces. As a corollary, the rational cohomology of the stable moduli space of Riemann surfaces is calculated.

The thesis starts off with an introduction to the topological theory of Riemann surfaces, defining mapping class groups and Teichmüller space in chapter 1. The Earle-Eells-Schatz theorem, showing the homotopy equivalence of diffeomorphism groups and mapping class groups of surfaces, is stated and proven. The mapping class group has a natural action on Teichmüller space, which is properly discontinuous with finite isotropy groups, and the quotient by this action is the moduli space. As Teichmüller space is contractible, this proves that moduli space and the classifying space for the mapping class group have the same rational homology and cohomology.

Chapter 2 is devoted to Harer stability. This theorem states that the homology of the mapping class group of a surface is independent of the genus and the amount of boundary components of the surface for homology degrees which are small with respect to the genus. The strictest currently known bound for the size of this degree is given, and the theorem is proved using arc complexes on surfaces, which have a natural action of the mapping class group. The stabilisers of this action correspond to mapping class groups of smaller genus, which make an induction argument possible, involving spectral sequences. The inductions argument requires a high connectivity of the arc complexes, the proof of which is the hardest part of Harer stability.

Chapter 3 deals with two homological constructions. Quillen's plus constructions modifies a space with perfect fundamental group to eliminate that fundamental group, keeping homology intact. The Group Completion Theorem states that for a topological monoid, its inclusion into the loop space of its classifying space induces a group completion on the monoid of connected components and a localisation of homology with respect to this last monoid.

Chapter 4 extends stable homotopy theory to the theory of spectra, essentially by enlarging the homotopy category of CW-complexes so that the reduced suspension functor becomes and auto-equivalence of categories. It is shown that spectra, generalised cohomology theories, and infinite loop spaces have essentially the same theories, and a recognition principle for infinite loop spaces is mentioned. As an example, Thom spectra and their connection to bordisms are explained in detail.

Finally, chapter 5 sees the complete statement of the Madsen-Weiss theorem, together with its proof, which occupies most of the chapter. The proof is given by considering different embedding spaces of surfaces in Euclidean space and connecting these embedding spaces via certain delooping maps. The proof that these deloopings are weak equivalences is quite messy, and is the hardest part of the current proof. In the end of the chapter, Mumford's conjecture, which is the origin of the Madsen-Weiss theorem, is confirmed.

Appendices A and B give essential background on classifying spaces and spectral sequences, respectively.

Populaire samenvatting

Voor elk gegeven geslacht en aantal randcomponenten bestaat er een zogenaamde moduliruimte die alle riemannoppervlakken met dat geslacht en dat aantal randcomponenten classificeert. Dit is een belangrijke ruimte, die topologisch zeer ingewikkeld is. Zo is de cohomologie van deze ruimte grotendeels onbekend. Wel is bekend dat de rationale cohomologie gelijk is aan die van een bepaalde groep gerelateerd aan de riemannoppervlakken.

Het blijkt dat als het geslacht groter wordt gemaakt, de cohomologie van de moduliruimte in lage graden stabiliseert. Dit zorgt ervoor dat er over de stabiele cohomologie van de moduliruimte kan worden gesproken. Het doel van deze scriptie is om de rationale versie van deze stabiele cohomologie te berekenen.

Contents

Introduction

Moduli spaces have long been objects of interest to geometers and topologists. They were originally introduced to classify certain classes of geometric spaces, but soon turned out to have interesting geometric properties of their own.

This thesis is concerned with moduli spaces of curves, i.e. of Riemann surfaces with possibly non-empty boundary. Riemann surfaces have two natural integral parameters, the genus g and the amount of boundary components r, together constituting the type (g, r). For every such type, the moduli space $\mathcal{M}_{g,r}$ is a $(6g - 6 + 3r)$-dimensional orbifold, which has highly non-trivial geometry.

Topologically, cohomology has long been established as one of the most natural and interesting invariants of spaces. Therefore, the cohomology of moduli spaces of curves has also generated a lot of interest over time. It has long been known that, up to torsion, the moduli space $\mathcal{M}_{g,r}$ is a classifying space for the mapping class group $\Gamma_{g,r}$, proving that the rational cohomology of $\mathcal{M}_{g,r}$ is isomorphic to that of $\Gamma_{g,r}$.

An important result on the cohomology of the mapping class group is Harer's stability theorem, [Har85], stating that the homology of the mapping class group is independent of the type in a range of degrees that goes to infinity if the genus goes to infinity. Contemporaneously, Mumford[Mum83] defined certain cohomology classes, κ_i in $H^{2i}(\mathcal{M}_g; \mathbb{Q})$, which are independent in degrees up to order $\sim g$. He then conjectured that in low degrees, $H^*(\mathcal{M}_g; \mathbb{Q})$ is a polynomial algebra in the κ_i. This has become known as Mumford's conjecture.

Mumford's conjecture has been an open question for nearly 25 years. In 1997, Ulrike Tillmann proved in [Til97] that the stable mapping class group of surfaces, $\Gamma_{\infty,1} = \operatorname{colim}_g \Gamma_{g,1}$ has a classifying space which, after applying Quillen's plus construction[Qui70; Qui73], is an infinite loop space. Madsen-Tillmann constructed a map from $B\Gamma^+_{g,1}$ to a known infinite loop space $\Omega^\infty_0 \mathbb{CP}^\infty_{-1}$ in [MT01] and conjectured that it is a homotopy equivalence. This was confirmed by Ib Madsen and Michael Weiss in the theorem now named after them in [MW07]. The rational cohomology of this infinite loop space was already known and together with the Madsen-Weiss theorem confirmed Mumford's conjecture.

This thesis gives a complete explanation of the Madsen-Weiss theorem, aimed at master-level students with some background in algebraic topology. We define all terms in the theorem and ingredients of the proof and explain and prove neccessary results. A complete proof of the theorem is given as well. We also give context for some of the terms and results.

This thesis has no claim to original research, but should rather be read as a literature study, giving an overview of the subject. All errors and mistakes are of course the responsability of the author.

Outline of the paper: chapter 1 gives an introduction to hyperbolic surfaces and moduli spaces. In section 1.1, the hyperbolic plane is defined and hyperbolic structures on surfaces are constructed from it. Section 1.2 introduces Teichmüller space by decomposing Riemann surfaces into pairs of pants. The moduli space of curves and the mapping class groups are defined in section 1.3 and the Earle-Eells-Schatz theorem and the action of the mapping class group on moduli space relate the cohomologies of these two objects.

Chapter 2 concerns the Harer stability theorem, proving stability of the cohomology of the mapping class groups in low degrees with respect to the genus. Section 2.1 defines arc complexes on surfaces and uses a natural action of mapping class groups on these complexes to relate mapping class groups of different types. This allows for an inductive proof of Harer stability in section 2.2.

Chapter 3 deals with two different homotopy-theoretical constructions. Section 3.1 explains Quillen's plus construction, eliminating a perfect fundamental group of a space without altering its homology. Section 3.2 is concerned with the Group Completion Theorem, which connects the homology of a topological monoid with that of the loop space of its classifying space.

Chapter 4 introduces spectra in section 4.1, the natural category for stable homotopy theory, and infinite loop spaces in definition 4.3.1, which generalise topological commutative groups to homotopy theory. As an aside, section 4.2 connects these notions with generalised cohomology theory.

Finally, in chapter 5, the Madsen-Weiss theorem is stated and proven. As a result, the rational cohomology of the stable mapping class group is calculated in section 5.2

Appendices A.1 and B give basic introductions to classifying spaces and spectral sequences, respectively.

A small note on notation:

Categories will be denoted in sans-serif. The category of groups will e.g. be denoted Grp. I will write Sp for the category of compactly generated Hausdorff spaces and CW for the category of CW-spaces. Pointed verions of these categories are denoted Sp_* and CW_*, respectively. A 'space' (when used in the topological sense) will always mean an object of Sp.

The (reduced) suspension of a pointed space $(X,*)$ will always mean $\Sigma(X,*) = (X,*) \wedge (S^1, 1) = (X \times S^1)/(X \times \{1\} \vee \{*\} \times S^1)$.

I would like to thank my supervisor, Sergey Shadrin, for his help and advise during this project.

Chapter 1 — Hyperbolic surfaces and their moduli space

Let $S_{g,r}$ be a fixed surface of type (g, r), i.e. a two-dimensional manifold with boundary which is obtained from a surface of genus g by cutting away r discs whose closures are disjoint, where $2g+r \geq 3$. Such a surface can be given a hyperbolic structure, a Riemannian metric with constant scalar curvature -1. However, this is a highly non-unique procedure: two resulting hyperbolic surfaces may not be isometric.

We will want to parametrise the different hyperbolic structures on $S_{g,r}$ by a space which is called the moduli space $\mathcal{M}_{g,r}$. To define this, we first need to look into hyperbolic surfaces.

Most of the theory in this section is well-known and is basic hyperbolic geometry. A reference would be [Bus92].

1.1 — Hyperbolic structures on a surface

1.1.1 — The hyperbolic plane

In order to define and analyse hyperbolic surfaces, it would be useful to have an example to fall back on. The basic model for hyperbolic surfaces, playing the same role as $\mathbb{R}^n$ for manifolds, is given by the following definition:

Definition 1.1.1. The *hyperbolic plane* is the Riemannian surface given by either of the following two models:

- The *Poincaré half-plane model*, denoted $\mathbb{H}$, which is given as a manifold by $\{z = x + iy \in \mathbb{C} \mid y > 0\}$, with metric
$$ds^2 = \frac{dx^2 + dy^2}{y^2} = \frac{-4dzd\bar{z}}{(z - \bar{z})^2}$$
- The *Poincaré disc model*, denoted $\mathbb{D}$, is given as a manifold by the open unit disc in $\mathbb{C}$, with metric
$$ds^2 = 4\frac{dx^2 + dy^2}{(1 - x^2 - y^2)^2} = \frac{4dzd\bar{z}}{(1 - z\bar{z})^2}$$

Remark 1.1.2. The notation $\mathbb{H}$ is also used to denote the abstract hyperbolic plane. In the rest of this thesis, this practice is adhered to.

Lemma 1.1.3. *The Poincaré half-plane model and the Poincaré disc model are isometric by the map* $f : \mathbb{H} \to \mathbb{D} : z \mapsto w := \frac{z-i}{z+i}$.

Proof. The map f is an element of $\mathrm{PGL}(2, \mathbb{C})$, hence a biholomorphism from $\mathbb{H}$ onto its image in $\mathbb{C}$. Since $f(0) = -1$, $f(\infty) = 1$, $f(1) = i$ and $f(i) = 0$, this image is indeed $\mathbb{D}$.

Now for the metric:

$$
\begin{aligned}
w &= \frac{z-i}{z+i} & \bar{w} &= \frac{\bar{z}+i}{\bar{z}-i} \\
dw &= \frac{(z+i)dz-(z-i)dz}{(z+i)^2} = \frac{2idz}{(z+i)^2} & d\bar{w} &= \frac{-2id\bar{z}}{(\bar{z}-i)^2}
\end{aligned}
$$

$$
\begin{aligned}
(ds')^2 &= \frac{4dwd\bar{w}}{(1-w\bar{w})^2} = \frac{16dzd\bar{z}}{(z+i)^2(\bar{z}-i)^2}\left(1-\frac{(z-i)(\bar{z}+i)}{(z+i)(\bar{z}-i)}\right)^{-2} \\
&= \frac{16dzd\bar{z}}{(z+i)^2(\bar{z}-i)^2}\left(\frac{(z+i)(\bar{z}-i)-(z-i)(\bar{z}+i)}{(z+i)(\bar{z}-i)}\right)^{-2} \\
&= \frac{16dzd\bar{z}}{(2i(\bar{z}-z))^2} = \frac{-4dzd\bar{z}}{(z-\bar{z})^2} = ds^2
\end{aligned}
$$

□

Lemma 1.1.4. *The metric on the Poincaré half-plane* $\mathbb{H}$ *has constant Gaussian curvature* -1.

Proof. This is just another calculation. Recall that the Gaussian curvature is given by

$$K = \frac{1}{2} g^{jl} R^i{}_{jil}$$

where R is the Riemannian curvature, given in terms of the metric, via the Christoffel symbols, by

$$
\begin{aligned}
R^i{}_{jkl} &= \partial_i \Gamma^k{}_{jl} - \partial_j \Gamma^k{}_{il} + \Gamma^h{}_{jl}\Gamma^k{}_{ih} - \Gamma^h{}_{il}\Gamma^l{}_{jh} \\
\Gamma^k{}_{ij} &= \frac{1}{2} g^{kl}\left(\partial_j g_{li} + \partial_i g_{lj} - \partial_l g_{ij}\right)
\end{aligned}
$$

Now, $g_{ij} = y^{-2}\delta_{ij}$ and $g^{ij} = y^2\delta^{ij}$. Hence, for the Christoffel symbol to be non-zero, one of the indices should be 2, and the other two should be equal. This yields:

$$
\begin{aligned}
\Gamma^1{}_{12} = \Gamma^1{}_{21} &= \frac{1}{2} y^2(\partial_1 0 + \partial_2 y^{-2} - \partial_1 0) = \frac{y^2}{2} \cdot -2y^{-3} = -y^{-1} \\
\Gamma^2{}_{22} &= \frac{1}{2} y^2 (2-1)\partial_2 y^{-2} = -y^{-1} \\
\Gamma^2{}_{11} &= \frac{1}{2} y^2 (0 + 0 - \partial_2 y^{-2}) = y^{-1}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
K &= \frac{1}{2} g^{jl} R^i{}_{jil} = \frac{1}{2} y^2 \delta^{jl}\left(\partial_i \Gamma^i{}_{jl} - \partial_j \Gamma^i{}_{il} + \Gamma^h{}_{jl}\Gamma^i{}_{ih} - \Gamma^h{}_{il}\Gamma^i{}_{jh}\right) \\
&= \frac{y^2}{2}\left(\partial_2 \Gamma^2{}_{jj} - \partial_2 \Gamma^i{}_{i2} + \Gamma^2{}_{jj}\Gamma^i{}_{i2} - \Gamma^1{}_{ij}\Gamma^i{}_{j1} - \Gamma^2{}_{ij}\Gamma^i{}_{j2}\right) \\
&= \frac{y^2}{2}\Big(\partial_2((1-1)y^{-1}) - \partial_2(-2y^{-1}) + ((1-1)y^{-1})(-2y^{-1}) \\
&\qquad - \Gamma^1{}_{12}\Gamma^1{}_{21} - \Gamma^1{}_{21}\Gamma^2{}_{11} - \Gamma^2{}_{11}\Gamma^1{}_{12} - \Gamma^2{}_{22}\Gamma^2{}_{22}\Big) \\
&= \frac{y^2}{2}\left(0 - 2y^{-2} + 0 - (-y^{-1})^2 - -y^{-1} \cdot y^{-1} - y^{-1} \cdot -y^{-1} - (-y^{-1})^2)\right) \\
&= \frac{1}{2}\big(- 2 - 1 + 1 + 1 - 1)\big) = -1
\end{aligned}
$$

□

The next thing to do is to determine the orientation-preserving isometries and the geodesics of $\mathbb{H}$. These two tasks are interrelated, and are therefore both handled here.

Proposition 1.1.5. *The real projective special linear group* $\mathrm{PSL}(2,\mathbb{R})$ *acts on the Poincaré half-plane* $\mathbb{H}$ *by orientation-preserving isometries via the assignment* $\rho : \mathrm{PSL}(2,\mathbb{R}) \longrightarrow \mathrm{Iso}^+(\mathbb{H})$ *given by Möbius transformations*

$$\rho\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(z) = \frac{az+b}{cz+d}$$

Proof. During this proof, we will fix an $f = \rho\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and show it is an isometry. Orientation preservation is direct, as it is holomorphic (with respect to the standard holomorphic structure on $\mathbb{C}$).

Let $\hat{\mathbb{C}}$ be the one-point compactification of $\mathbb{C}$. Then it is clear that f, interpreted as a map $f : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$, preserves the extended real line and

$$\begin{aligned} f(i) &= \frac{ai+b}{ci+d} = \frac{(ai+b)(-ci+d)}{c^2+d^2} \\ &= \frac{ac+bd+(ad-bc)i}{c^2+d^2} = \frac{ac+bd+i}{c^2+d^2} \in \mathbb{H} \end{aligned}$$

so f is indeed a diffeomorphism of $\mathbb{H}$.

To prove it is an isometry, we calculate

$$\begin{aligned} df(z) &= d\left(\frac{az+b}{cz+d}\right) = \frac{(cz+d)\cdot adz - (az+b)\cdot cdz}{(cz+d)^2} \\ &= \frac{(ad-bc)dz}{(cz+d)^2} = \frac{dz}{(cz+d)^2} \end{aligned}$$

Therefore,

$$\begin{aligned} (ds')^2 &= \frac{-4df(z)d\overline{f(z)}}{(f(z)-\overline{f(z)})^2} \\ &= \frac{-4dzd\bar{z}}{(cz+d)^2(c\bar{z}+d)^2}\left(\frac{az-b}{cz-d} - \frac{a\bar{z}+b}{c\bar{z}+d}\right)^{-2} \\ &= \frac{-4dzd\bar{z}}{(cz+d)^2(c\bar{z}+d)^2}\left(\frac{(az-b)(c\bar{z}+d)-(a\bar{z}+b)(cz+d)}{(cz-d)(c\bar{z}+d)}\right)^{-2} \\ &= \frac{-4dzd\bar{z}}{((ad-bc)(z-\bar{z}))^2} = \frac{-4dzd\bar{z}}{(z-\bar{z})^2} = ds^2 \end{aligned}$$

Hence, f is indeed an orientation-preserving isometry. □

Corollary 1.1.6. *The geodesics in* $\mathbb{H}$ *are exactly those circles and straight lines that meet the real axis of* $\mathbb{C}$ *orthogonally (in the Euclidean sense).*

Proof. The straight lines parallel to the imaginary axis are geodesics, because all non-zero Christoffel symbols have at least two y-coordinates (see the proof of lemma 1.1.4), so if $\dot{y} = 0$ at a point, it will remain so, via the geodesic equation.

Now, isometries, and hence the action of $\mathrm{PSL}(2,\mathbb{R})$, preserve geodesics, and $\mathrm{PSL}(2,\mathbb{R})$ acts conformally on $\mathbb{C}$. Also, it acts transitively on generalised circles orthogonal to the real line, as it is the stabiliser of that oriented line in $\mathrm{PGL}(2,\mathbb{C})$. Thus, all generalised circles meeting the real line orthogonally are geodesics.

As a generalised circle is determined by a line on which its diameter lies and two points not on that diameter, there is a unique geodesic of this kind through any two points. In particular, this is true for any point p and all points in a small neighbourhood of p. Hence, via the exponential map, these are all geodesics. □

Corollary 1.1.7. *The action map* $\rho : \mathrm{PSL}(2,\mathbb{R}) \longrightarrow \mathrm{Iso}^+(\mathbb{H})$ *is an isomorphism.*

Proof. The action is clearly faithful, so ρ is injective. It is surjective, because an isometry must send geodesics to geodesics, hence generalised lines to generalised lines, and must therefore be an element of $\mathrm{PGL}(2,\mathbb{C})$. As it must also send the real line to itself, preserving orientation, it must lie in $\mathrm{PSL}(2,\mathbb{R})$. □

1.1.2 — Hyperbolic surfaces

The hyperbolic plane is indeed a model for hyperbolic surfaces, as is stated in the next lemma:

Lemma 1.1.8. *A Riemannian surface S, possibly with boundary, has constant curvature* -1 *if and only if its Riemannian structure contains an atlas* $\mathcal{A} = (U_\alpha, \varphi_\alpha)_{\alpha \in A}$ *with each* $\operatorname{Im} \varphi_\alpha \subset \mathbb{H}$ *and* $\varphi_\alpha \circ \varphi_\beta^{-1} = \rho(B)|_{\varphi_\beta(U_\beta)}$ *for some* $B \in \mathrm{PSL}(2, \mathbb{R})$*, such that* $\varphi_\alpha(U_\alpha)$ *is one of:*

- *A disc, if* U_α *lies in the interior of S;*
- *A disc intersected with either one or two half-planes of* $\mathbb{H}$*, if* U_α *contains part of the boundary of S.*

Proof. Trivial. □

Remark 1.1.9. The last part of the lemma states in effect that the boundaries of hyperbolic surfaces are piecewise geodesics.

Definition 1.1.10. A *Riemann surface* of type (g, r), is a compact one-dimensional complex manifold with genus g and r holes, such that all boundary components are geodesics.

Remark 1.1.11. A Riemann surface determines a real, oriented two-dimensional manifold with Riemannian metric by identifying $\mathbb{C}$ with $\mathbb{R}^2$ on charts and setting $g_z(v, w) = \Re(v\bar{w})$. Conversely, a Riemannian metric on an oriented surface determines an almost complex structure by defining $J(v)$ to be a vector of the same length and orthogonal to v such that the pair $(v, J(v))$ is a positively oriented basis. As all almost complex structures on a surface are integrable, this determines a complex manifold structure on the surface.

By the uniformisation theorem, any Riemann surface with negative Euler characteristic, i.e. $2g + r \geq 3$, is a hyperbolic manifold. Hence, from now on we will use the term 'Riemann surface' for compact hyperbolic surfaces with geodesic boundary components.

1.2 — Teichmüller space

We want to find a space, the moduli space $\mathcal{M}_{g,r}$, which parametrises Riemann surfaces of type (g, r) up to isometry. As the space of all Riemann surfaces of type (g, r) is obviously extremely large and unwieldy, and $\mathcal{M}_{g,r}$ itself is also quite complicated, it turns out to be convenient to first construct an intermediate space, the Teichmüller space $\mathcal{T}_{g,r}$, which parametrises surfaces up to 1-isotopic isometries. The actual moduli space can then be obtained as a quotient of this Teichmüller space.

The way to parametrise Riemann surfaces is by glueing together certain fundamental building blocks, called Y-pieces. This will allow us build the Teichmüller space by parametrising hyperbolic structures on its constituent Y-pieces and the glueing conditions. Therefore we will first consider those Y-pieces and glueing.

1.2.1 — Y-pieces

Definition 1.2.1. A *geodesic polygon* is a hyperbolic submanifold of the hyperbolic plane $\mathbb{H}$ with piecewise geodesic boundary and interior angles of less than or equal to π at each vertex.

Proposition 1.2.2. *Up to isometry, there exists exactly one right-angled hexagon with prescribed length of alternating sides. I.e., if the sides of the hexagon are* $a, \gamma, b, \alpha, c, \beta$ *(in that order), then the lengths of* a, b, c *determine the lengths of* α, β, γ*.*

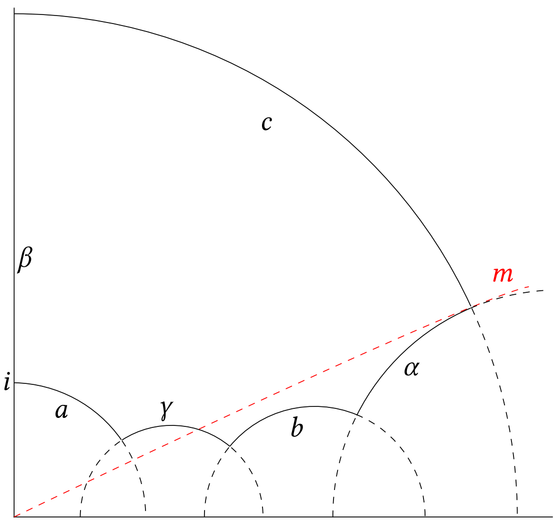


Figure 1.1: The hexagon.

Proof. Let us first construct such a hexagon in the first quadrant in $\mathbb{H}$. The construction is shown in figure 1.1. Choose the point $i \in \mathbb{H}$ as the vertex between a and β and let a be the unique geodesic of the prescribed length in the first quadrant perpendicular to the imaginary axis. Then β will lie along the imaginary axis, above i, and γ will lie along the unique perpendicular to a at its other endpoint, such that the interior angles are at the same side.

The line m of points at distance $l(c)$ from the imaginary axis is a straight Euclidean line through the origin, as dilatations are isometries. Consider the set A of geodesics having m as a tangent line. Letting $\alpha \in A$ slide from α_0 intersecting γ at $\partial\mathbb{H}$ (so $d(\alpha_0, \gamma) = 0$) towards the right, there is a unique α which has distance $l(b)$ to γ.

Taking as our geodesic the one having a, γ, b, α, the geodesic c from α to the imaginary axis and the cut-off part of the imaginary axis, we have constructed the required hexagon.

Now for uniqueness. Assume we have a hexagon fulfilling the conditions. By definition, this hexagon is embedded in $\mathbb{H}$. Now, $\mathrm{PSL}(2, \mathbb{R})$ is transitive on the set of oriented geodesics and one point on them, so there is an isometry placing the chosen hexagon so that its side a coincides with that of the hexagon constructed before. Possibly after applying the map $z \mapsto -\bar{z}$, their interiors are on the same side. As there were no choices in our construction above after the choice of a, this proves the two hexagons are isometric. □

We also need a lemma about the results of glueing:

Lemma 1.2.3. *Let a surface F be obtained by glueing m compact hyperbolic surfaces $S_1, \dots, S_m$ along (part of) their edges. Assume the following hold:*

- *At each interior point of F obtained by glueing edges, the sum of interior angles equals 2π;*
- *At each boundary point of F obtained by glueing edges, the sum of interior angles is less than or equal to π;*
- *Geodesic boundaries that are identified under the glueing have the same lengths;*
- *F is connected, all S_i are complete as metric spaces, and any pair of non-adjacent glueing edges of any S_i has positive distance.*
- *The glueing is of the form*

$$\gamma_i(t) \sim \gamma_i'(a_i - t)$$

for some $a_i \in \mathbb{R}$, where all γ_i and γ_i' are distinct oriented edges of the S_j, parametrised at unit speed.

Then F has a unique complete hyperbolic structure such that $q : \coprod_{i=1}^m S_i \to \bigcup_{i=1}^m S_i = F$ is a local isometry and its boundary is piecewise geodesic.

Proof. The existence of the structure is just a local argument on charts, while the completeness is immediate. □

Definition 1.2.4. A *Y-piece* or *pair of pants* is a Riemann surface of type $(0, 3)$.

Proposition 1.2.5. *The universal covering space of any hyperbolic surface S is isometric to a convex domain in $\mathbb{H}$ with piecewise geodesic boundary.*

Proof. According to the uniformisation theorem, this holds for hyperbolic surfaces without boundary. If the boundary of S is non-empty, S can be isometrically embedded in a hyperbolic surface without boundary S^*: for any geodesic boundary component, glue a one-side infinite cylinder with boundary geodesic of the right length to it, and for any boundary component with vertices, glue one-side infinite strips with right angles with the finite side to the edges (choosing the strips so that those edges match up), and then glue infinite sectors in the remaining gaps of the vertices. This works, as all interior angles are less than π.

The universal cover of S^* is given by $\pi : \mathbb{H} \to S^*$. Any connected component $\tilde{S} \subset \pi^{-1}(S)$ will have interior angles of less than π, hence is convex, and in particular simply connected. Therefore, $\pi|_{\tilde{S}}$ is a universal cover of S. □

Corollary 1.2.6. *Let S be a hyperbolic surface. Any homotopy class γ of curves with both endpoints either fixed or gliding on a connected component of ∂S contains a geodesic, unique if the endpoint sets are disjoint, which has minimal length among curves in that class.*

Proof. Let A and B denote the sets to which the endpoints of γ are constrained. If $A \cap B \neq \emptyset$, the geodesic will be any constant curve with value in $A \cap B$.

If A and B are disjoint, take consistent lifts $\tilde{A}$, $\tilde{B}$, and $\tilde{\gamma}$ of A, B, and γ respectively, i.e. the boundary of $\tilde{\gamma}$ should lie on $\tilde{A}$ and $\tilde{B}$. As $\tilde{S}$ is convex, there does exist a unique geodesic in $\tilde{\gamma}$, whose push-down is the sought-after geodesic in γ.

The characterisation of minimal length is a standard fact about geodesics. □

Proposition 1.2.7. *Let S be a hyperbolic surface and γ a homotopy class of closed curves. Then there exists a geodesic c in γ, of minimal length in this class, and the following hold:*

- *If γ is homotopically trivial, c is constant;*
- *If γ is not homotopically trivial, c is unique (up to reparametrisation);*
- *Either $c \subset \partial S$ or $c \cap \partial S = \emptyset$;*

Proof. If γ is homotopically trivial, this statement is trivial and so is point one.

If γ is not homotopically trivial, the existence and uniqueness follow as in the previous corollary, using any base point, which splits into different start and end points on the universal cover.

In general, if two geodesics intersect, they must either cross or coincide. The first is impossible for a boundary component, so if c intersects the boundary, it must coincide with a geodesic part of it. It cannot leave the boundary, as all vertex angles are less than π. This proves the third point.

□

Remark 1.2.8. Because of corollary 1.2.6 and proposition 1.2.7, we will often use geodesics and their homotopy classes interchangeably. It will be clear from the context which is meant.

Proposition 1.2.9. *A Y-piece is determined, up to isometry, by the lengths of its boundary geodesics.*

Proof. According to corollary 1.2.6, there is a unique geodesic from any boundary component of the Y-piece to another. These do not intersect, because otherwise one of them is not of minimal length in its class. They must also clearly be perpendicular to the boundary. Cutting along these three geodesics, we obtain two right-angled hexagons. So any Y-piece can be obtained by glueing together two right-angled hexagons at alternating edges. These edges need to have pairwise the same lengths in order for lemma 1.2.3 to apply, so by proposition 1.2.2, the lengths of the other three sides are also pairwise equal. Hence the two hexagons must be isometric.

Given the lengths of the boundary geodecis of the Y-piece, we can therefore obtain an isometric Y-piece by glueing two hexagons in exactly one way: take two isomorphic hexagons with three edges having half the length of the boundary geodesics of the Y-piece (again, this determines the other three edges), and glue them along the other three edges. □

1.2.2 — Constructing hyperbolic structures

Any Riemann surface with $2g + r \geq 3$ can be obtained by glueing pairs of pants. However, such a decomposition is in general not unique. Therefore, will have to mod out non-trivial isometries if we want to find a parameter space.

So fix, for the rest of this thesis, for every type (g, r) a surface $S_{g,r}$, a smooth orientable compact 2-manifold with smooth boundary of that given type. We also choose a decomposition of $S_{g,r}$ into $2g - 2 + r$ pairs of pants.

Definition 1.2.10. The space of all hyperbolic metrics on $S_{g,r}$ that make $S_{g,r}$ a Riemann surface is denoted $\mathcal{H}_{g,r}$.

The space $\mathrm{Diff}(S_{g,r})$ of all orientation-preserving diffeomorphisms from $S_{g,r}$ to itself, fixing the boundary pointwise, acts on $\mathcal{H}_{g,r}$ by pullback:

$$\mathcal{H}_{g,r} \times \mathrm{Diff}(S_{g,r}) \to \mathcal{H}_{g,r} : (g, \varphi) \mapsto \varphi^* g$$

The topology on $\mathrm{Diff}(S_{g,r})$ means that the identity component $\mathrm{Diff}^0(S_{g,r})$ is the space of all diffeomorphisms isotopic to the identity.

Definition 1.2.11. The *Teichmüller space* of type (g, r), denoted $\mathcal{T}_{g,r}$, is defined as the quotient space $\mathcal{T}_{g,r} \coloneqq \mathcal{H}_{g,r}/\mathrm{Diff}^0(S_{g,r})$.

Remark 1.2.12. If $r = 0$, we will often omit it, writing e.g. $S_g \coloneqq S_{g,0}$ or $\mathcal{T}_g \coloneqq \mathcal{T}_{g,0}$.

Proposition 1.2.13. *Teichmüller space $\mathcal{T}_{g,r}$ is homeomorphic to $\mathcal{R} = \mathbb{R}_+^{3g-3+2r} \times \mathbb{R}^{3g-3+r}$ with the Euclidean topology.*

Proof. Define a map $f : \mathcal{R} \to \mathcal{T}_{g,r}$ by sending a tuple to a hyperbolic surface S having the first coordinates as the lengths of the boundary geodesics of the Y-pieces and the last coordinates as the twists of the glueing (see lemma 1.2.3) and then pulling back the hyperbolic structure h of S along a diffeomorphism $\varphi : S_{g,r} \to S$ preserving the pair of pants decomposition. Because the $2g - 2 + r$ pairs of pants have $3(2g - 2 + r) = 6g - 6 + 3r$ boundary components, of which $3g - 3 + r$ pairs are glued together, S is a surface of the right type. If (S, φ) and (S', φ') are two choices of surface and diffeomorphism, there is an isometry $m : S \to S'$ preserving the geodesics and twists, and $(\varphi')^{-1} \circ m \circ \varphi : (S_{g,r}, \varphi^* h) \to (S_{g,r}, (\varphi')^* h')$ then is an isometry that sends boundary geodesics of the Y-pieces to themselves pointwise (as it preserves the twists) and is therefore an isotopy, as all independent Y-pieces have no nontrivial isometries fixing the boundary pointwise. This shows that f is well-defined.

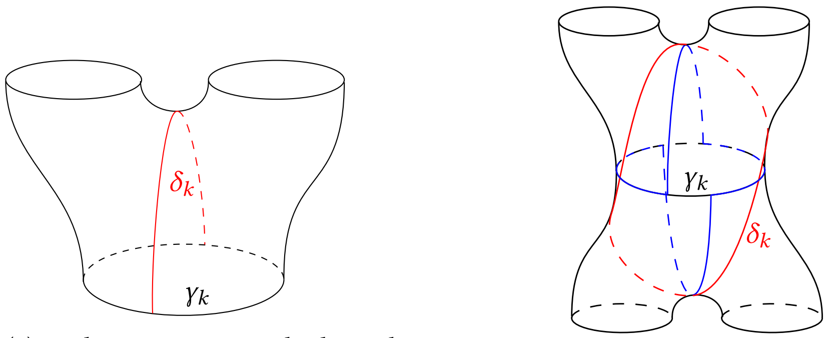
Now to define an inverse to f. A hyperbolic structure $h \in \mathcal{H}_{g,r}$ determines the unique geodesics homotopic to the cutting curves of the pants decomposition, see proposition 1.2.7. Furthermore, it determines the lengths of these boundary geodesics. Because homotopy classes are preserved under isotopy, the lengths are independent of isotopy. This gives the first coordinates of f^{-1}.

To recover the twist parameters, first find the values $\bar{a}_i \in \mathbb{R}/(\mathbb{Z} \cdot l_i) \cong S^1$ such that the glueing condition lemma 1.2.3 is satisfied. Taking a reference metric h' with twist parameters $a'_i \in [0, l_i)$ such that $a'_i \equiv \bar{a}_i \mod l_i$ (i.e. $h' = f(l_i, a'_i)$), the Riemann surfaces $(S_{g,r}, h)$ and $(S_{g,r}, h')$ are isometric. However, this isometry is via multiples of the right Dehn twists $\tau_i^{m_i}$ around the boundary geodesics, which are not isotopies. Hence, the $\bar{a}_i \in S^1$ lift via the $m_i \in \mathbb{Z}$ to $a_i \in \mathbb{R}$, according to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 0$. Again, this argument is independent of isotopy, and gives an inverse to f. □

Definition 1.2.14. The *standard curve system* associated to a hyperbolic metric $h \in \mathcal{T}_{g,r}$ is given by $\Omega := \{\gamma_k, \delta_k\}$, $1 \leq k \leq 3g - 3 + 2r$, where the γ_k are the boundary geodesics of the pair of pants decomposition and the δ_k are one of two cases:

- If $\gamma_k \subset \partial S$, δ_k is a geodesic with endpoints on γ_k and going once between the other two boundary geodesics of the relevant pair of pants, see figure 1.2a;
- If $\gamma_k \subset S^\circ$, δ_k is the geodesic obtained by taking the two δ_k of the case of γ_k being a boundary part, joining them, and twisting the resulting curve by a 'right Dehn twist of order a_i around γ_k', see figure 1.2b.

Remark 1.2.15. The term 'standard curve system' is not at all a standard term.



(a) In the case γ_k is on the boundary.

(b) In the case γ_k is in the interior. Here δ_k is the geodesic homotopic to the blue curve.

Figure 1.2: The curves of definition 1.2.14.

Proposition 1.2.16. *If $\varphi : S \to S$ is a diffeomorphism of a surface S preserving the homotopy classes of all curves in Ω, then φ is isotopic to the identity.*

Proof. By changing φ within its isotopy class, we may assume it preserves the actual curves in Ω pointwise.

As we assume that either $r > 0$ or $g > 2$, for each pair of pants Y, there are two boundary curves that do not both bound a different Y' as well. Hence, each Y-piece is fixed under φ.

As φ preserves the orientations of the γ_k, it is orientation-preserving. Hence it is isotopic to a product of Dehn twists around the γ_k. As it fixes the δ_k as well, it must be isotopic to the identity. □

For the proof of theorem 1.3.6, we will need the following definition of a certain metric on Teichmüller space, compatible with the topology.

Definition 1.2.17. For a given $q \geq 1$, a *q-quasi isometry* is a map $\varphi : A \to B$ between metric spaces such that

$$\frac{1}{q} \cdot d(x, y) \leq d(\varphi(x), \varphi(y)) \leq q \cdot d(x, y) \quad \forall x, y \in A$$

The infimum over q for which φ is a q-quasi isometry is called the *maximal length distortion*, denoted $q[\varphi]$.

Definition 1.2.18. Let $h, h' \in \mathcal{T}_{g,r}$. The distance δ on $\mathcal{T}_{g,r}$ is defined by

$$\delta(h, h') := \inf \log q[\varphi]$$

where φ runs through all quasi-isometries $\varphi : (S_{g,r}, h) \to (S_{g,r}, h')$ isotopic to the identity on $S_{g,r}$.

Proposition 1.2.19. *The topology on Teichmüller space $\mathcal{T}_{g,r}$ is compatible with the metric δ.*

Proof. As the isotopy class is determined by the lengths of all geodesics in the canonical curve system, it is clear that a sequence converges in the one topology if and only if it converges in the other. □

1.3 — The moduli space and the mapping class group

The Teichmüller space is a good step towards defining and handling the actual moduli space. However, we still need to mod out isotopically non-trivial diffeomorphisms. We do this as follows:

Definition 1.3.1. The *mapping class group*, abbreviated *MCG*, is defined by

$$\Gamma_{g,r} := \mathrm{Diff}(S_{g,r})/\mathrm{Diff}^0(S_{g,r})$$

Definition 1.3.2. The *moduli space of curves* of type (g, r) is defined by

$$\mathcal{M}_{g,r} := \mathcal{H}_{g,r}/\mathrm{Diff}(S_{g,r}) = \mathcal{T}_{g,r}/\Gamma_{g,r}$$

The two groups $\mathrm{Diff}(S_{g,r})$ and $\Gamma_{g,r}$, used in the equivalent definitions of the moduli space, both have their own advantages and disadvantages. For example, principal $\mathrm{Diff}(S_{g,r})$-bundles correspond to $S_{g,r}$-bundles, giving a practical interpretation of its classifying space $B\,\mathrm{Diff}(S_{g,r})$ (see proposition A.1.8). On the other hand, the topological group $\mathrm{Diff}(S_{g,r})$ is very large, while $\Gamma_{g,r}$ is discrete, and in fact finitely generated.

Therefore, we would like to relate the one to the other. The following theorem, originally proven by Earle and Eells[EE69] for closed surfaces and extended by Earle and Schatz[ES70] to surfaces with boundary components, provides this link. Their proof used the analysis of Beltrami equations. Shortly after, Gramain[Gra73] found a purely topological proof, which is essentially the one presented here, slightly adapted along [Hat14]. (Note, however, that the proof of [Hat14] contains some small mistakes.)

Theorem 1.3.3 (Earle-Eells-Schatz)**.** *For any surface S of type (g, r) with $g > 1$ or $r > 0$, the connected component of diffeomorphisms isotopic to the identity,* $\mathrm{Diff}^0(S)$*, is contractible.*

Proof. The proof consists of three steps: first the contractibility of a certain space of paths is proven, then this is used to prove the theorem in case $r > 0$, and finally the case $r = 0$ is deduced from this.

Step 1: Assume $r > 0$, i.e. ∂S is non-empty. Recall that an arc is an embedding $I = [0, 1] \to S$. Choose two points $p, q \in \partial S$ and an arc α from p to q. Let $A(S, \alpha)$ be the space of all arcs isotopic to α fixing the endpoints. We want to show $A(S, \alpha)$ is contractible.

There are two cases: p and q either do or do not lie on the same boundary component. We will handle the second case first. Define T by plugging the boundary component of q with a disc D. We take the following fibration of embedding spaces

$$\begin{array}{ccc} \mathrm{Emb}(I, S) & \longrightarrow & \mathrm{Emb}(I \cup D, T) \\ & & \downarrow \\ & & \mathrm{Emb}(D, T^\circ) \end{array}$$

where the embeddings are required to send $0 \in I$ to p and the rest of the space to the interior, and $I \cup D$ is obtained by identifying $1 \in I$ with a point on the boundary of D.

The total space of this fibration is evidently contractible. The base space is the total space of another fibration

$$\begin{array}{ccc} \mathrm{Emb}((D, x_0), (T^\circ, x_0)) & \longrightarrow & \mathrm{Emb}(D, T^\circ) \\ & & \downarrow \mathrm{ev}_{x_0} \\ & & T^\circ \end{array}$$

By tubular neighbourhood theory, $\mathrm{Emb}((D, x_0), (T^\circ, x_0)) \to \mathrm{GL}(\mathbb{R}^2) : i \mapsto (Di)_{x_0}$ is a homotopy equivalence, so $\mathrm{Emb}((D, x_0), (T^\circ, x_0)) \simeq S^1$. The Serre long exact sequence of the second fibration implies, using $\pi_i(T^\circ) = 0$ for $i > 1$, that $\pi_i \,\mathrm{Emb}(D^2, T^\circ) = 0$ for $i > 1$, which together with the long exact sequence of the first embedding shows $\mathrm{Emb}(I, S)$ has contractible components, one of which is $A(S, \alpha)$.

If p and q lie on the same boundary component $\partial_0 S$ of S, the situation is more difficult. In this case, let β and γ be subarcs of the two different components of $\partial_0 S \setminus \{p, q\}$ and glue them together. Call the resulting surface T. Then the previous case applies, showing $A(T, \alpha)$ is contractible.

Let $\tilde{S}$ be the universal covering space of S, and define a covering space $\tilde{T}$ of T by first taking S, glueing β to a lift of γ in a copy of $\tilde{S}$, and vice versa for γ, glueing new copies of $\tilde{S}$ in the same way to all other lifts of β and γ of the other copies of $\tilde{S}$, et cetera ad infinitum. Clearly, $\tilde{T}$ is the covering corresponding to $\pi_1(\tilde{T}) = \pi_1(S) \subset \pi_1(T) = \pi_1(S) * \mathbb{Z}$.

Denoting by $\tilde{\alpha}$ the lift of α in $S \subset \tilde{T}$, any arc in $A(T, \alpha)$ lifts uniquely to an arc in $A(\tilde{T}, \tilde{\alpha})$, forming a subspace $\tilde{A}(T, \alpha) \cong A(T, \alpha)$ in $A(\tilde{T}, \tilde{\alpha})$. The inclusion $i : A(S, \alpha) \hookrightarrow A(\tilde{T}, \tilde{\alpha})$ factors through $A(T, \alpha)$, so $i_* : \pi_i A(S, \alpha) \to \pi_i A(\tilde{T}, \tilde{\alpha})$ factors through zero, hence is zero.

Because $\tilde{S}$ is contractible and all of its copies in $\tilde{T}$ are glued via contractible glueings, there is a deformation retraction of $\tilde{T}$ onto S preserving boundary and interior along the way. This yields a map $r : A(\tilde{T}, \tilde{\alpha}) \to A(S, \alpha)$ such that $ri \simeq 1_{A(S,\alpha)}$. Therefore, $i_* : \pi_i A(S, \alpha) \to \pi_i A(\tilde{T}, \tilde{\alpha})$ is injective. As it is also zero, we conclude that $\pi_i A(S, \alpha) = 0$ for all i, i.e. $A(S, \alpha)$ is contractible.

Step 2: If, for $A \subset S$, $\mathrm{Diff}(S, A)$ is the space of diffeomorphisms restricting to the identity on A, and similarly for $\mathrm{Diff}^0(S, A)$, then clearly $\mathrm{Diff}^0(S, \alpha(I)) \simeq \mathrm{Diff}^0(S')$, where S' is obtained by cutting S open along α.

Evaluation on $\alpha(I)$ gives a fibration

$$\begin{array}{ccc} \mathrm{Diff}^0(S, \alpha(I)) & \longrightarrow & \mathrm{Diff}^0(S) \\ & & \downarrow \\ & & A(S, \alpha) \end{array}$$

As the complex $A(S, \alpha)$ is contractible, the Serre long exact sequence yields (weak) homotopy equivalences $\mathrm{Diff}^0(S) \simeq \mathrm{Diff}^0(S, \alpha(I)) \simeq \mathrm{Diff}^0(S')$. Induction yields $\mathrm{Diff}^0(S) \simeq \mathrm{Diff}^0(D^2)$.

Now let α be the 'equator' and D^2_+ the upper half of D^2. Define $\mathrm{Emb}(D^2_+, D^2)$ to be the space of smooth embeddings of D^2_+ in D^2, fixing their common boundary and sending the rest of the boundary into the interior. By construction, we get a fibration

$$\begin{array}{ccc} \mathrm{Diff}(D^2_+) & \longrightarrow & \mathrm{Emb}(D^2_+, D^2) \\ & & \downarrow \\ & & A(D^2, \alpha) \end{array}$$

Because the total space and the base space are contractible, so is the fibre. Hence, $* \simeq \mathrm{Diff}(D^2_+) \simeq \mathrm{Diff}(D^2) \simeq \mathrm{Diff}^0(D^2) \simeq \mathrm{Diff}^0(S)$.

Step 3: Now let $r = 0$. We will fix a point x_0 in S and cut away a disc D around this, keeping track of the homotopy group of Diff.

There is a fibration

$$\begin{array}{ccc} \operatorname{Diff}(S, x_0) & \longrightarrow & \operatorname{Diff}(S) \\ & & \downarrow \mathrm{ev}_{x_0} \\ & & S \end{array}$$

The Serre long exact sequence then yields that $\pi_i \operatorname{Diff}(S) \cong \pi_i \operatorname{Diff}(S, x_0)$ for $i > 1$. The tail of the long exact sequence yields

$$\pi_2(S, x_0) = 0 \longrightarrow \pi_1 \operatorname{Diff}(S, x_0) \longrightarrow \pi_1 \operatorname{Diff}(S) \longrightarrow \pi_1(S, x_0) \xrightarrow{\partial} \pi_0 \operatorname{Diff}(S, x_0) = \Gamma_S$$

Here, ∂ sends a homotopy class γ to the isotopy class τ_γ of a Dehn twist around a representative of γ. The natural map $\rho : \Gamma_S \to \operatorname{Aut}(\pi_1(S, x_0))$ sends τ_γ to conjugation by γ. As the center of $\pi_1(S, x_0)$ is zero, $\rho \circ \partial$ is injective. Hence, ∂ is injective and $\pi_1 \operatorname{Diff}(S, x_0) \cong \pi_1 \operatorname{Diff}(S)$.

Let $D \ni x_0$ be a closed disc in S and set $S_0 = S \setminus D^\circ$. Then $\operatorname{Diff}(S, D) \simeq \operatorname{Diff}(S_0) \simeq *$ by step 2. We have a fibration

$$\begin{array}{ccc} \operatorname{Diff}(S, D) & \longrightarrow & \operatorname{Diff}(S, x_0) \\ & & \downarrow \\ & & \operatorname{Emb}((D, x_0), (S, x_0)) \end{array}$$

As in step 1, $\operatorname{Emb}((D, x_0), (S, x_0)) \simeq S^1$ and the Serre exact sequence gives $\pi_i(\operatorname{Diff}(S, x_0)) = *$ for $i > 1$, and the tail yields

$$0 \longrightarrow \pi_1 \operatorname{Diff}(S, x_0) \longrightarrow \pi_1 \operatorname{Emb}((D, x_0), (S, x_0)) \cong \mathbb{Z} \xrightarrow{\partial} \pi_0 \operatorname{Diff}(S, D) = \Gamma_{S_0}$$

The $\mathbb{Z}$ term is generated by a full rotation of D around x_0, and ∂ again sends this to a Dehn twist, this time around a curve parallel to ∂D. These again act on $\pi_1(S_0)$ by conjugation, showing ∂ is injective. Hence $\pi_i \operatorname{Diff}(S) \cong \pi_i \operatorname{Diff}(S, x_0) \cong *$ for all $i > 0$, proving the theorem. □

Corollary 1.3.4. *The mapping class group $\Gamma_{g,r}$ is homotopy equivalent to the diffeomorphism group $\operatorname{Diff}(S_{g,r})$ unless $g \in \{0, 1\}$ and $r = 0$. In the same cases, the same holds for their classifying spaces: $B\Gamma_{g,r} \simeq B \operatorname{Diff}(S_{g,r})$.*

Proof. The homotopy equivalence is made by contracting all connected components, which is possible, because $\operatorname{Diff}(S_{g,r})$ is a topological group, hence a homogeneous space. The classifying space functor B preserves homotopy equivalences, showing the second statement. □

We have, by definition 1.3.2 and proposition 1.2.13, that the moduli space is a quotient of a contactible space by a group action. If this action were free, the moduli space would be a classifying space for the mapping class group, giving us a good hold on it. However, this is not quite true. We will get close though, as the following results show.

Lemma 1.3.5. *There are only finitely many geodesics on a surface S with length smaller than a prescribed L.*

Proof. Parametrise all geodesics on S with the unit interval.

Suppose there are inifinitely many geodesics on S with length smaller than L. Then we can find a subsequence of these geodesics such that their initial points, tangent directions at the initial point, and lengths all converge, using compactness of S and of the interval $[0, L]$. Also by compactness, there is an $r > 0$ such that $B(x, r)$ forms a compact neighbourhood for each $x \in S$. The convergent subsequence then gives two distinct geodesics γ_i, γ_j such that $d(\gamma_i(t), \gamma_j(t)) < r$ for all t, so γ_i and γ_j are homotopic, which is in contradiction with proposition 1.2.7. □

Theorem 1.3.6. *The mapping class group $\Gamma_{g,r}$ acts properly discontinuously on Teichmüller space $\mathcal{T}_{g,r}$.*

Proof. Clearly, $\Gamma_{g,r}$ acts by isometries with respect to the distance δ of definition 1.2.18. Therefore, we only need to prove discontinuity: for each $h \in \mathcal{T}_{g,r}$, there is an open $U \ni h$ such that $\varphi(U)$ meets U for only finitely many $\varphi \in \Gamma_{g,r}$.

Take an h and set $U = B_\delta(h, \log 2)$, a δ-ball around h. Let λ the the maximum length of all geodesics in the standard curve system Ω from definition 1.2.14. The set of closed geodesics C in (S,h) of length $\leq 4\lambda$ is finite by lemma 1.3.5, and if φ is such that $U \cap \varphi(U) \neq \emptyset$, then there are $h_1, h_2 \in U$ such that $\varphi(S, h_1) \longrightarrow (S, h_2)$ is an isometry. But h_1 and h_2 are 4-quasi isometric via a quasi-isometry isotopic to the identity, so the lengths of all $\beta \in \Omega$ in (S, h_1) and (S, h_2) differ by no more than a factor 4. By finiteness of C and proposition 1.2.16, there are only finitely many isotopy classes of this kind. □

Theorem 1.3.7 (Hurwitz[Hur92])**.** *The orientation-preserving isometry group* $\mathrm{Iso}^+(S)$ *of a closed Riemann surface S of genus g is finite and bounded by* $84(g-1)$.

Proof. Let γ be a given figure-eight geodesic in S, i.e. a closed geodesic having exactly one self-intersection. By lemma 1.3.5, there are only finitely many geodesics with the same length as γ. As at most two orientation-preserving isometries can fix γ (they must preserve the self-intersection point and can only permute the two loops) and there are only finitely many possible images of γ, this shows $\mathrm{Iso}^+(S)$ is finite.

By the Gauss-Bonnet theorem, the area of S is equal to $A_S = -2\pi\chi(S) = 4\pi(g-1)$. A fundamental domain of the isometry action will be a polygon defining a tiling of $\mathbb{H}$, the universal cover of S. Hence its corners must have angles π/v_i, where $v_i \in \mathbb{N}$. The area of such a hyperbolic n-gon is given by the formula

$$A_P = (n-2)\pi - \sum_{i=1}^{n} \frac{\pi}{v_i} = \pi\left(n - 2 - \sum_{i=1}^{n} \frac{1}{v_i}\right)$$

For this to be positive and as small as possible, simple consideration of cases yields $n = 3$, with $\{v_i\} = \{2,3,7\}$, so $A_{P,\min} = \frac{\pi}{42}$. Therefore, $|\mathrm{Iso}(S)| = \frac{A_S}{A_P} \leq \frac{4\pi(g-1)}{A_{P,\min}} = 168(g-1)$. As $\mathrm{Iso}^+(S)$ has index 2 in $\mathrm{Iso}(S)$, its order is bounded by $84(g-1)$. □

Corollary 1.3.8. *The action of the mapping class group $\Gamma_{g,r}$ on Teichmüller space $\mathcal{T}_{g,r}$ has finite isotropy groups.*

Proof. The isotropy group of a hyperbolic metric $h \in \mathcal{T}_{g,r}$ is the group of isotopy classes of diffeomorphisms φ such that $(S_{g,r}, h)$ is isometric to $(S_{g,r}, \varphi^* h)$ by a 1-isotopic isometry ψ. Hence, $\psi^*\varphi^* h = h$, and $\psi \circ \varphi$ is an isometry of $(S_{g,r}, h)$ isotopic to φ. By proposition 1.2.16, this identifies the isotropy group of h with the isometry group of $(S_{g,r}, h)$. For closed surfaces, the corollary now follows from the Hurwitz theorem. For a non-closed surface, the first part of the proof of the Hurwitz theorem still goes through, which is enough for this corollary to hold. □

Corollary 1.3.9. *There are rational homology equivalences between the moduli space of curves of type (g,r) and the classifying space of mapping class group of the same type:*

$$H^*(\mathcal{M}_{g,r}; \mathbb{Q}) \cong H^*(B\Gamma_{g,r}; \mathbb{Q}) =: H^*(\Gamma_{g,r}; \mathbb{Q})$$

Proof. The first isomorphism of the statement follows from theorem 1.3.6 because the isotropy groups are torsion by theorem 1.3.7. The final part is definition A.3.1. □

Chapter 2 — Harer stability

The cohomology of the mapping class group is far from being well-understood. However, there are some maps between mapping class groups of different types inducing isomorphisms in degrees which are small with respect to the genus. This allows us to define the homology of the stable mapping class group, the limit as g goes to infinity. This is the object we will want to compute in the rest of this thesis.

The existence of these induced isomorphisms is known as Harer stability, after John Harer, who proved, in [Har85], the existence of certain isomorphisms

$$\begin{aligned}
\Phi_* : H_*(\Gamma_{g,r}) &\xrightarrow{\sim} H_*(\Gamma_{g,r+1}) && \text{if} \quad * \leq \frac{g+2}{3},\ r \geq 1 \\
\Psi_* : H_*(\Gamma_{g,r}) &\xrightarrow{\sim} H_*(\Gamma_{g+1,r-1} && \text{if} \quad * \leq \frac{g-1}{3},\ r \geq 2 \\
\eta_* : H_*(\Gamma_{g,r}) &\xrightarrow{\sim} H_*(\Gamma_{g+1,r-2}) && \text{if} \quad * \leq \frac{g}{3},\ r \geq 2
\end{aligned}$$

This bound was soon improved by Ivanov to an approximate range of $* \leq \frac{g}{2}$ for surfaces with boundary in [Iva89] and for closed surfaces in [Iva93]. Recently, Boldsen[Bol12] proved a $\frac{2g}{3}$ range, based on an unpublished paper by Harer[Har93]. The current best range, due to Randal-Williams[Ran14] is given in the following theorems. This range is known to be at most one off the best possible bound.

Definition 2.1. Let $\Phi : \Gamma_{g,r} \longrightarrow \Gamma_{g,r+1}$ for $r \geq 1$ be the map induced by glueing a pair of pants to one boundary component, $\Psi : \Gamma_{g,r} \longrightarrow \Gamma_{g+1,r-1}$ for $r \geq 2$ the map induced by glueing a pair of pants to two boundary components, and $\delta : \Gamma_{g,r} \longrightarrow \Gamma_{g,r-1}$ for $r \geq 1$ the map induced by plugging one boundary component with a disc.

Remark 2.2. The map η above is now redundant. It is induced by glueing two boundary components to each other.

Theorem 2.3 (Harer stability for surfaces with boundary). *Let $r \geq 1$. The induced map on homology $H_*(\Phi) : H_*(\Gamma_{g,r}) \longrightarrow H_*(\Gamma_{g,r+1})$ is always injective and an isomorphism for $* \leq \frac{2g}{3}$.*

The map $H_*(\Psi) : H_*(\Gamma_{g,r+1}) \longrightarrow H_*(\Gamma_{g+1,r})$ is surjective for $* \leq \frac{2g+1}{3}$ and an isomorphism for $* \leq \frac{2g-2}{3}$.

Theorem 2.4 (Harer stability for closed surfaces). *Suppose $g \geq 2$. The map $H_*(\delta) : H_*(\Gamma_{g,1}) \longrightarrow H_*(\Gamma_{g,0})$ is surjective for $* \leq \frac{2g+3}{3}$ and an isomorphism for $* \leq \frac{2g}{3}$.*

The proof we will give is an amalgamation of the proofs of the different versions of the stability. It is derived from Wahl[Wah12].

The injectivity of $H_*(\Phi)$ is simple: $H_*(\delta)$ is a left inverse. Note that this does *not* show surjectivity of δ in theorem 2.4, as $H_*(\Phi)$ cannot be defined if $r = 0$.

The proof goes as follows: in section section 2.1, certain complexes of arcs are defined and the maps of the first theorem are related to the stabilisers of these complexes. In section 2.1.1, a connectivity bound of the arc complexes is proven. In section 2.2.1, the complexes are used in a double spectral sequence argument to prove theorem 2.3. In section 2.2.2, this result is combined with another spectral sequence argument to prove theorem 2.4.

These theorems suggest very strongly we define some kind of limit of the mapping class group for the genus going to infinity. We will give the definition here.

Definition 2.5. For a given $r > 0$, we define $\Gamma_{\infty,r}$ to be the colimit over the infinite diagram

$$\Gamma_{1,r} \xrightarrow{\Psi\circ\Phi} \Gamma_{2,r} \xrightarrow{\Psi\circ\Phi} \Gamma_{3,r} \xrightarrow{\Psi\circ\Phi} \cdots$$

If we only care about the homology type of this space, we write Γ_∞ for $\Gamma_{\infty,r}$.

Remark 2.6. By Harer stability, Γ_∞ is a well-defined homology type and it is the stable homology of Γ_g as g goes to infinity. However, it is not an actual space, as it cannot be defined as a limit over the mapping class groups Γ_g themselves. Therefore, for explicit constructions, one should take an $r > 0$ (often, r is chosen to be either 1 or 2) and apply the construction to $\Gamma_{\infty,r}$ instead. In this thesis, we will stick to writing Γ_∞.

2.1 — Arc complexes on a surface

In this section, we will define certain arc complexes on an oriented surface S with non-empy boundary and consider the action of its mapping class group $\Gamma(S)$ on these complexes. We will then see how this action interacts with the maps Φ and Ψ, cast in a slightly different form.

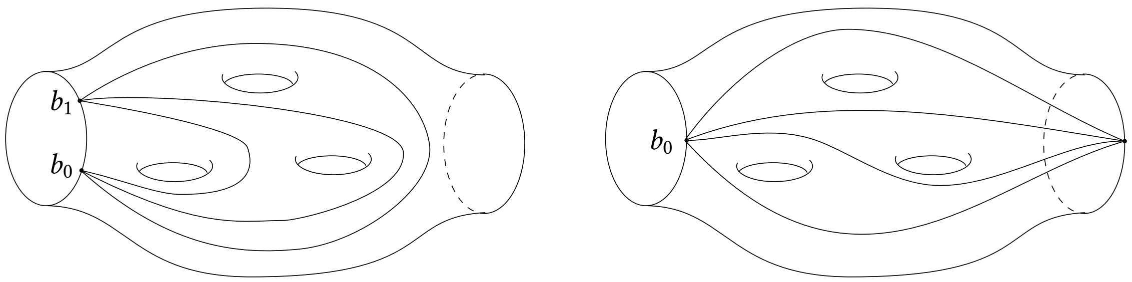
An arc will always mean an embedded arc, intersecting ∂S transversally and only at its endpoints. Isotopy classes of arcs will fix the endpoints.

Definition 2.1.1. Let S be an oriented surface with boundary. A tuple of isotopy classes of arcs $\langle a_0, \dots, a_p\rangle$ is called *non-separating* if there are representatives $\alpha_i \in a_i$ with disjoint interior such that $S \setminus (\alpha_0 \cup \cdots \cup \alpha_p)$ is connected.

Let $b_0, b_1 \in \partial S$ be distinct points. We define the *arc complex* $O(S, b_0, b_1)$ to be the simplicial complex whose p-simplices are non-separating $(p+1)$-tuples of isotopy classes of arcs $\langle a_0, \dots, a_p\rangle$ such that the clockwise ordering of their germs at b_0 equals their anticlockwise ordering at b_1.

If b_0 and b_1 lie on the same boundary component of S, we will write $O^1(S) := O(S, b_0, b_1)$. If they lie on different components, we define $O^2(S) := O(S, b_0, b_1)$. If it does not matter whether b_0 and b_1 lie on the same component, we will write $O(S)$.

Remark 2.1.2. All possible complexes $O^1(S)$ are clearly isomorphic, as are all $O^2(S)$. Hence this notation will not lead to confusion.



(a) A 2-simplex in $O^1(S_{3,2})$.

(b) A 3-simplex in $O^2(S_{3,2})$.

Figure 2.1: Simplices in arc complexes.

Proposition 2.1.3. *The natural action of the mapping class group $\Gamma_{g,r}$ on the arc complex $O(S_{g,r})$ is transitive on p-simplices for all p.*

For any p-simplex σ_p of $O(S_{g,r})$, there are isomorphisms

$$\mathrm{Stab}_{O^1}(\sigma_p) \xrightarrow{\sim} \Gamma_{g-p-1,r+p+1}$$
$$\mathrm{Stab}_{O^2}(\sigma_p) \xrightarrow{\sim} \Gamma_{g-p,r+p-1}$$

Proof. Let $\sigma = \langle a_0, \ldots, a_p \rangle$, so it is represented by the arcs a_i. The surface $S \setminus \sigma$ has Euler characteristic $\chi(S) + p + 1$, as a cell decomposition gains two extra 0-cells and one 1-cell per cut arc.

As $S \setminus \sigma$ is connected, we get

$$2 - 2g_\sigma - r_\sigma = 2 - 2g - r + p + 1$$

If b_0 and b_1 lie on the same boundary component, $\partial(S \setminus \sigma)$ has $r - 1$ components inherited from S, plus the extra $p + 2$ components $[\partial_0^+ S * a_0^-], [a_0^+ * a_1^-], \ldots, [a_p^+ * \partial_0^- S]$, where $a_i^\pm$ are the two copies of a_i and $\partial_0^\pm S$ are the two parts of ∂S between b_0 and b_1. Hence, $r_\sigma = r + p + 1$, and $g_\sigma = g - p - 1$.

If b_0 and b_1 lie on different boundary components, $\partial(S \setminus \sigma)$ has $r - 2$ components inherited from S, plus the extra $p + 1$ components $[\partial_0 S * a_0^- * \partial_1 S * a_p^+], [a_0^+ * a_1^-], \ldots, [a_{p-1}^+ * a_p^+]$, where $a_i^\pm$ are the two copies of a_i and $\partial_i S$ are the two components of ∂S associated to b_0 and b_1, respectively. Hence, $r_\sigma = r + p - 1$, and $g_\sigma = g - p$.

Now, if σ and σ' are both p-simplices, it follows that $S \setminus \sigma$ is diffeomorphic to $S \setminus \sigma'$, as g_σ and r_σ only depend on p. Labeling the boundary components of $S \setminus \sigma$ and $S \setminus \sigma'$ according to the arcs of the simplices and choosing the diffeomorphism appropriately, we can glue both $S \setminus \sigma$ and $S \setminus \sigma'$ to get a diffeomorphism of S sending σ to σ'. This proves the first statement.

For the second statement, the argument at the start of step 2 of the proof of theorem 1.3.3 shows the isomorphism between $\Gamma(S \setminus \sigma)$ and the pointwise stabiliser of a representative σ for σ a vertex. Induction yields an isomorphism between $\Gamma(S \setminus \sigma)$ and the pointwise stabiliser of a representative $\langle a_0, \ldots, a_p \rangle$ of σ for any p-simplex σ. Note that $S \setminus \sigma$ has the correct type for the proposition by the first part of this proof.

So it is left to show that any φ stabilising σ is isotopic to a a map fixing σ pointwise.

As φ is the identity on the boundary, it is isotopic to the identity near b_0 and b_1. Therefore, it respects the ordering of the a_i, and stabilises each homotopy class $[a_i]$. So there are isotopies $\varphi(a_i) \simeq_{h_i} a_i$ for all i.

By the isotopy extension theorem, there is an isotopy $H_0 : S \times I \to S$ such that $H_0(-, 0) = \mathrm{id}$ and $H_0(\varphi(a_0), t) = h_0(\varphi(a_0), t)$.

Then $\varphi_1 = H_0(\varphi(-), 1)$ is isotopic to φ (via H_0) and fixes a_0 pointwise. We will keep writing h_i for the other isotopies.

Suppose we have a φ_i isotopic to φ fixing a_j, $j < k$ pointwise. Then there is a commutative diagram

$$\begin{array}{ccc} \partial I^2 & \longrightarrow & S \setminus \bigcup_{j<k} a_j \\ \big\downarrow & \overset{\tilde{h}_k}{\nearrow} & \big\downarrow \\ I^2 & \xrightarrow{\;h_k\;} & S \end{array}$$

Since $\pi_2(S \setminus \bigcup_{j<k} a_j) = 0$, there is a dotted arrow as in the diagram, making it commute up to homotopy. As homotopic arcs are isotopic, we can again use the isotopy extension theorem to find an isotopy $H_k : S \setminus \bigcup_{j<k} a_j \times I \to S \setminus \bigcup_{j<k} a_j$ such that $H_k(-, 0) = \mathrm{id}$ and H_k extends $\tilde{h}_k$. Extending H_k to all of S by glueing and defining $\varphi_{k-1} = H_k(\varphi_k(-), 1)$, this fixes a_j for $j \leq k$. By induction, we get the required result. □

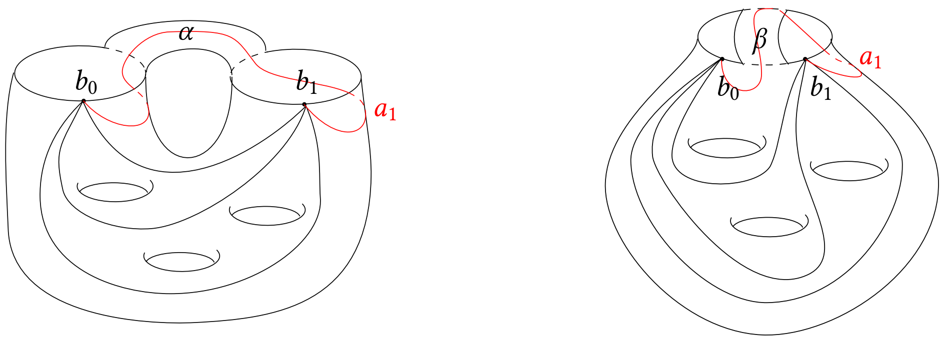
Definition 2.1.4. Define the map $\alpha : S_{g,r+1} \to S_{g+1,r}$ by glueing a strip with both ends to two different boundary components of $S_{g,r}$, thereby linking them. Similarly, define $\beta : S_{g,r} \to S_{g,r+1}$ by glueing a strip with both ends to the same boundary component. See figure 2.2.

There are induced maps $\alpha_g : \Gamma_{g,r+1} \to \Gamma_{g+1,r}$ and $\beta_g : \Gamma_{g,r} \to \Gamma_{g,r+1}$.

If α connects the two boundary components containing b_0 and b_1, there is an induced map $\alpha : \mathcal{O}^2(S_{g,r+1}) \to \mathcal{O}^1(S_{g+1,r})$. If b_0 and b_1 lie on the same boundary component and β separates them by glueing the strip with its two ends to the two arcs from b_0 to b_1, there is an induced map $\beta : \mathcal{O}^1(S_{g,r}) \to \mathcal{O}^2(S_{g,r+1})$.

Remark 2.1.5. The maps α_g and β_g are the same as Ψ and Φ from the first part of this chapter, respectively. We will stick to α and β from now on.

Clearly, the maps of complexes are equivariant with respect to their corresponding group homomorphisms.



(a) The map α.

(b) The map β.

Figure 2.2: The stabilising maps.

Proposition 2.1.6. *Under the isomorphisms in proposition 2.1.3, the restricition of $\alpha_g : \Gamma_{g,r+1} \longrightarrow \Gamma_{g+1,r}$ to the stabilisers of a p-cell corresponds to $\beta_{g-p} : \Gamma_{g-p,r+p} \longrightarrow \Gamma_{g-p,r+p+1}$.*

Similarly, the restriction of $\beta_g : \Gamma_{g,r} \longrightarrow \Gamma_{g,r+1}$ corresponds to $\alpha_{g-p-1} : \Gamma_{g-p-1,r+p+1} \longrightarrow \Gamma_{g-p,r+p}$.

Proof. Using the notation from the proof of proposition 2.1.3, the map α glues a strip to $\partial_0 S$ and $\partial_1 S$. In $S \setminus \sigma_p$, this corresponds to a glueing to the single boundary component $[\partial_0 S * a_0^- * \partial_1 S * a_p^+]$, one end to either side of b_0. Hence, this is a β-map.

Similarly, the map β glues a strip to $\partial_0^+ S$ and $\partial_0^- S$. In $S \setminus \sigma_p$, this corresponds to a glueing to the two boundary components $[\partial_0^+ S * a_0^-]$ and $[a_p^+ * \partial_0^-]$. Hence, this is an α-map.

□

Proposition 2.1.7. *Let S be a surface, and set S_α and S_β to be the codomains of α and β from S, respectively. Then $\alpha : \Gamma(S) \longrightarrow \Gamma(S_\alpha)$ and $\beta : \Gamma(S) \longrightarrow \Gamma(S_\beta)$ are injective.*

There is an arc $a_1 \in S_\alpha$ or $a_1 \in S_\beta$ from b_0 to b_1 such that for any vertex $\sigma \in \mathcal{O}(S)$, there is a representative a of σ such that conjugation by Dehn twists $t_{a_1 * a}$ yields commutative diagrams

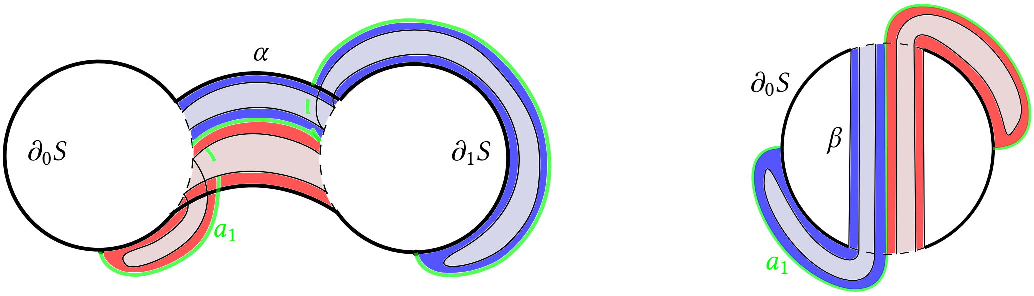
$$\begin{array}{ccc} \mathrm{Stab}_{\mathcal{O}^2}(\sigma) & \hookrightarrow & \mathrm{Stab}_{\mathcal{O}^1}(\alpha(\sigma)) \\ \downarrow & \swarrow_{t_{a_1*a}} & \downarrow \\ \Gamma(S) & \hookrightarrow & \Gamma(S_\alpha) \end{array} \qquad \begin{array}{ccc} \mathrm{Stab}_{\mathcal{O}^1}(\sigma) & \hookrightarrow & \mathrm{Stab}_{\mathcal{O}^2}(\beta(\sigma)) \\ \downarrow & \swarrow_{t_{a_1*a}} & \downarrow \\ \Gamma(S) & \hookrightarrow & \Gamma(S_\beta) \end{array}$$

Proof. Set S_i to be either S_α or S_β, let σ be represented by a, and let the arc $a_1 \in S_i$ be given as in figure 2.2. Any neighbourhood of $\partial S_i \cup a_1$ can be deformed, using a 1-isotopic $\varphi \in \mathrm{Diff}(S_i)$, to a neighbourhood of $\partial S \cup (S_i \setminus S)$, see figure 2.3.

The elements of $\mathrm{Stab}_{\mathcal{O}}(a_1)$ can be assumed to fix a neighbourhood of $a_1 \cup \partial S_i$ and are hence conjugated by φ to elements fixing a neighbourhood of $\partial S \cup (S_i \setminus S)$, which lie in the image of $\Gamma(S)$ in $\Gamma(S_i)$. But φ is 1-isotopic, so this conjugation is the identity on $\Gamma(S_i)$. Therefore, $\mathrm{Im}\,\Gamma(S) = \mathrm{Stab}_{\mathcal{O}}(a_1)$. However, $\Gamma(S) \cong \mathrm{Stab}(a_1)$, by proposition 2.1.3. The map $\Gamma(S) \longrightarrow \Gamma(S_i)$ must therefore be injective, with image $\mathrm{Stab}(a_1)$.

The diagrams of the proposition hence have the form:

$$\begin{array}{ccc} \mathrm{Stab}(\langle a, a_1 \rangle) & \hookrightarrow & \mathrm{Stab}(a) \\ \downarrow & \swarrow_{t_{a_1*a}} & \downarrow \\ \mathrm{Stab}(a_1) & \hookrightarrow & \Gamma(S_i) \end{array}$$

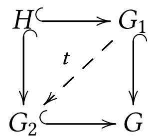


(a) S_α (b) S_β

Figure 2.3: The neighbourhood of $a_1 \cup \partial S_i$ can be deformed to contain a neighbourhood of $\partial S \cup (S_i \setminus S)$. Black is part of the boundary of S_i, green is a_1, dark red and blue are part of a typical neighbourhood of $a_1 \cup \partial S_i$, which can be deformed by a 1-isotopic diffeomorphism to the lighter blue and red neighbourhood of $\partial S \cup (S_i \setminus S)$.

This diagram commutes, because $t_{a_1 * a}(a) = a_1$ by elementary Dehn twist theory, and this Dehn twist lives in a neighbourhood of $\langle a, a_1 \rangle$, hence is trivial on $\mathrm{Stab}(\langle a, a_1 \rangle)$. □

Lemma 2.1.8. *Given a commutative diagram of groups*



where the diagonal map is conjugation by an element $t \in G$, the map induced by the vertical inclusions, $i_* : H_k(G_1, H) \to H_k(G, G_2)$, factors as

$$H_k(G_1, H) \xrightarrow{d} H_{k-1}(H) \xrightarrow{-\times t} H_k(G, G_2)$$

Where

$$(h_1, \ldots, h_{k_1}) \times t = \sum_{i=0}^{k-1} (-1)^i (h_1, \ldots, h_i, t, h_{i+1}, \ldots, h_{k-1})$$

Proof. For a general $c \in H_k(G_1, H)$, define

$$(c_1, \ldots, c_k) \times t = \sum_{i=0}^{k} (-1)^i (c_1, \ldots, c_i, t, t^{-1} c_{i+1} t, \ldots, t^{-1} c_k t)$$

and extend linearly. Computing $d(c \times t)$, all terms $(-1)^{2i+1}(c_1, \ldots, c_i, t \cdot t^{-1} c_{i+1} t, \ldots, t^{-1} c_k t)$ and $(-1)^{2i+2}(c_1, \ldots, c_{i+1} \cdot t, t^{-1} c_{i+1} t, \ldots, t^{-1} c_k t)$ add up to zero, leading to $d(c \times t) = t^{-1} c t + dc \times t - c$.

Because c is a cycle in G_1, which is conjugated into G_2, $[t^{-1} c t] = 0 \in H_k(G, G_2)$. Therefore, $i_*(c) = [dc \times t]$ in $H_k(G, G_2)$. □

Corollary 2.1.9. *Let $\sigma \in O^2(S)$ be a vertex. Then the map*

$$H_*(\mathrm{Stab}_{O^1}(\alpha(\sigma)), \mathrm{Stab}_{O^2}(\sigma)) \to H_*(\Gamma(S_\alpha), \Gamma(S))$$

induced by the vertical inclusions of the diagram of proposition 2.1.7 is zero.

Proof. By proposition 2.1.7 and lemma 2.1.8, it suffices to show that the assigment $(-) \times t_{a_1 * a}$ maps $H_{k-1}(\mathrm{Stab}_{O^2}(\sigma))$ into $H_k(\Gamma(S))$.

Let U be a neighbourhood of $\partial S \cup a \cup (S_\alpha \setminus A)$ and let c be a curve in the interior of U, isotopic to $\partial_0 S$. Then both c and $a_1 * a$ are non-separating in U, so there is a diffeomorphism φ of U sending $a_1 * a$ to c and fixing the boundary. Then φ can be extended by the identity to S_α, and there clearly commutes with $\mathrm{Stab}_{O^2}(\sigma) = \mathrm{Stab}(\langle a, a_1 \rangle)$, using notation from proposition 2.1.7.

Now take a $[b] \in H_{k-1}(\mathrm{Stab}_{O^2}(\sigma))$. Then, using the computations of $d(c \times t)$ from the proof of lemma 2.1.8,

$$\begin{aligned} d(b \times t_{a_1 * a} \times \varphi) &= \varphi^{-1}(b \times t_{a_1 * a})\varphi + d(b \times t_{a_1 * a}) \times \varphi - b \times t_{a_1 * a} \\ &= b \times t_c + (t_{a_1 * a}^{-1} b t_{a_1 * a} + db \times t_{a_1 * a} - b) \times \varphi - b \times t_{a_1 * a} \\ &= b \times t_c - b \times t_{a_1 * a} \end{aligned}$$

where the second equality holds because φ commutes with b and sends $a_1 * a$ to c and the third equality holds because b fixes $a_1 * a$, and therefore is stable under conjugation, and $db = 0$.

As both b and t_c have support in S, $[b \times t_{a_1 * a}] = [b \times t_c]$ lies in $H_k(\Gamma(S))$, and the claim follows. □

Corollary 2.1.10. *Let $\sigma \in O^1(S)$ be a vertex, represented by an arc a. Let a' be a non-trivial loop in S intersecting a exactly once, and let S' be the complement of a neighbourhood U of $(S_\beta \setminus S) \cup \partial S \cup a \cup a'$. Then the composition*

$$H_{*-1}(\Gamma(S')) \xrightarrow{i_*} H_{*-1}(\mathrm{Stab}(\sigma)) \xrightarrow{-\times t_{a_1 * a}} H_*(\Gamma(S_\beta), \Gamma(S))$$

is zero.

Proof. In U, again $a_1 * a$ and a curve c isotopic to $\partial_0 S$ are non-separating, so the proof is identical to that of corollary 2.1.9 □

2.1.1 — Connectivity of the arc complexes

The final ingredient needed before we can prove the stability theorems is the high connectivity of the arc complexes. In order to prove this we need to introduce some more complexes and show their connectivity first.

Definition 2.1.11. Let $\Delta \subset \partial S$ be a non-empty finite set. An arc with endpoints in Δ is called *trivial* if it is isotopic, relative to its endpoints, to a segment of ∂S meeting Δ only in its endpoints.

Let $\mathcal{A}(S, \Delta)$ be the simplicial complex whose p-simplices are $p+1$-tuples of distinct isotopy classes of non-trivial arcs with boundary in Δ, representable by arcs with disjoint interiors.

Given disjoint non-empty discrete $\Delta_0, \Delta_1 \subset \partial S$, let $\mathcal{B}(S, \Delta_0, \Delta_1) \subset \mathcal{A}(S, \Delta_0 \cup \Delta_1)$ be the subcomplex where all arcs must go from Δ_0 to Δ_1, and write $\Delta = \Delta_0 \cup \Delta_1$.

Define $\mathcal{B}_0(S, \Delta_0, \Delta_1) \subset \mathcal{B}(S, \Delta_0, \Delta_1)$ to be the subcomplex of arcs from Δ_0 to Δ_1 such that all simplices are non-separating.

Given Δ_0, Δ_1, ∂S decomposes into vertices, the points of Δ_0 and Δ_1, edges between vertices, and closed circles. An edge is *pure* if both ends are in the same Δ_i. Otherwise, it is *impure.* A boundary component containing points of $\Delta_0 \cup \Delta_1$ is *pure* if the points are all in either Δ_0 or Δ_1, equivalently, if all its edges are pure.

Remark 2.1.12. The complexes $O(S, b_0, b_1)$ of the previous part of this chapter are the subcomplexes of $\mathcal{B}_0(S, \{b_0\}, \{b_1\})$ such that the orderings of the arcs near b_0 and b_1 are opposite.

In order to prove the connectivity bound of $O(S, b_0, b_1)$, we will first give a connectivity bound for $\mathcal{A}(S, \Delta)$ in theorem 2.1.16, and use that to get connectivity bounds on first $\mathcal{B}(S, \Delta_0, \Delta_1)$ and then $\mathcal{B}_0(S, \Delta_0, \Delta_1)$ in theorems 2.1.18 and 2.1.23, respectively. From the last, the connectivity bound of $O(S, b_0, b_1)$ will finally be deduced in theorem 2.1.24.

Here and later on, we will also need the following definition:

Definition 2.1.13. For a simplicial complex A and a simplex $a \in A$, the *star of* a, $\mathrm{Star}(a)$, is the subcomplex of all simplices having a as a face.

The *link of* a, $\mathrm{Link}(a)$, is the subcomplex of $\mathrm{Star}(a)$ consisting of simplices disjoint from a.

A *piecewise linear triangulation* of an n-manifold is one for which the link of a p-simplex is an $(n-p)$-ball if the simplex is contained in the boundary and an $(n-p)$-sphere else.

Remark 2.1.14. Both the k-simplex Δ^k and its boundary $\partial\Delta^k$ are piecewise linear, and so are subdivisions of piecewise linear complexes.

Remark 2.1.15. Note for future reference that $\mathrm{Star}(a) = a \star \mathrm{Link}(a)$, the join of a with its link.

Theorem 2.1.16. *Let $q = |\Delta|$, the order of Δ. If S is a disc or an annulus with Δ concentrated in one boundary component, $\mathcal{A}(S, \Delta)$ is $(q + 2r - 7)$-connected. In all other cases, it is contractible.*

Before we give the proof of this theorem, we will first need the following lemma:

Lemma 2.1.17. *Suppose $\mathcal{A}(S, \Delta)$ is d-connected, $d \geq -1$, and $\Delta' = \Delta \sqcup \{q\}$, where the extra point lies on a boundary component already containing points of Δ. Then $\mathcal{A}(S, \Delta')$ is $(d + 1)$-connected.*

Proof. Let q be adjacent to some p on ∂S and draw the arcs I and I', where the other endpoints are again adjacent points in Δ' at the other side—they may be q and p again—, see figure 2.4.

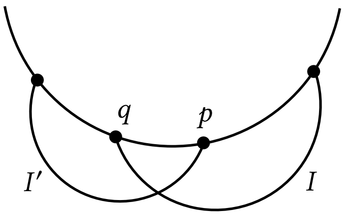


Figure 2.4: The arcs I and I'.

Then $\mathcal{A}(S, \Delta') = \mathrm{Star}(I) \cup_{\mathrm{Link}(I)} X$, where X consists of all simplices not containing I.

There is a retraction from X to $\mathrm{Star}(I')$, defined by moving arcs from q to p, as seen in figure 2.5. This retraction can clearly be made inside the geometric realisation, as all intermediate steps (the middle three pictures of the figure) are still simplices of X. As I is not in X, no arc becomes trivial under this retraction. Now, the star of a vertex is clearly contractible, as it can be contracted onto that vertex.

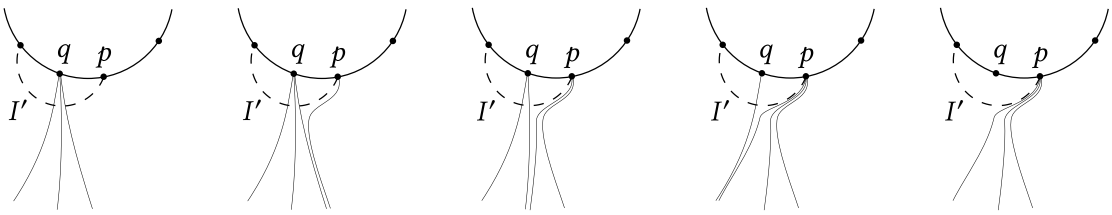


Figure 2.5: The retraction of X to $\mathrm{Star}(I')$, moving three arcs in this example, as seen from left to right.

Using the decomposition $\mathcal{A}(S, \Delta') = \mathrm{Star}(I) \cup_{\mathrm{Link}(I)} X$ and the above, if $d = -1$, i.e. $\mathcal{A}(S, \Delta)$ is non-empty, $\mathcal{A}(S, \Delta')$ is connected, proving the lemma in this case. In the case $d \geq 0$, the Seifert-Van Kampen theorem applied to this decomposition shows $\mathcal{A}(S, \Delta')$ is simply connected. Hence, by the Hurewicz isomorphism, connectivity can be tracked on homology. The lemma follows with the Mayer-Vietoris sequence. □

Proof of theorem 2.1.16. In the case $S = D^2$, $r = 4$, the theorem holds because $\mathcal{A}(D^2, \Delta) \neq \emptyset$, as it contains an arc from two opposite points. For $S = S^1 \times I$, $r = 2$, there is a loop around the cylinder based at one point, so again $\mathcal{A}(S^1 \times I, \Delta) \neq \emptyset$. The first part of the theorem then follows with lemma 2.1.17.

For the second part, lemma 2.1.17 allows us to assume Δ has at most one point in each boundary component. If $|\Delta| \geq 2$, $\mathcal{A}(S, \Delta)$ is clearly non-empty, as any arc connecting two points on different boundary components is non-trivial. If $|\Delta| = 1$, this is still true, as either $g \geq 1$ or $r \geq 3$, and in both cases there is a non-trivial loop: either looping though one of the holes or between two of the boundary components not containing Δ.

Choose a $p \in \Delta$ and an arc $a \in \langle a \rangle \in \mathcal{A}(S, \Delta)$ terminating at p. We will construct a retraction from $\mathcal{A}(S, \Delta)$ to Star(a). Let $\sigma = \langle a_0, \ldots, a_q \rangle \in (\mathcal{A}(S, \Delta))_q$, where the a_i are chosen so that they intersect a transversely and minimally in their isotopy class. The retraction is then given by cutting all a_i open along a and moving the endpoints to p, as seen in figure 2.6. If a_i intersects a, this results in two new arcs, $R(a_i)$ and $L(a_i)$, being the right and left arc, respectively. If either is trivial, we forget it. They cannot both be trivial, as a_i is either an arc with one endpoint on a different boundary component or a loop at p, which does not intersect a with trivial part by minimality. Therefore, this construction does result in a new simplex.

This retraction is well-defined, because isotopic sets of arcs with minimal transverse intersection are isotopic through minimal transverse intersection. □

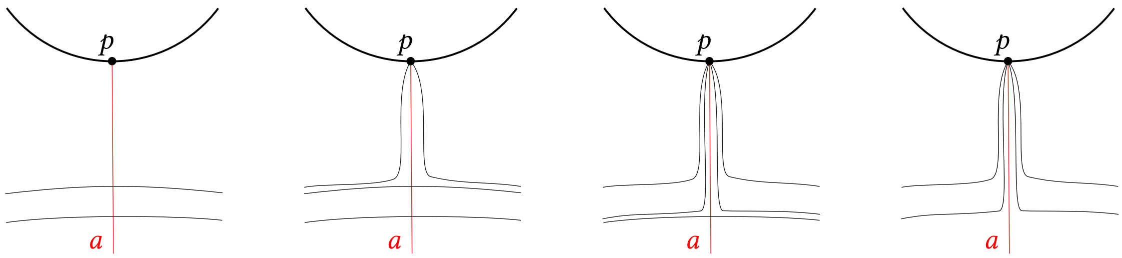


Figure 2.6: The retraction of $\mathcal{A}(S, \Delta)$ to Star(a), moving two arcs in this example, as seen from left to right.

Theorem 2.1.18. *Let r' be the number of components of ∂S containing points of Δ, m the number of pure edges, and l half the number of impure edges. Then the complex $\mathcal{B}(S, \Delta_0, \Delta_1)$ is $(4g+r+r'+l+m-6)$-connected.*

We will need a couple of lemma's to reduce the complexity of the theorem.

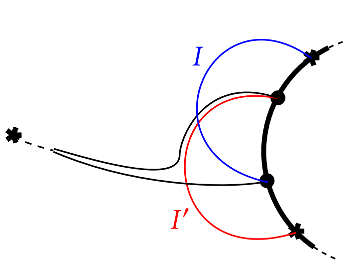
Lemma 2.1.19. *If the surface S has two neighbouring pure edges, which may be equal, the arc complex $\mathcal{B}(S, \Delta_0, \Delta_1)$ is contractible. In particular, this holds if S has a pure boundary component.*

Proof. Choose an arc a with an endpoint p between two pure edges. Without loss of generality, assume $p \in \Delta_0$. Then $\mathcal{B}(S, \Delta_0, \Delta_1)$ is non-empty, as there is either a point $p' \in \Delta_1$ on a different boundary component as p or on the same component, but with a point of Δ_0 in between on either side, as p lies between pure edges. An arc from p to p' gives a non-trivial vertex in either case.

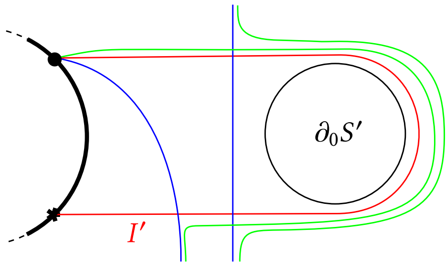
As in the proof of theorem 2.1.16, there is a contraction of $\mathcal{B}(S, \Delta_0, \Delta_1)$ onto Star(a). In this case, exactly one of $R(a_i)$ and $L(a_i)$ is admitted in the retraction, as the other has both endpoint in one of the Δ_i. The admitted arc is then non-trivial, as the points of Δ next to p at either side are also in Δ_0. □

Lemma 2.1.20. *If $\mathcal{B}(S, \Delta_0, \Delta_1) \neq \emptyset$, the addition of a pure edge between two impure edges increases its connectivity by one.*

Proof. This is exactly the argument of lemma 2.1.17: compare figure 2.7a with figures 2.5 and 2.6. □



(a) In lemma 2.1.20.



(b) In lemma 2.1.21. The retraction is a path from the blue lines to the green ones

Figure 2.7: Retractions onto Star(I').

Lemma 2.1.21. *Suppose (S, Δ_0, Δ_1) has an impure edge, $S = S' \cup_{\partial_0 S'} D^2$, and $\mathcal{B}(S, \Delta_0, \Delta_1) \neq \emptyset$. In particular, $\Delta_0 \cup \Delta_1$ is disjoint from $\partial_0 S'$. Then the connectivity of $\mathcal{B}(S', \Delta_0, \Delta_1)$ is one higher than that of $\mathcal{B}(S, \Delta_0, \Delta_1)$, i.e. adding a boundary component disjoint from Δ increases connectivity by one.*

Proof. Call an arc in $\mathcal{B}(S', \Delta_0, \Delta_1)$ *special* if it is trivial in $\mathcal{B}(S, \Delta_0, \Delta_1)$, i.e. its endpoints are adjacent points in Δ and it separates a cylinder off of S', of which one boundary component is $\partial_0 S'$. Let $X \subset \mathcal{B}(S', \Delta_0, \Delta_1)$ be the subcomplex of simplices without special arcs. Then

$$\mathcal{B}(S', \Delta_0, \Delta_1) = X \bigcup_{\substack{\mathrm{Link}(I) \\ I \text{ special}}} \mathrm{Star}(I)$$

Clearly, $\mathrm{Link}(I) \cong \mathcal{B}(S, \Delta_0, \Delta_1)$ for any special arc I.

Choosing a special I', there is a retraction from $X \cup_{\mathrm{Link}(I')} \mathrm{Star}(I')$ onto $\mathrm{Star}(I')$, analogous to the retraction in lemma 2.1.17, as seen in lemma 2.1.21. This is well-defined, as only special arcs become trivial, and these are not in X.

Again using the Seifert-Van Kampen theorem, the Hurewicz isomorphism and the Mayer-Vietoris sequence for

$$\mathcal{B}(S', \Delta_0, \Delta_1) = \left(X \cup_{\mathrm{Link}(I')} \mathrm{Star}(I')\right) \bigcup_{\substack{\mathrm{Link}(I) \\ I \text{ special}, I \neq I'}} \mathrm{Star}(I)$$

the lemma follows. □

We also state a standard lemma about connectivity of joins of complexes, which we will need in the proof of the theorem.

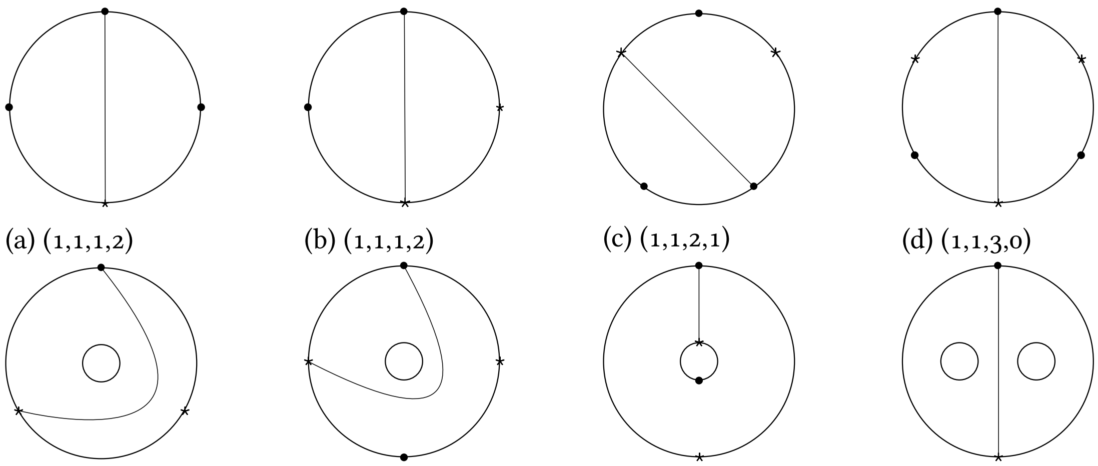
Lemma 2.1.22 ([Mil56])**.** *If spaces or simplicial complexes X_i are n_i-connected, where $1 \leq i \leq k$, then their join $X_1 \star \cdots \star X_k$ is $\left(\sum_{i=1}^{k}(n_i + 2) - 2\right)$ connected.* □

Proof of theorem 2.1.18. Using lemmas 2.1.19 to 2.1.21, we may assume $m = 0$ and $r = r'$ after we have proven (-1)-connectedness (i.e. non-emptiness) for the relevant complexes. The dependence on these two parameters then follows from those lemma's. Therefore, it is left to prove in that case $\mathcal{B}(S, \Delta_0, \Delta_1)$ is $(4g + 2r + l - 6)$-connected.

We will use induction on the lexicographically ordered triples (g, r, l). The basis step, which is (-1)-connectivity or non-emptiness, must be checked for $g = 0$, where (r, r', l, m) is one of the cases $(1, 1, 1, 2)$, $(1, 1, 2, 1)$, $(1, 1, 3, 0)$, $(2, 1, 1, 1)$, $(2, 1, 2, 0)$, $(2, 2, 1, 0)$, or $(3, 1, 1, 0)$, as $r' \leq r$ by definition and $r' \leq l$ by lemma 2.1.19. These are all checked in figure 2.8.

Now, the induction starts with $(g, r, l) = (0, 1, 3)$. Consider a surface (S, Δ_0, Δ_1), with $r = r'$, $m = 0$ (and $l \geq 3$ if $(g, r) = (0, 1)$).

Let $k \leq 4g + 2r + l - 6$ and $f : S^k \longrightarrow \mathcal{B}(S, \Delta_0, \Delta_1)$. By the simplicial approximation theorem, we may assume f to be simplicial for some piecewise linear triangulation of S^k.



(a) (1,1,1,2) (b) (1,1,1,2) (c) (1,1,2,1) (d) (1,1,3,0)

(e) (2,1,1,1) (f) (2,1,2,0) (g) (2,2,2,0) (h) (3,1,1,0)

Figure 2.8: The basis step of the induction.

By theorem 2.1.16, $\mathcal{A}(S,\Delta)$ is contractible unless $(g,r) = (0,1)$ (as $r = r'$), and in the second case, we have $q = 2l \geq 6$, so $4g + 2r + l - 6 \leq 2r + q - 7$. Therefore, f extends to a simplicial map $\hat{f} : D^{k+1} \longrightarrow \mathcal{A}(S,\Delta)$ with respect to a piecewise linear triangulation of D^{k+1} extending that of S^k. We will do surgery on D^{k-1} and $\hat{f}$ to remove all simplices that do not have image in $\mathcal{B}(S,\Delta_0,\Delta_1)$.

So let us call a simplex $\sigma \in D^{k+1}$ *bad* if $\hat{f}(\sigma)$ and $\mathcal{B}(S,\Delta_0,\Delta_1)$ are disjoint in $\mathcal{A}(S,\Delta)$, i.e. if all arcs of $\hat{f}(\sigma)$ are pure. We have a decomposition into connected components of the cut surface $S' := S \setminus \hat{f}(\sigma)$:

$$(S,\Delta_0,\Delta_1) \setminus \hat{f}(\sigma) = \bigcup_{i=1}^{c}(X^i,\Delta_0^i,\Delta_1^i) \cup \bigcup_{j=1}^{d}(Y^j,\Gamma^j)$$

Where the Δ_ε^i are non-empty subsets of X^i inherited from Δ_ε, and Γ^j is the only non-empty one of the Δ_ε^j in Y^j.

Set $Y_\sigma = \bigcup_{j=1}^{d} \iota_j(Y^j)$, where all $\iota_j : Y^j \longrightarrow S$ are inclusions (they need not be injective on the boundaries). We call σ *regular* if $\bigcup_{j=1}^{d} \iota_j : \bigcup_{j=1}^{d} Y^j \longrightarrow Y_\sigma$ is injective away from the Γ_j.

Set an ordering on pairs (Y_σ, p) by defining $(Y_\sigma, p) < (Y_{\sigma'}, p')$ if either $Y_\sigma \subsetneq Y_{\sigma'}$ or $Y_\sigma = Y_{\sigma'}$ and $p < p'$, and let σ be a regular bad p-simplex, maximal with respect to this ordering. If $\tau \in \mathrm{Link}(\sigma)$ is a simplex, none of the arcs of $\hat{f}(\tau)$ can equal any of $\hat{f}(\sigma)$, by maximality of p. Therefore, they all lie in the X_i or the Y_j. If they lie in the X_i, they must be impure by maximality of p again. Hence $\hat{f}$ restricts to

$$S^{k-p} \cong \mathrm{Link}(\sigma) \longrightarrow J_\sigma := \mathcal{B}(X^1,\Delta_0^1,\Delta_1^1) \star \cdots \star \mathcal{B}(X^c,\Delta_0^c,\Delta_1^c) \star \mathcal{A}(Y^1,\Gamma^1) \star \cdots \star \mathcal{A}(Y^d,\Gamma^d)$$

We want to extend this map to D^{k-p+1}, so we will show J_σ is $(k-p)$-connected.

If any of the Y^j is not a disc, contractibility of J_σ follows from theorem 2.1.16 combined with lemma 2.1.22. Hence assume all Y^j are discs.

Let indices i and j be used for relevant parameters of X^i and Y^j, respectively. Let p' be the degree of $\hat{f}(\sigma)$. Then, as $p'+1$ arcs and $2(p'+1)$ vertices are doubled,

$$\chi(S') = \sum_{i=1}^{c}(2 - 2g_i - r_i) + d = 2 - 2g - r + p' + 1 = \chi(S) + p' + 1$$

$$\sum_{i=1}^{c}(2g_i + r_i) = 2g + r - p' + 2c + d - 3$$

Because we assumed $m = 0$, we get that the number of pure edges in S' is equal to

$$2p' + 2 = \sum_{i=1}^{c} m_i + \sum_{j=1}^{d} q_j$$

Since ∂Y^j is pure, half the number of impure edges of S' is given by

$$l = \sum_{i=1}^{c} l_i$$

Since $m = 0$ and ∂Y^j is pure, the edges of Y^j are arcs of σ, and any non-trivial Y^j has at least three edges. As σ is regular, no edge can be used twice for the Y^j's. Therefore, $3d \le p + 1$, so $d \le p$.

As $(g_i, r_i, l_i) < (g, r, l)$, induction gives that $\mathcal{B}(X^i, \Delta_0^i, \Delta_1^i)$ is $(4g_i + 2r_i + l_i + m_i - 6)$-connected. And by theorem 2.1.16, $\mathcal{A}(Y^j, \Gamma^j)$ is $(q_j - 5)$-connected. Therefore, combining all equalities together and using lemma 2.1.22, the connectivity of J_σ is at least

$$\begin{aligned} &\sum_{i=1}^{c}(4g_i + 2r_i + l_i + m_i - 4) + \sum_{j=1}^{d}(q_j - 3) - 2 \\ &= (4g + 2r - 2p' + 4c + 2d - 6) + l + (2p' + 2) - 4c - 3d - 2 \\ &= 4g + 2r + l - d - 6 \ge 4g + 2r + l - p - 6 \ge k - p \end{aligned}$$

This proves there exists a $(k - p + 1)$-disc K with $\partial K = \mathrm{Link}(\sigma)$ and a map $F : K \to J_\sigma$ extending $\hat{f}|_{\mathrm{Link}(\sigma)}$.

As $\mathrm{Star}(\sigma) = \sigma \star \mathrm{Link}(\sigma)$ is a $(k + 1)$-disc with boundary $\partial\sigma \star \mathrm{Link}(\sigma)$, we can surger it in the triangulation of D^{k+1} to $\partial\sigma \star K$, which has the same boundary, and surger $\hat{f}$ to $\hat{f} \star F$ on $\partial\sigma \star K$.

Now let $\tau = \alpha \star \beta$ be a new regular bad simplex of D^{k+1}. Then $\alpha \in \partial\sigma$ is a proper face and β maps to J_σ. As τ is bad, the arcs of β lie in the $\mathcal{A}(Y^j, \Gamma^j)$, so $Y_\tau \subseteq Y_\sigma$. If $Y_\tau = Y_\sigma$, then $\tau = \alpha$, by regularity of σ and τ, and $\dim(\tau) < p$. Hence $(Y_\tau, \dim(\tau)) < (Y_\sigma, p)$, which shows we have reduced the number of maximal regular bad simplices. Repeating this, we eliminate all regular bad simplices. As any bad simplex has a regular bad subsimplex, we eliminate all of those as well. Therefore, we have surgered $\hat{f}(D^{k+1})$ into $\mathcal{B}(S, \Delta_0, \Delta_1)$, proving the connectivity. □

Theorem 2.1.23. *The non-separating arc complex $\mathcal{B}_0(S, \Delta_0, \Delta_1)$ is $(2g + r' - 3)$-connected.*

Proof. We use induction on (g, r, q), which is the same as in theorem 2.1.18, because there $q = 2l$. For $g = 0$ and $r' \le 1$, the statement is vacuous. For $(g, r') = (0, 2)$, the theorem holds as well, as there must be a point of Δ_0 and a point of Δ_1 that lie on different boundary components. Any arc connecting them would be non-separating.

As $q \ge 2$, we can now fix an (S, Δ_0, Δ_1) such that $(g, r, q) \ge (0, 3, 2)$. Then either $r \ge 3$, so $2g + r' - 3 \le 4g + r + r' + l + m - 6$, or $g \ge 1$, in which case again $2g + r' - 3 \le 4g + r + r' + l + m - 6$, as clearly $r \ge 1$. Therefore, $\mathcal{B}(S, \Delta_0, \Delta_1)$ is at least $(2g + r' - 3)$-connected.

The induction step is similar to that of the proof of theorem 2.1.18: we take a $k \le 2g + r' - 3$, a map $f : S^k \to \mathcal{B}_0(S, \Delta_0, \Delta_1)$, which we may take to be simplicial with respect to a piecewise linear triangulation of S^k, and extend it to a simplicial map $\hat{f} : D^{k+1} \to \mathcal{B}(S, \Delta_0, \Delta_1)$, where D^{k+1} is again piecewise linear simplical. We now call a simplex σ of D^{k+1} *regular bad* if $\hat{f}(\sigma) = \langle a_0, \dots, \alpha_p \rangle$ and each a_i separates $S \setminus \bigcup_{j \ne i} a_j$ and take such a simplex σ of maximal dimension p. The arcs of $\hat{f}(\sigma)$ are impure, so each connected component X^i of $S' := S \setminus \hat{f}(\sigma)$ has points from both Δ_ε. Again,

$$\hat{f}|_{\mathrm{Link}(\sigma)} : S^{k-p} \cong \mathrm{Link}(\sigma) \to J_\sigma := \mathcal{B}_0(X^1, \Delta_0^1, \Delta_1^1) \star \cdots \star \mathcal{B}_0(X^c, \Delta_0^c, \Delta_1^c)$$

The Euler characteristic gives

$$\chi(S') = \sum_{i=1}^{c}(2 - 2g_i - r_i) = 2 - 2g - r + p' + 1 = \chi(S) + p' + 1$$

$$\sum_{i=1}^{c}(2g_i + r_i') = 2g + r' + p' + 2c - 3$$

Using $\sum_{i=1}^{c}(r_i - r_i') = r - r'$ and induction to get that $\mathcal{B}_0(X^i, \Delta_0, \Delta_1)$ is $(2g_i + r_i' - 3)$-connected, lemma 2.1.22 yields that the connectivity of J_σ is at least

$$\sum_{i=1}^{c}(2g_i + r_i' - 3 + 2) - 2 = 2g + r' - p' + c - 5 \geq 2g + r' - p - 3$$

because $p' \leq p$ and $c \geq 2$.

Therefore, we can again extend $\hat{f}|_{\mathrm{Link}(\sigma)}$ to some simplicial $F : K \longrightarrow J_\sigma$, where K is a $(k-p+1)$-disc and $\partial K = \mathrm{Link}(\sigma)$, and again surger $\hat{f}$ on $\mathrm{Star}(\sigma)$ to $\hat{f} \star F$ on $\partial\sigma \star K$.

Given a regular bad simplex $\alpha \star \beta$ in $\partial\sigma \star K$, β must be trivial, as it does not separate $S \setminus \hat{f}(\alpha)$—it does not even separate $S \setminus \hat{f}(\sigma)$. Therefore, $\alpha \star \beta = \alpha$, which is a face of σ. As in the proof of theorem 2.1.18, the theorem follows. □

Theorem 2.1.24. *The arc complexes $O(S_{g,r})$ are $(g-2)$-connected.*

Proof. We will use induction on g, in a similiar way to the proofs of the previous two theorems.

For $g = 0$, the theorem is vacuous. For $g = 1$, non-emptiness of $O(S, b_0, b_1)$ follows from non-emptiness of $\mathcal{B}_0(S, \{b_0\}, \{b_1\})$ (see theorem 2.1.23 and use $1 - 2 \leq 2 \cdot 1 + r' - 3$), as both complexes have the same set of vertices.

So now assume $g \geq 2$, take a $k \leq g - 2$ and a $f : S^k \longrightarrow O(S, b_0, b_1)$, which we may assume to be piecewise linear simplicial. As $g - 2 \leq 2g + r' - 3$, this map extends to a map $\hat{f} : D^{k+1} \longrightarrow \mathcal{B}_0(S, \{b_0\}, \{b_1\})$, again assumed to be piecewise linear simplicial.

Let $\sigma = \langle a_0, \ldots, a_p \rangle \in \mathcal{B}_0(S, \{b_0\}, \{b_1\})$ be such that $a_i < a_j$ for $i < j$ in the anticlockwise ordering at b_0. Then the clockwise ordering at b_1 starts with $a_0 < \cdots < a_i$ (where possibly $i = -1$) and we define the *good* part of σ to be $\sigma^g = \langle a_0, \ldots, a_i \rangle$. Define the *bad* part σ^b by $\sigma = \sigma^g \star \sigma^b$. If $\sigma = \sigma^g$, it lies in $O(S, b_0, b_1)$, and if $\sigma = \sigma^b$, we call it *purely bad.*

Suppose b_0 and b_1 lie on the same boundary component. Then the genus of $S \setminus \sigma$, for σ a p-simplex, is at least $g - p$: the first arc cut will not reduce the genus and the other cuts can only reduce the genus by one per cut: $\chi(S \setminus a) = 2 - 2g' - r' = \chi(S) + 1 = 3 - 2g - r$ and $r' \leq r + 1$, so $g' \geq g - 1$.

If b_0 and b_1 lie on different boundary components, and σ is purely bad, this still holds: given two arcs a and a' ordered the same way at b_0 and b_1, $S \setminus (a \cup a')$ has the same amount of boundary components as S, and by the Euler characteristic has genus $g - 1$. Again, extra arc cuts only reduce the genus by at most one per cut.

Let now σ be a p-simplex in D^{k+1} such that $\hat{f}(\sigma)$ is purely bad and p is maximal for this. Then any arc in a simplex of $\mathrm{Link}(\sigma)$ must be before all arcs in σ for both the counter-clockwise ordering at b_0 and the clockwise ordering at b_1, so $\hat{f}(\mathrm{Link}(\sigma))$ maps to $O(S \setminus \hat{f}(\sigma), b_0', b_1')$, where the b_i' are the copies of b_i 'before' the first arc in the respective orderings. As σ is bad, $p' \geq 1$, so $g' < g$, and $O(S \setminus \hat{f}(\sigma), b_0', b_1')$ is $(g' - 2)$-connected by induction.

As $\mathrm{Link}(\sigma) \cong S^{k-p}$ we can use that $g' - 2 \geq g - p' - 2 \geq g - p - 2 \geq k - p$ to extend $\hat{f}|_{\mathrm{Link}(\sigma)} \rightarrow O(S, b_0, b_1)$ to F over a $(k - p + 1)$-disc K and replace $\hat{f}$ on $\mathrm{Star}(\sigma)$ by $\hat{f} \star F$ on $\partial\sigma \star K$.

The new purely bad simplices are in $\partial\sigma$, as simplices in K are good, so the result again follows by induction. □

2.2 — Proof of Harer stability

2.2.1 — Proof for non-empty boundary

Write $(\tilde{C}_\bullet(X), \partial)$ for the augmented simplicial chain complex of a simplicial set X_* and, if G is a group, $(E_\bullet G, d)$ for the complex of simplicial chains of EG, see definition A.1.3. It is a free $\mathbb{Z}[G]$-resolution of $\mathbb{Z}$ by lemma A.3.2. If X_* and Y_* are simplicial sets with G- and H-actions, respectively, $\varphi : G \longrightarrow H$ is a homomorphism, and $f : X \longrightarrow Y$ is a φ-equivariant simplicial map (i.e. $f(gx) = \varphi(g)f(x)$), there is an induced map of double chain complexes

$$F : \tilde{C}_p(X) \otimes_G E_q G \longrightarrow \tilde{C}_p(Y) \otimes_H E_q H$$

Definition 2.2.1. In the above situation, define the double complex $C_{p,q}$ to be the mapping cone of the map F in the q-direction:

$$\begin{aligned} C_{p,q} &= (\tilde{C}_p(X) \otimes_G E_{q-1}G) \oplus (\tilde{C}_p(Y) \otimes_H E_q H) \\ d^h(a \otimes b, a' \otimes b') &= (\partial a \otimes b, \partial a' \otimes b') \\ d^v(a \otimes b, a' \otimes b') &= (a \otimes db, a' \otimes db' + F(a \otimes b)) \end{aligned}$$

This double complex has two associated spectral sequences, the vertical an horizontal one, denoted ${}^vE^r_{pq}$ and ${}^hE^r_{pq}$ respectively, see example B.2.7.

Lemma 2.2.2. *In the above situation, if X is $(n-1)$-connected and Y is n-connected, the ${}^hE^1$-sheet is zero for $p+q \le n$. Therefore, the total homology is zero for $* \le n$ and ${}^vE$ converges to zero for $p+q \le n$ as well.* □

Lemma 2.2.3. *If the actions of G on X and H on Y are transitive on p-simplices for all p,*

$${}^vE^1_{pq} = H_q(\mathrm{Stab}(f(\sigma_p)), \mathrm{Stab}(\sigma_p))$$

where $\sigma \in X_p$.

Proof. By definition of relative homology, and because the columns $C_{p,*}$ are the mapping cones of $F_{p,*}$,

$${}^vE^1_{pq} = H_q\big(\tilde{C}_p(Y) \otimes_H E_\bullet H, \tilde{C}_p(X) \otimes_G E_\bullet G\big)$$

The right version of Shapiro's lemma, corollary A.4.3, directly implies the lemma. □

Writing $H_*(A_{g,r}) := H_*(\Gamma_{g+1,r}, \Gamma_{g,r+1}; \mathbb{Z})$ and $H_*(B_{g,r}) := H_*(\Gamma_{g,r+1}, \Gamma_{g,r}; \mathbb{Z})$, defined via the pant-attaching maps α and β, respectively, and using the long exact sequence of the pair, we can restate theorem 2.3 as follows:

Theorem 2.2.4. *Suppose $r \ge 1$. We have that*

a) If $i \le \frac{2g+1}{3}$, then $H_i(A_{g,r}) = 0$;
b) If $i \le \frac{2g}{3}$, then $H_i(B_{g,r}) = 0$.

Hence, the homology of the mapping class group $\Gamma_{g,r}$ stabilises in this range.

Proof. We will use a double induction argument on g. So let (1_g) and (2_g) denote the statements of the theorem for the given g. Then (1_0), (2_0), and (2_1) are trivially true, as they only concern H_0. This proves the basis step.

The induction steps are as follows:

(a) Let $g \geq 1$. If $(2_{\leq g})$ holds, then so does (1_g).
(b) Let $g \geq 2$. If $(1_{<g})$ and (2_{g-1}) hold, then so does (2_g).

First we will prove (a). In the situation above, we take $G = \Gamma_{g,r+1}$, $X = \mathcal{O}^2(S_{g,r+1})$, $H = \Gamma_{g+1,r}$, and $Y = \mathcal{O}^1(S_{g+1,r})$. Both $\varphi : G \to H$ and $f : X \to Y$ are induced by α as in definition 2.1.4. Then we consider the associated spectral sequences. By proposition 2.1.3, the actions of G on X and H on Y are both transitive, so we can apply lemma 2.2.3: ${}^vE^1_{pq} = H_q(\mathrm{Stab}(f(\sigma_p)), \mathrm{Stab}(\sigma_p))$.

If $p = -1$, we get ${}^vE^1_{-1,q} = H_q(\Gamma_{g+1,r}, \Gamma_{g,r+1}) =: H_q(A_{g,r})$. These are the groups we want to calculate.

By theorem 2.1.24 we know that X is $(g-2)$-connected and Y is $(g-1)$-connected. Hence, by lemma 2.2.2, ${}^vE^\infty_{-1,i} = 0$ for $i \leq g$. In particular, this holds for $i \leq \frac{2g+1}{3}$.

If $p \geq 0$, we get ${}^vE^1_{p,q} = H_q(\Gamma_{g-p,r+p+1}, \Gamma_{g-p,r+p}) =: H_q(B_{g-p,r+p})$, by proposition 2.1.6. By the induction hypothesis, these are zero for $q \leq \frac{2(g-p)}{3}$. If $i \leq \frac{2g+1}{3}$, this means that all sources of differentials into ${}^vE^r_{-1,i}$ are zero unless $i = \frac{2g+1}{3}$ and $p = 0$: they must lie on the line $p + q = i$, with $p \geq 0$ (so $q = i - p \leq \frac{2g+1-3p}{3} \leq \frac{2g-2p}{3}$).

For $i < \frac{2g+1}{3}$, ${}^vE^\infty_{-1,q} = 0$ and all differentials into ${}^vE^r_{-1,i}$ are zero for $r \geq 1$, thereby showing that $H_i(A_{g,r}) = {}^vE^1_{-1,i} = 0$.

For $i = \frac{2g+1}{3}$, the only possibly non-zero map into ${}^vE^r_{-1,i}$ is $d^1 : H_i(B_{g,r}) = {}^vE^1_{0,i} \to {}^vE^1_{-1,i} = H_i(A_{g,r})$. But corollary 2.1.9 states exactly that this map is zero. This concludes the proof of (a).

For (b), the argument is similiar. Here, we take $G = \Gamma_{g,r}$, $X = \mathcal{O}^1(S_{g,r})$, $H = \Gamma_{g,r+1}$, and $Y = \mathcal{O}^2(S_{g,r+1})$. The maps φ and f are induced by β this time, again as in definition 2.1.4. Then proposition 2.1.3 and lemma 2.2.3 still apply, so once again ${}^vE^1_{pq} = H_q(\mathrm{Stab}(f(\sigma_p)), \mathrm{Stab}(\sigma_p))$.

In this case, we get, for $p = -1$, that ${}^vE^1_{-1,q} = H_q(\Gamma_{g,r+1}, \Gamma_{g,r}) =: H_q(B_{g,r})$.

Again by theorem 2.1.24 we know that X and Y are $(g-2)$-connected. Hence, by lemma 2.2.2, ${}^vE^\infty_{-1,i} = 0$ for $i \leq g-1$. In particular, this holds for $i \leq \frac{2g}{3}$.

If $p \geq 0$, we get ${}^vE^1_{p,q} = H_q(\Gamma_{g-p,r+p}, \Gamma_{g-p-1,r+p+1}) =: H_q(A_{g-p-1,r+p})$, by proposition 2.1.6. By the induction hypothesis, these are zero for $q \leq \frac{2(g-p-1)+1}{3}$. If $i \leq \frac{2g}{3}$, this means that all sources for differentials into ${}^vE^r_{-1,i}$ are zero unless $i = \frac{2g}{3}$ and $p = 0$, as they must lie on the line $p + q = i$, with $p \geq 0$ (so $q = i - p \leq \frac{2g-3p}{3} \leq \frac{2g-2p-1}{3}$).

For $i < \frac{2g}{3}$, ${}^vE^\infty_{-1,q} = 0$ and all differentials into ${}^vE^r_{-1,i}$ are zero for $r \geq 1$, showing that $H_i(A_{g,r}) = {}^vE^1_{-1,i} = 0$.

For $i = \frac{2g}{3}$, only the map $d^1 : H_i(B_{g,r}) = {}^vE^1_{0,i} \to {}^vE^1_{-1,i} = H_i(A_{g,r})$ into ${}^vE^r_{-1,i}$ can be non-zero. The proof that this is zero is slightly more involved than in step (a), and works by induction.

This map d^1 is exactly the map $H_q(\mathrm{Stab}(\beta(\sigma)), \mathrm{Stab}(\sigma)) \to H_q(\Gamma(S_\beta), \Gamma(S))$ induced by proposition 2.1.7, where $S = S_{g,r}$ and $S_\beta = S_{g,r+1}$. This map factors through $- \times t_{a_1 * a}$ by lemma 2.1.8. Using the notation from corollary 2.1.10, the map

$$H_{i-1}(\Gamma(S')) \cong H_{i-1}(\Gamma_{g-1,r}) \to H_{i-1}(\Gamma_{g-1,r+1}) \cong H_{i-1}(\mathrm{Stab}(\sigma))$$

is clearly a β-map. By the induction hypothesis (2_{g-1}), $H_{i-1}(B_{g-1,r}) = 0$, so by the long exact sequence of the pair, this β-map is an epimorphism. Inserting this into corollary 2.1.10 yields that $- \times t_{a_1 * a} = 0$, so $d^1 = (d-) \times t_{a_1 * a} = 0$.

This concludes the proof of (b).

□

2.2.2 — Proof for closed surfaces

In the case $r > 1$, the map $\delta : H_*(\Gamma_{g,r}) \to H_*(\Gamma_{g,r-1})$ is a left inverse to β, so it is an isomorphism in the appropriate range. If $r = 1$, this is not the case anymore. Therefore, we will construct two spectral sequences and a morphism between them, such that the sequences converge to the homology groups of theorem 2.4 and the E^1-terms all involve mapping class groups of surfaces with boundary. Then we can invoke theorem 2.3 to prove the morphism of spectral sequences is an isomorphism or epimorphism in certain ranges, using a version of theorem B.1.10 to link this to the morphism of the limit homologies.

The spectral sequences in the proof are associated to double complexes defined using the following semi-simplicial space:

Definition 2.2.5. Define the semi-simplicial space $B\Gamma_{g,\bullet+r}$ by setting $(B\Gamma_{g,\bullet+r})_k = B\Gamma_{g,k+r}$, the classifying space of the mapping class group of $S_{g,k+r}$, and the face maps $d_i : B\Gamma_{g,k+r} \longrightarrow B\Gamma_{g,k-1+r}$ defined by glueing a disc on the i-th boundary circle of the surface $S_{g,k+r}$ (counting boundaries from 0).

To make use of this semi-simplicial space, we need to know its geometric realisation as well, as that will correspond to the total complex of the double complex. It is given in the following theorem. Note, however, that this uses the actual topological group $\mathrm{Diff}(S_{g,r})$ in stead of the mapping class group. For high genus we know this is the same by the Earle-Eells-Schatz theorem 1.3.3, but the proof uses the topology in an important way.

Theorem 2.2.6. *[Ran14] For any r, there is a homotopy equivalence $\|B\Gamma_{g,\bullet+r+1}\| \simeq B\,\mathrm{Diff}(S_{g,r})$.*

Proof. Define $C_{k,g,r}$ to be the space of configurations of k ordered points in the interior of $S_{g,r}$, where each point has a co-oriented framing. Then $\mathrm{Diff}(S_{g,r})$ acts transitively on $C_{k,g,r}$ for each k, and the stabiliser of any point is isomorphic to $\mathrm{Diff}(S_{g,k+r})$. By corollary A.4.4, we get

$$C_{k,g,r}/\!\!/_{\mathrm{Diff}} := C_{k,g,r} \times_{\mathrm{Diff}(S_{g,r})} E\,\mathrm{Diff}(S_{g,r}) \simeq B\,\mathrm{Diff}(S_{g,k+r})$$

By theorem 1.3.3, this is homotopy equivalent to $B\Gamma_{g,r}$ if $k + r > 0$. If we define a semi-simplical space $C_{\bullet+1,g,r}$ by letting the i-th boundary map forget the i-th point, this is compatible with the $\mathrm{Diff}(S_{g,r})$-action, yielding an equivalence of semi-simplicial spaces $(C_{\bullet+1,g,r})/\!\!/_{\mathrm{Diff}} \simeq B\Gamma_{\bullet+r+1}$.

If we can show $C_{\bullet+1,g,r}$ is contractible, we get that $C_{\bullet+1,g,r} \times E\,\mathrm{Diff}(S_{g,r})$ is contractible, with free action of $\mathrm{Diff}(S_{g,r})$, so $\|C_{\bullet+1,g,r}/\!\!/_{\mathrm{Diff}}\| \simeq B\,\mathrm{Diff}(S_{g,r})$, which would show the map is a weak homotopy equivalence. As both spaces are (homotopy equivalent to) CW-complexes, the Whitehead theorem would then imply this is an actual homotopy equivalence.

First we assume $r > 0$. In this case, $C_{k,g,r}$ is homotopy equivalent to $\bar{C}_{k,g,r}$, a similar space where all points are outside a small neighbourhood U of one chosen boundary component, by deformation retracting $S_{g,r}$ onto $S_{g,r} \setminus U$. Choosing a framed point $p \in U$ and regarding it as a 0-simplex in $C_{k,g,r}$, we get an equivalence $\bar{C}_{\bullet+1,g,r} \star p \cong C(\bar{C}_{\bullet+1,g,r})$ the cone of $\bar{C}_{\bullet+1,g,r}$. This clearly gives a retraction of $\bar{C}_{\bullet+1,g,r}$ onto p inside $C_{\bullet+1,g,r}$, showing contractibility of $\bar{C}_{\bullet+1,g,r}$ and therefore of $C_{\bullet+1,g,r}$.

Now take $r = 0$. The glueing of a disc $S_{g,1} \hookrightarrow S_{g,0}$ induces a simplicial map $C_{\bullet+1,g,1} \longrightarrow C_{\bullet+1,g,0}$ which is an inclusion in each degree, and therefore a cofibration. In degree k the cofibre is given by

$$C_{k+1,g,0}/C_{k+1,g,1} = \bigvee_{i=0}^{k} \left(S(D^2) \times C_{k,g,1}\right) \Big/ \left(S(\partial D^2) \times C_{k,g,1}\right)$$

Where the wedge tracks which point is closest to the centre of the disc (at distance r, say), $S(D^2)$ is the sphere bundle of the disc with radius r tracking the framing of that point, $C_{f,g,1}$ tracks the framing of the k other point, outside the disc with radius r, and all configurations disjoint from D^2 or with several points closest to the centre are identified with the base point.

If for any spaces X, Y, with ∂X non-empty, we let $X/\partial X$ be pointed by ∂X and set Y_+ to be Y union a disjoint base point, we get

$$\begin{aligned} X/\partial X \wedge \bigvee_{i=0}^{k} Y_+ &= X/\partial X \wedge \Big(\coprod_{i=0}^{k} Y\Big)_+ = \coprod_{i=0}^{k} (X/\partial X \times Y) \Big/ \coprod_{i=0}^{k} (Y \times \partial X) \\ &= \bigvee_{i=0}^{k} (X \times Y)/\partial X \times Y \end{aligned}$$

Hence the cofibre is $\left(S(D^2)/S(\partial D^2)\right) \wedge \bigvee_{i=0}^{k}(C_{k,g,1})_+$ on the degree k level and the cofibre simplicial space is

$$\left(S(D^2)/S(\partial D^2)\right) \wedge \bigvee_{i=0}^{\bullet}(C_{\bullet,g,1})_+$$

With induced boundary maps on $\bigvee_{i=0}^{\bullet}(C_{\bullet,g,1})_+$

$$d_i : (p_0, \dots, p_j, \dots, p_k) \mapsto \begin{cases} (p_0, \dots, p_i, \dots, p_j, \dots, p_k) & i \neq j \\ * & i = j \end{cases}$$

As in the case $r > 0$, there is a retraction $\bigvee_{i=0}^{\bullet}(C_{\bullet,g,1})_+ \simeq \bigvee_{i=0}^{\bullet}(\bar{C}_{\bullet,g,1})_+ \simeq *$. So we have a cofibration

$$C_{\bullet+1,g,1} \longrightarrow C_{\bullet+1,g,0} \longrightarrow \left(S(D^2)/S(\partial D^2)\right) \wedge \bigvee_{i=0}^{\bullet}(C_{\bullet,g,1})_+$$

in which both the subspace and the cofibre are contractible, which shows the total space is contractible, using the long exact sequence in homology, proving the theorem. □

Proof of theorem 2.4. Consider, for $B\Gamma_{g,\bullet+2}$, the Segal spectral sequence (example B.2.17), using theorem 2.2.6:

$$E^1_{pq} = H_q(B\Gamma_{g,p+2}) \Rightarrow H_{p+q}(B\,\mathrm{Diff}(S_{g,1})) = H_{p+q}(\mathrm{Diff}(S_{g,1}))$$

According to Earle-Eells-Schatz, theorem 1.3.3, this limit is isomorphic to $H_{p+q}(\Gamma_{g,1})$.

There is a map of spectral sequences, starting at sheet 1,

$$f^1 : E^1_{pq} = H_q(\Gamma_{g,p+2}) \longrightarrow \tilde{E}^1_{pq} = H_q(\Gamma_{g,p+1})$$

induced on k-simplices by glueing a disc to the last boundary component of $S_{g,p+2}$.

This map, δ on all terms, does have right inverse β, as all surfaces have non-empty boundary. Therefore, it is an epimorphism on the first sheet. Also, by theorem 2.3, it is an isomorphism for $q \leq \frac{2g}{3}$.

For non-zero terms total degree $p + q \leq \frac{2g}{3}$, all non-zero differentials going in and out remain in the region $q \leq \frac{2g}{3}$ (because it is a first quadrant spectral sequence), so the isomorphism persists to the infinity sheet: $f^\infty : {}^vE^\infty_{pq} \xrightarrow{\sim} {}^v\tilde{E}^\infty_{pq}$ for $q \leq \frac{2g}{3}$. This shows that $\delta : H_{p+q}(\Gamma_{g,1}) \longrightarrow H_{p+q}(\Gamma_{g,0})$ is an isomorphism for $p + q \leq \frac{2g}{3}$ by a simple extension of theorem B.1.10.

For the line $p+q = \lfloor \frac{2g}{3} \rfloor + 1$, the isomorphism still holds for $p > 0$, and for the term ${}^vE_{0q}$, there is an epimorphism on the first sheet, which together with the epimorphic edge homomorphisms gives a commutative diagram

$$\begin{array}{ccc} {}^vE^1_{0q} & \twoheadrightarrow & {}^vE^\infty_{0q} \\ \downarrow f^1 & & \downarrow f^\infty \\ {}^v\tilde{E}^1_{0q} & \twoheadrightarrow & {}^v\tilde{E}^\infty_{0q} \end{array}$$

which shows that $f^\infty : {}^vE^\infty_{0q} \longrightarrow {}^v\tilde{E}^\infty_{0q}$ is an epimorphism, hence so is the map $\delta : H_{p+q}(\Gamma_{g,1}) \longrightarrow H_{p+q}(\Gamma_{g,0})$. □

Chapter 3 — Altering homotopy and homology groups

3.1 — Quillen's plus construction

For a connected space X such that $H_1(X) = 0$, there exists a so-called plus construction—famously used by Daniel Quillen in [Qui70; Qui73] to define higher algebraic K-theory—which eliminates its fundamental group without altering its homology. In the most general version, it eliminates a perfect subgroup N of $\pi_1(X)$, but as we will apply it to classifying spaces of mapping class groups $B\Gamma_{g,r}$ with $N = \pi_1(B\Gamma_{g,r})$, we will not need this generality here.

Definition 3.1.1. A group G is *perfect* if it equals its own commutator subgroup, $G = [G, G]$. Equivalently, its abelianisation is trivial.

Proposition-definition 3.1.2. For a connected cell complex X for which $H_1(X) = 0$, there exists a space X^+ together with a map $i : X \to X^+$, called the *plus construction*, such that the fundamental group $\pi_1(X^+)$ is trivial and the induced map on homology $i_* : H_*(X) \to H_*(X^+)$ is an isomorphism.

Proof. The space X^+ will be constructed by first attaching 2-cells to kill all generators of $\pi_1(X, *)$ and then attaching 3-cells to restore the homology to its original state.

So let $W = \bigvee S^1$ be a wedge of k circles, one for each of a chosen set of generators of $\pi_1(X)$, and let the map $f : W \to X$ send each circle to a representative of its corresponding generator. Define $j : X \to X^*$ to be the mapping cone of f. Obviously, $\pi_1(X^*) = 0$. We also have a long exact sequence

$$\cdots \longrightarrow H_n(W) \xrightarrow{f_*} H_n(X) \xrightarrow{j_*} H_n(X^*) \xrightarrow{\delta} H_{n-1}(W) \longrightarrow \cdots \longrightarrow H_2(X^*) \xrightarrow{\delta} H_1(W) \xrightarrow{f_*} H_1(X) \xrightarrow{j_*} H_1(X^*) \xrightarrow{\delta} \tilde{H}_0(W) \xrightarrow{f_*} \tilde{H}_0(X) \xrightarrow{j_*} \tilde{H}_0(X^*) \longrightarrow 0$$

As we have that $H_n(W) = 0$ for $n > 1$, we get that $j_* : H_n(X) \to H_n(X^*)$ is an isomorphism for $n > 2$ and an injection for $n = 2$. Therefore, the non-trivial part of the sequence is

$$0 \longrightarrow H_2(X) \xrightarrow{j_*} H_2(X^*) \xrightarrow{\delta} H_1(W) \xrightarrow{f_*} H_1(X) = 0 \xrightarrow{j_*} H_1(X^*) = 0 \longrightarrow 0$$

Hence, $H_2(X^*) = H_2(X) \oplus H_1(W) = H_2(X) \oplus \mathbb{Z}^k = \pi_2(X^*)$, using the Hurewicz theorem. The sequence splits because $\mathbb{Z}^k$ is a free, hence projective, abelian group.

Now, let $V = \bigvee S^2$ be a wedge of k 2-spheres, one for each new generator of $H_2(X^*)$, and define a map $g : V \to X^*$ by sending each sphere to a representative of its corresponding generator in X^*. Again let $h : X^* \to X^+$ be the mapping cone of g. As we are only adding 3-cells, the fundamental group does not change, so $\pi_1(X^+) = \pi_1(X^*) = 0$. Also, we get a similar long exact sequence as above, whose interesting part is (as $\tilde{H}_n(V) = \mathbb{Z}^k$ for $n = 2$ and 0 else)

$$0 \longrightarrow H_3(X^*) \xrightarrow{h_*} H_3(X^+) \xrightarrow{\delta} H_2(V) \xrightarrow{g_*} H_2(X^*) = H_2(X) \oplus \mathbb{Z}^k \xrightarrow{h_*} H_2(X^+) \longrightarrow 0$$

Now, g has been constructed such that $g_* : H_2(V) \longrightarrow H_2(X^*)$ is injective with image the direct summand $\mathbb{Z}^k$, so on the one hand $\delta_3 = 0$ and $H_3(X^+) \cong H_3(X^*) \cong H_3(X)$, and on the other hand $H_2(X^+) \cong H_2(X^*)/\operatorname{Im} g_* \cong (H_2(X) \oplus \mathbb{Z}^k)/\mathbb{Z}^k \cong H_2(X)$, both via the maps h_* and j_*. This means that $i = h \circ j : X \longrightarrow X^+$ is the required map. □

The plus construction has a universality property, given in proposition 3.1.5. To prove it, we will first need the following lemma, given in [Ste51]:

Definition 3.1.3. Let $L \subset K$ be finite CW-complexes and Y a path-connected space. For a map $f : L \longrightarrow Y$, we define a sequence of relative cohomology classes $c_n(f) \in H^{n+1}(K, L; \pi_n(Y))$ as follows: for any singular $(n+1)$-chain $\sigma : (D^{n+1}, \partial D^{n+1}) \longrightarrow (K, L)$ in $C_{n+1}(K, L)$, define $c_n(f)(\sigma) := f \circ \partial\sigma : S^n = \partial D^{n+1} \longrightarrow L \longrightarrow Y$. This defines a class in $\pi_n(Y)$. If $\sigma = 0 \in C_{n+1}(K, L)$, then the image of σ lies in L entirely and hence $c_n(f)(\sigma) = f \circ \partial\sigma$ can be contracted to a point through $f \circ \sigma : D^{n+1} \longrightarrow L \longrightarrow Y$. It gives a cocycle in $H^{n+1}(K, L; \pi_n(Y))$, as $dc_n(f)(\tau) = c_n(f)(\partial\tau) = f \circ \partial^2\tau = 0$ for any $(n+2)$-chain t.

Lemma 3.1.4. *Let K be a finite CW-complex and $L \subset K$ a subcomplex. Then a map $f : L \longrightarrow Y$ to a path-connected Y can be extended to a map $\tilde{f} : K \longrightarrow Y$ if and only if $c_n(f) = 0 \in H^{n+1}(K, L; \pi_n(Y))$ for all n.*

Proof. First assume $c(f) = 0$. The only possible problem lies in extending the $f|_{\partial\sigma}$ to σ, for the cells $\sigma \in K \setminus L$. For zero-cells, define g arbitrarily. As Y is path-connected, clearly $c(g|_{L\cup K_0}) = 0$.

Assume now g has been constructed up to the p-skeleton and $c(g|_{L\cup K_p}) = 0$, and let σ be a $(p+1)$-cell in $K \setminus L$. If there is a $(p+1)$-chain $\tilde{\sigma}$ with $\partial\tilde{\sigma} = \partial\sigma$ on which g is already defined, set $g|_\sigma = g|_{\tilde{\sigma}}$. Otherwise, use that σ defines an element of $H_{p+1}(K, L \cup K_p)$. Then $c(g|_{L\cup K_p})(\sigma) = 0$, so there is a contracting homotopy $h : S^p \times I \longrightarrow Y$, with $h_0 = f|_{\partial\sigma}$ and $h_1 = c$, a constant function. This defines an extension $g|_\sigma = \tilde{h} : D^{p+1} \cong S^p \times I/(S^p \times \{1\}) \longrightarrow Y$. By the first case, $c(g|_{L\cup K_p \cup\sigma}) = 0$.

Induction yields the first direction of the lemma. Conversely, given an extension $g : K \longrightarrow Y$, $f|_{\partial\sigma}$ is contractible for any $\sigma \in H_{n+1}(K, L)$, by the map $g|_\tau$. Therefore, $c(f) = 0$. □

Proposition 3.1.5. *Suppose X is a connected cell complex with $H_1(X) = 0$. Then the plus contruction is universal among pairs (Y, f), where $f : X \longrightarrow Y$ is a homotopy class of maps, Y is path-connected, and $f_* = 0 : \pi_1(X) \longrightarrow \pi_1(Y)$.*

Proof. The proof is due to Loday[Lod76].

Let $f : X \longrightarrow Y$ be a map as in the proposition. We want to extend it to $f^+ : X^+ \longrightarrow Y$, which can be constructed cell by cell. By lemma 3.1.4, the obstructions for the n-cells lie in $H^n(X^+, X; \pi_{n-1}(Y))$, which are zero, as $H^n(X^+, X; \mathbb{Z}) = 0$ by definition and the action of $\pi_1(X)$ on $\pi_{n-1}(Y)$ via $\pi_1(f)$ is trivial by assumption, making it a free $\mathbb{Z}$-module. Therefore, f^+ exists.

If f_1^+ and f_2^+ are two such extensions, the obstructions for a homotopy between them lie in $H^n(X^+, X; \pi_n(Y))$ (by an argument similar to that of lemma 3.1.4), which are again zero. □

As we are interested in the cohomology of the stable mapping class group Γ_∞ (see definition 2.5), which is defined via its classifying space (see definition A.3.1) we could consider $B\Gamma_\infty^+ := (B\Gamma_\infty)^+$ as well. The Madsen-Weiss theorem states that this space is homotopy equivalent to the identity component of a certain infinite loop space (see chapter 4), whose cohomology can, and will, be computed explicitly.

3.2 — The group completion theorem

By Harer stability, the stable cohomology $H_*(\Gamma_{\infty,2})$ of the mapping class group of two-holed surfaces is well-defined and finite-dimensional in each degree, as the map $\Psi_* \circ \Phi_* : H_*(\Gamma_{g,2}) \longrightarrow H_*(\Gamma_{g+1,2})$

stabilises for every degree. This operation is induced from glueing a two-holed torus to a boundary component of a representative surface.

This operation can be extended to a monoid operation on a certain space of embeddings of two-holed surfaces in Euclidean space. In this topological monoid, we would then like to identify the cohomology of the limit space under multiplication by a genus one object. For details, see chapter 5.

The tool to compute such a cohomology is the Group Completion Theorem, a well-known theorem which was first proven by Barratt and Priddy[BP72]. It is a generalisation of the standard fact that for a group G, there is a weak homotopy equivalence between G and ΩBG, the loop space of its classifying space, see corollary A.1.7. We give several versions of the Group Completion Theorem here.

Definition 3.2.1. Let M be an H-space. Then the multiplication map $\mu : M \times M \longrightarrow M$ induces a multiplication $\mu_* : H_*(M) \otimes H_*(M) \longrightarrow H_*(M \times M) \longrightarrow H_*(M)$, using the Künneth map. This μ_* is called the *Pontrjagin product*. If M is homotopy associative, the graded homology group of M becomes a graded ring with multiplication μ_*, called the *Pontrjagin ring*.

Theorem 3.2.2 (Abstract Group Completion Theorem)**.** *Let M be a topological monoid such that both M and the loop space of its classifying space, ΩBM, are* admissable, *i.e. right multiplication by any element is homotopic to left multiplication by that element. Then the natural inclusion $j : M \longrightarrow \Omega BM$ induces a localisation of Pontrjagin rings $j_* : H_*(M) \longrightarrow H_*(\Omega BM) \cong H_*(M)[\pi_0(M)^{-1}]$.*

Theorem 3.2.3 (Group Completion Theorem)**.** *Let M be a homotopy-commutative topological monoid with $\pi_0(M)$ generated by the classes of elements $m_1, \ldots, m_r$, set $m = m_1 \cdot \ldots \cdot m_r$, and let*

$$M^\infty := \qquad \operatorname{hocolim}\left(M \xrightarrow{-\cdot m} M \xrightarrow{-\cdot m} M \xrightarrow{-\cdot m} M \xrightarrow{-\cdot m} \cdots\right)$$

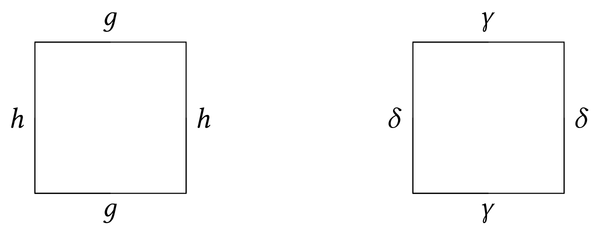
(The homotopy colimit in this case is just the mapping telescope.)

Then there is a map

$$M^\infty \longrightarrow \Omega BM$$

which induces an isomorphism in homology.

Proof (assuming theorem 3.2.2). Given elements $\gamma, \delta \in \Omega BM$, they are homotopic to one-cells of BM, which are given by elements $g, h \in B_1 M = M$, possibly with opposite orientation (say $\gamma \sim \pm g$, $\delta \sim \pm h$). By homotopy commutativity of M, gh and hg are homotopic in BM, so we have $g * h \sim gh \sim hg \sim h * g$. The boundaries of the squares



are homotopic in BM, and the left one can be filled, so this shows ΩBM is homotopy commutative. Hence, the conditions of theorem 3.2.2 are satisfied, and $H_*(M)[\pi_0(M)^{-1}] \cong H_*(\Omega BM)$.

Because M is homotopy-commutative, we get that $H_*(M^\infty) = H_*(M)[\pi_0(M)^{-1}]$: multiplication by any of the m_i is a homotopy equivalence on M^∞.

The map $j : M \longrightarrow \Omega BM$ must therefore factor through M^∞, and the isomorphism on homology is clear. □

Corollary 3.2.4 (Basic Group Completion Theorem)**.** *Let M be a homotopy-commutative topological monoid, with $\pi_0(M) \cong \mathbb{Z}_{\geq 0}$. Let $m \in M_1$, the component of M corresponding to $1 \in \mathbb{Z}_{\geq 0}$, and set*

$$M_\infty := \qquad \lim\left(M_0 \xrightarrow{-\cdot m} M_1 \xrightarrow{-\cdot m} M_2 \xrightarrow{-\cdot m} M_3 \xrightarrow{-\cdot m} \cdots\right)$$

Then there is a map

$$M_\infty \longrightarrow \Omega_0 BM$$

inducing an isomorphism on homology, where $\Omega_0 BM$ is the identity component of ΩBM.

Proof (assuming theorem 3.2.3). As M_∞ is a limit of connected spaces, it is connected. As clearly $M^\infty = M_\infty \times \mathbb{Z}$ in this situation, we get $(M^\infty)_0 = M_\infty$, so $H_*(M_\infty) = H_*((M^\infty)_0) \cong H_*(\Omega_0 BM)$. □

The approach to the proof of theorem 3.2.2 taken here is mainly derived from chapter 15 of [May75] and works with explicit homology and spectral sequences. It is based on [Qui71]. Other approaches to slightly different versions of the the theorem are also known, see e.g. [MS76] (via homology fibrations), [Jar89] (via bisimplicial sets), and [Hat14] (via quasifibrations).

First, we will give a couple of preliminary lemma's and definitions, before starting on the proof proper. The first of these states j at least works as expected on π_0.

Lemma 3.2.5. *In the situation of theorem 3.2.2, the induced map $\pi_0 j : \pi_0 M \longrightarrow \pi_0 \Omega BM$ is a group completion.*

Proof. As ΩBM is admissable, $\pi_0(\Omega BM)$ is commutative, so $\pi_0(\Omega BM) = \pi_1(BM) = H_1(BM)$. Using the chain complex associated to the nerve B_*M, $H_1(BM)$ is given by $\mathbb{Z} \cdot \pi_0(M)/\langle g - gh + h \mid g, h \in M\rangle$, which is the group completion of $\pi_0(M)$. The map $\pi_0 j$ clearly induces this. □

Next, we will give some useful manipulations with functors on categories that interest us.

Definition 3.2.6. A simplicial space X_* is *proper* if for any p, the inclusion of degenerate faces $\bigcup_{i=0}^{p} s_i(X_{p-1}) \hookrightarrow X_p$ is a cofibration.

A proper simplicial space X_* is *reduced* if X_0 is a single point and it is *special* if moreover the map $(\delta^0, \dots, \delta^p) : X_p \longrightarrow X_1^p$ is a homotopy equivalence, where $\delta^i = \partial_0 \cdots \partial_{i-1}\partial_{i+2} \cdots \partial_p$. (This sends a simplex to the path through all its vertices in order.) See figure 3.1.

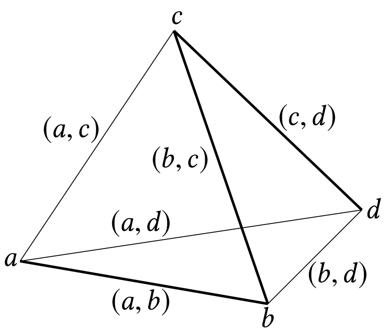


Figure 3.1: A 3-simplex (a, b, c, d) with the sides it is mapped to thickened. In this simplex labeling, ∂_i forgets the ith label.

Write $\mathcal{S}^+\mathrm{Sp}$ for the category of special proper simplicial compactly generated Hausdorff spaces and simplicial continuous maps.

Define the *pointed singular chains functor* $\tilde{C}_* : \mathrm{Sp}_* \longrightarrow \mathcal{S}^+\mathrm{Sp}$ (for this proof) on spaces by $\tilde{C}_p(X, *) = (X, *)^{(\Delta_p, (\Delta_p)_0)}$ and inducing faces and degeneracies from the Δ_p.

Remark 3.2.7. For any topological monoid M, the morphism $B_p MG \longrightarrow (B_1 M)^p$ for its classifying space is the tautological isomorphism $1 : M^p \xrightarrow{\sim} M^p$, so nerves of monoids are special.

Lemma 3.2.8. *The pointed singular chains functor is right adjoint to geometric realisation, $|\cdot| \dashv \tilde{C}_*$:*

$$\mathcal{S}^+\mathrm{Sp}(Y_*, \tilde{C}_* X) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathrm{Sp}_*(|Y_*|, X)$$

Proof. For $y \in Y_p$, $u \in \Delta_p$, $f : Y_* \to \tilde{C}_*X$ and $g : |Y_*| \to X$, define $\varphi(f)[y,u] = (fy)(u)$ and $(\psi g)(y)(u) = [y,u]$. □

We will write $\varepsilon_{Y_*} = \psi(1_{|Y_*|}) : Y_* \to \tilde{C}_*|Y_*|$ and $\eta_X = \varphi(1_{\tilde{C}_*X}) : |\tilde{C}_*X| \to X$ for the unit and the counit of the adjunction, respectively.

Lemma 3.2.9. *Define a natural transformation $\xi : BM \to 1_{\mathrm{Sp}_*}$, where M is the Moore loop space functor, by*

$$\xi_X\big([(\gamma_1,\dots,\gamma_p),(t_0,\dots,t_p)]\big) = (\gamma_1 * \cdots * \gamma_p)\Big(\sum_{i=1}^{p}\sum_{j=0}^{i-1} t_j \ell(\gamma_i)\Big)$$

Then the composite

$$\Omega X \xhookrightarrow{i_X} MX \xrightarrow{j_{MX}} \Omega BMX \xrightarrow{\Omega\xi_X} \Omega X$$

is the identity. Hence, if X is connected, ξ_X is a weak equivalence.

Proof. The image of $j_{MX} \circ i_X$ consists of one-cells of BMX of length one, by definition. On the one-skeleton, ξ_X simplifies to $\xi_X : [\gamma,(t,1-t)] \to \gamma(t)$, hence indeed $\Omega\xi_X \circ j_{MX} \circ i_X = 1_{\Omega X}$. □

Proposition 3.2.10. *For any $X \in \mathrm{Sp}_*$, the counit of the adjunction, $\eta_X : |\tilde{C}_*X| \to X$ is a weak homotopy equivalence.*

For any $Y_* \in \mathcal{S}^+\mathrm{Sp}$, the geometric realisation of the unit of the adjunction $|\varepsilon_Y| : |Y_*| \to \big|\tilde{C}_*|Y_*|\big|$ is a weak homotopy equivalence.

Proof. By standard category theory,

$$BMX = |B_*MX| \xrightarrow{|\psi(\xi_X)|} |\tilde{C}_*X| \xrightarrow{\eta_X} X$$

is equal to $\xi_X : BMX \to X$.

Clearly, $\psi_1(\xi_X) : MX \to \tilde{C}_1X \cong \Omega X$ is the standard retraction map. As both B_*MX and $\tilde{C}_*X$ are special, all $\psi_p(\xi_X)$ are homotopy equivalences, wherefore $|\psi_p(\xi_X)|$ is too. By lemma 3.2.9, η_X is a weak equivalence.

As $\eta_{|Y_*|} \circ |\varepsilon_{Y_*}| = 1_{|Y_*|}$, and $|Y_*|$ is connected for special Y_*, the second statement follows as well. □

Lemma 3.2.11. *Let E^r_{pq} and $\bar{E}^r_{pq}$ be spectral sequences, converging weakly to filtered graded groups H and $\bar{H}$, respectively. Let $f^r_{pq} : E^r_{pq} \to \bar{E}^r_{pq}$ be a morphism of spectral sequences compatible with a morphism of filtered graded groups $g : H \to \bar{H}$. Suppose*

1. *g is an isomorphism;*
2. *$E^2_{0q} = \bar{E}^2_{0q} = 0$ for $q > 0$;*
3. *There is an $n \geq 0$ such that f^2_{pq} is an isomorphism for all $q < n$ and all p.*

Then f^2_{1n} is an isomorphism and f^2_{2n} is an epimorphism.

Proof. We will only consider sheets two and higher in the spectral sequence.

Because g is an isomorphism and compatible with f, f^∞ is an isomorphism as well. For any objects E_{pq}, $\bar{E}_{pq}$ on the line $p+q = n+2$, $q < n$, all differentials into E_{pq} and $\bar{E}_{pq}$ and all differentials on sheet $\leq p-2$ out of E_{pq} and $\bar{E}_{pq}$ are isomorphisms, and all differentials on sheet $\geq p$ map to zero. As they are isomorphisms on sheet two, this shows f^r_{pq} is an isomorphism for all $r \geq 2$ (inducting from below to $r = p-1$ and from above to $r = p$). Hence, the differentials out of E_{pq} on sheet $p-1$ must be isomorphic as well. These are exactly all differentials into E_{1n}. As f^∞_{1n} is an isomorphism and all differentials out of E_{1n} map to zero, this means f^2_{1n} must be an isomorphism too.

For the line $p + q = n + 3$, again all differentials into E_{pq} and $\bar{E}_{pq}$ are isomorphic under f. Similarly, all differentials out of E_{pq} and $\bar{E}_{pq}$ on sheet $\neq p-2, p-1$ are isomorphic (using that all $f^r_{p,n+2-p}$ are isomorphisms). As all differentials 'into' are isomorphic, the E^r_{pq} and $\bar{E}^r_{pq}$ are subobjects of $E^2_{pq}/B^r_{pq} \xrightarrow{\sim}_f \bar{E}^2_{pq}/\bar{B}^r_{pq}$. As the isomorphism on sheet two restricts to a morphism $f^r_{pq} : E^r_{pq} \to \bar{E}^r_{pq}$, this must be a monomorphism, so $Z^{p-2}_{pq} \hookrightarrow_f \bar{Z}^{p-2}_{pq}$.

Using downward induction on p starting at $p = n + 3$, this proves there are monomorphisms $B^{p-2}_{2n} \hookleftarrow_{f^{-1}} B^{p-2}_{2n}$ up until $p = 4$. As all $Z^r_{2n} = E^\infty_{2n} \cong_f \bar{E}^\infty_{2n} = \bar{Z}^r_{2n}$, this shows $f^2_{2n} : Z^2_{2n}/B^2_{2n} \to \bar{Z}^2_{2n}/\bar{B}^2_{2n}$ is an epi. □

Proof of theorem 3.2.2. If we prove the theorem for coefficients in $\mathbb{Q}$ and in $\mathbb{F}_p$, it follows for coefficients in $\mathbb{Z}$ as well. So we will assume coefficients in a field k.

Suppose $Y_* \in \mathcal{S}^+\mathrm{Sp}$, writing Y for its geometric realisation $|Y_*|$, and consider its Segal spectral sequence with coefficients in k, see example B.2.17:

$$E^1_{pq}Y_* = H_q(Y_p; k) \Rightarrow H_{p+q}(Y; k)$$

Then we get, using the Künneth theorem and the fact that Y_* is special, that

$$\begin{aligned} E^1_{pq}Y_* &= H_q(Y_p; k) = H_q(Y_1^p; k) \\ &= \left(H_*(Y_1; k)^{\otimes p}\right)_q \end{aligned}$$

It is clear in this description that for $p = 0$, i.e. an empty product of copies of H_*, we get $E^1_{00}Y_* = k$ and $E^1_{0q}Y_* = 0$ for $q > 0$, and therefore the same holds for $r = 2$. Therefore, condition 2 of lemma 3.2.11 is satisfied.

Now consider the adjunction unit $\varepsilon_{B_*M} : B_*M \to \tilde{C}_*|B_*M| = \tilde{C}_*BM$. This induces a morphism of spectral sequences $\{E^r_{pq}\varepsilon_{B_*M}\} : \{E^r_{pq}B_*M \to E^r_{pq}\tilde{C}_*BM\}$. As $|\varepsilon_{B_*M}|$ is a weak homotopy equivalence by proposition 3.2.10, conditions 1 of lemma 3.2.11 is satisfied as well. Furthermore, on 1-simplices, the definitions show $\varepsilon_{B_1M} = j : B_1M = M \to \tilde{C}_1BM = \Omega BM$, which is the map we are interested in. Therefore, we will write $E^r_{pq}j = E^r_{pq}\varepsilon_{B_*M}$.

By lemma 3.2.5, we can factor $j_* : H_*(M) \to H_*(\Omega BM)$ as $j_* : H_*(M) \overset{i}{\hookrightarrow} H_*(M)[\pi_0(M)^{-1}] \overset{\bar{j}}{\to} H_*(\Omega BM)$. Write $\pi = \pi_0(M)$, $\bar{\pi}$ its group completion, $A := H_*(M)$, $\bar{A} := H_*(M)[\pi^{-1}]$, and $B := H_*(\Omega BM)$. The last three are all Hopf algebras, and $\bar{A} = \bar{A}_e \otimes k[\bar{\pi}]$ and $B = B_e \otimes k[\bar{\pi}]$ by lemma 3.2.5, where X_e is the connected component of the unit, as $\bar{\pi}$ is central in either by assumption. Then $\bar{j} = \bar{j}_e \otimes 1_{k[\bar{\pi}]}$.

In degree $q = 0$, $\bar{j}_e$ is then trivially an iso. Take as induction hypothesis that $\bar{j}_e$ is an isomorphism for degrees $< n$. Then so is $E^2_{pq}\bar{j}_e$ for $q < n$, so by lemma 3.2.11, $E^2_{1n}\bar{j}_e$ is an iso and $E^2_{2n}\bar{j}_e$ is an epi. In a diagram, we have (using $n > 0$)

$$\begin{array}{lccccccc} \text{degree} & : & 0 & & 1 & & 2 & \\[1em] E^1_{*n}\bar{A}_e & : & 0 & \longleftarrow & (\bar{A}_e)_n & \longleftarrow & \bigoplus_{q=0}^n (\bar{A}_e)_q \otimes_k (\bar{A}_e)_{n-q} & \longleftarrow \cdots \\ \downarrow{\scriptstyle \bar{j}_e} & & & & \downarrow & & \downarrow & \\ E^1_{*n}B_e & : & 0 & \longleftarrow & (B_e)_n & \longleftarrow & \bigoplus_{q=0}^n (B_e)_q \otimes_k (B_e)_{n-q} & \longleftarrow \cdots \end{array}$$

Which induces an isomorphism on H_1, an epimorphism on H_2, and is an isomorphism on all $(\bar{A}_e)_q$ for $0 < q < n$. By section 7 of [Wal60], this means $\bar{j}_e$ induces a bijection between minimal sets of generators in degree n and a surjection between minimal sets of relations, so it is an isomorphism for degrees $\leq n$. By induction, $\bar{j}_e$, and hence $\bar{j}$, is an isomorphism for all degrees. □

Chapter 4 — Infinite loop spaces

This chapter deals with three very intimately linked types of objects: Infinite loop spaces, spectra, and generalised cohomology theories. The first of these, infinite loop spaces, is a central theory in the study of homotopy-commutative H-spaces, in the sense that they are E_∞- or homotopy everything spaces, those H-spaces with homotopy inverse such that all higher associativity and commutativity conditions exist.

The theory of spectra originated in the theory of stable homotopy theory, and indeed spectra are a certain kind of stabilisations of spaces such that their ordinary homotopy groups correspond to stable homotopy groups for spaces.

Generalised cohomology theories are cohomology theories that do not neccessarily satisfy the dimension axiom. The individual cohomology functors are representable by Brown representability, and the connection maps induce maps between these representing spaces. It turns out this makes the collection of spaces for a given generalised cohomology theorem a spectrum.

My main references for this chapter will be by J. F. Adams[Ada78; Ada95].

4.1 — Spectra

The theory of infinite loop spaces and the related theory of spectra start out by considering the loop space functor $\Omega : \mathrm{Sp}_* \to \mathrm{Sp}_*$ and its left adjoint, the reduced suspension functor $\Sigma : \mathrm{Sp}_* \to \mathrm{Sp}_*$. The subject starts out by the realisation that, although Σ is neither full nor faithful in general, the Freudenthal suspension theorem implies that for $X, Y \in \mathrm{Sp}_*$, the sequence

$$[X, Y] \to [\Sigma X, \Sigma Y] \to [\Sigma^2 X, \Sigma^2 Y] \to [\Sigma^3 X, \Sigma^3 Y] \to \cdots$$

induces isomorphisms on homotopy groups in higher and higher degrees, and in particular stabilises in case X and Y are finite-dimensional. This theorem led to the definition and study of stable homotopy theory, which also ushered in the study of generalised cohomology theories. It is therefore a natural step to give the following definition.

Definition 4.1.1. Given finite cell complexes X, Y, define the group of *stable maps* between them as

$$\{X, Y\} := \mathrm{colim}_n [\Sigma^n X, \Sigma^n Y]$$

It is an abelian group, as any $[\Sigma^n X, \Sigma^n Y]$ for $n \geq 2$ is.

Taking a category of spaces with this definition of morphisms, the suspension functor becomes a fully faithful endofunctor of the category. However, it is not essentially surjective. One reason to consider spectra is to remedy this.

Definition 4.1.2. A *spectrum* is a sequence $(X_n)_{n \in \mathbb{Z}}$ of pointed CW complexes together with *suspension maps* $\varepsilon_n : \Sigma X_n \to X_{n+1}$ which are isomorphisms onto a subcomplex of X_{n+1}.

It is *connective* if $X_n = *$ for $n \ll 0$.

A spectrum is called a Σ-*spectrum* if the suspension maps $\varepsilon_n : \Sigma X_n \to X_{n+1}$ are weak equivalences for $n \gg 0$.

A spectrum is called an Ω-*spectrum* if the adjoints of all suspension maps, $\bar{\varepsilon}_n : X_n \to \Omega X_{n+1}$, are weak homotopy equivalences.

Remark 4.1.3. If a spectrum is written $(X, Y, Z, \dots)$ it is a connective spectrum with the first denoted space, X, in degree zero.

Of course, we would like to form a category of spectra, tentatively denoted Spec. However, it is not entirely clear what morphisms to take. A first idea can be formed from the following definition.

Definition 4.1.4. For a pointed CW complex X, define its *suspension spectrum*, $\Sigma^\infty X$, to be given by $\Sigma^\infty X := (X, \Sigma X, \Sigma^2 X, \Sigma^3 X, \dots)$ with all suspension maps the identity.

Example 4.1.5. The *sphere spectrum* $\mathbb{S}$ is defined as $\mathbb{S} := \Sigma^\infty S^0$. Its nth space is S^n.

Example 4.1.6. The *Eilenberg-Maclane spectrum* for an abelian group A is the Ω-spectrum $HA := (A, BA, B^2A, \dots)$, with suspension maps $\Sigma B^n A \to B^{n+1}A$ adjoint to the weak equivalences $B^n A \to \Omega B^{n+1} A$, cf. appendix A.2. We write $H = H\mathbb{Z}$.

We would like the assignment $X \mapsto \Sigma^\infty X$ to become a functor $\mathrm{CW}_* \to \mathrm{Spec}$. Therefore our first definition, of functions.

Definition 4.1.7. For two spectra X and Y a *function* $\varphi : X \to Y$ is a set of cellular functions $\varphi_n : X_n \to Y_n$ such that all diagrams

$$\begin{array}{ccc} \Sigma X_n & \xrightarrow{\varepsilon_{X,n}} & X_{n+1} \\ {\scriptstyle \varphi_n}\downarrow & & \downarrow{\scriptstyle \varphi_{n+1}} \\ \Sigma Y_n & \xrightarrow[\varepsilon_{Y,n}]{} & Y_{n1} \end{array}$$

commute. Composition of functions is defined degree-wise.

Definition 4.1.8. Given a spectrum X, its *suspension* $\Sigma X = X[-1]$ is defined by $(\Sigma X)_n = X_{n+1}$. Its inverse, *desuspension* $\Sigma^{-1} X = X[1]$ is defined by $(\Sigma^{-1}X)_n = X_{n-1}$. Both of these assignments have obvious extensions to functions.

However, in light of stability viewpoint at the start of this section, this does not seem to be enough. For example, we would like the Hopf map $\eta : S^3 \to S^2$ to define a map $\eta : \mathbb{S} \to \mathbb{S}[1]$. However, there are no maps $S^2 \to S^1$ or $S^1 \to S^0$ making this a function of spectra. To define a better-suited notion of morphisms, we need some more definitions.

Definition 4.1.9. A *subspectrum* A of a spectrum X is a sequence of sub-CW-complexes $A_n \subseteq X_n$ such that all $\varepsilon_n : \Sigma X_n \to X_{n+1}$ restrict to $\varepsilon_n|_{A_n} : \Sigma A_n \to A_{n+1}$. These $\varepsilon_n|_{A_n}$ are part of the subspectrum structure.

A subspectrum $A \subseteq X$ is *cofinal* if if for every compact subspace $K \subseteq X_n$ there exists an m such that the natural map $\Sigma^m K \to X_{n+m}$ factors through A_{n+m}.

Definition 4.1.10. A *map* between two spectra X and Y is an equivalence class of functions $\varphi : X' \to Y$, where $X' \subseteq X$ is cofinal and two such maps $\varphi : X' \to Y$ and $\psi : X'' \to Y$ are equivalent if there exists a cofinal $X''' \subseteq X' \cap X''$ such that $\varphi|_{X'''} = \psi|_{X'''}$. Spectra and maps clearly make up a category.

It is trivial to see that this is indeed an equivalence relation, and that maps can be composed by picking small enough representatives of the first cofinal set.

The maps just defined encapsulate a lot of the properties of stable maps for spaces. However, those were defined as stable *homotopy classes* of maps. Therefore, it would make sense to introduce homotopies on spectra as well.

Definition 4.1.11. Let I_+ be the interval with a disjoint basepoint (as a CW-complex it has three vertices and one edge). For a spectrum X, define its *cylinder* $\mathrm{Cyl}(X)$ to have spaces $\mathrm{Cyl}(X)_n = I_+ \wedge X_n$ and suspension maps $\mathrm{id} \wedge \varepsilon_n : \Sigma(I_+ \wedge X_n) = I_+ \wedge X_n \wedge S^1 \to I_+ \wedge X_{n+1}$. Then Cyl can be extended to a functor. There are two obvious inclusion maps $i_0, i_1 : X \to \mathrm{Cyl}(X)$.

Given two maps of spectra $f, g : X \to Y$, a *homotopy* between them is a map $H : \mathrm{Cyl}(X) \to Y$ such that $Hi_0 = f$ and $Hi_1 = g$. Being homotopic is clearly an equivalence relation on morphisms.

Definition 4.1.12. Define the *category of spectra*, Spec, to have spectra as objects and homotopy classes of maps as morphisms. It is well-defined as a quotient of the category of spectra and maps by the equivalence relation of homotopy between maps. Denote $[X, Y] := \mathrm{Spec}(X, Y)$.

Remark 4.1.13. For any spectrum $X = (X_n)_{n\in\mathbb{Z}}$, the connective spectrum $X' = (X_0, X_1, X_2, \dots)$ is clearly cofinal in X. This gives us the equivalence between spectra and the subcategory of connective spectra.

Definition 4.1.14. The suspension spectrum assignment of definition 4.1.4 defines a functor $\Sigma^\infty : \mathrm{CW}_* \to \mathrm{Spec}$.

Similarly, suspension and desuspension from definition 4.1.8 define quasi-inverse autofunctors $\Sigma, \Sigma^{-1} : \mathrm{Spec} \to \mathrm{Spec}$. In this context, they are also called *shift functors*.

We have defined a wealth of suspension functors and suspension maps. To make sense of all this, one would like them to all be compatible. The next lemma gives us just that.

Lemma 4.1.15. *The functors $\Sigma^\infty \circ \Sigma$ and $\Sigma \circ \Sigma^\infty : \mathrm{CW}_* \to \mathrm{Spec}$ are naturally isomorphic. Here, the first $\Sigma : \mathrm{CW}_* \to \mathrm{CW}_*$ is the usual reduced suspension, the second $\Sigma : \mathrm{Spec} \to \mathrm{Spec}$ is the shift functor defined in definition 4.1.14, and both $\Sigma^\infty : \mathrm{CW}_* \to \mathrm{Spec}$ are the suspension spectrum functor.*

Proof. On objects, $\Sigma^\infty \circ \Sigma(X) = (\Sigma X, \Sigma^2 X, \Sigma^3 X, \dots)$ and $\Sigma \circ \Sigma^\infty(X) = (X, \Sigma X, \Sigma^2 X, \dots)[1]$. The second spectrum is a cofinal subspectrum of the first, giving the isomorphism. On morphisms, the exact same thing occurs. ◻

4.1.1 — Homotopy groups of spectra

For any spectrum X, there are induced maps on homotopy groups

$$\pi_{n+r}(X_n) \to \pi_{n+r+1}(\Sigma X_n) \xrightarrow{(\varepsilon_n)_*} \pi_{n+r+1}(X_{n+1})$$

Hence, there is a natural definition of homotopy groups for spectra.

Definition 4.1.16. For a spectrum X define its *homotopy groups* to be $\pi_r(X) := \mathrm{colim}_n\, \pi_{n+r}(X_n)$.

Similarly, for a pair of spectra (X, A), its *relative homotopy groups* are defined to be $\pi_r(X, A) := \mathrm{colim}_n\, \pi_{n+r}(X_n, A_n)$.

Lemma 4.1.17. *For a CW-complex X, the homotopy groups of its suspension spectrum are given by its stable homotopy groups: $\pi_r(\Sigma^\infty X) = \mathrm{colim}_n\, \pi_{n+r}(\Sigma^n X) = \pi_r^s(X)$.*

Proof. Trivial. ◻

Lemma 4.1.18. *Homotopy groups of a spectrum are given by $\pi_r(X) = [\mathbb{S}, X[r]]$.*

Proof. There is a chain of isomorphisms

$$\begin{aligned}\pi_r(X) &:= \mathrm{colim}_n\, \pi_{r+n}(X_n) := \mathrm{colim}_n [S^{r+n}, X_n] = \mathrm{colim}_n [\Sigma^n S^r, X_n] \\ &\cong [\Sigma^\infty S^r, X] = [\Sigma^\infty(\Sigma^r S^0), X] \cong [\Sigma^r \Sigma^\infty S^0, X] = [\mathbb{S}, X[r]]\end{aligned}$$

◻

Example 4.1.19. For an Ω-spectrum X, $\pi_{n+r+1}(X_{n+1}) \cong \pi_{n+r+1}(\Omega X_n) \cong \pi_{n+r}(X_n)$ for $n + r \geq 1$. Hence, all homotopy groups can be computed on finite stages of the spectrum.

Proposition 4.1.20. *For a pair of spectra (X, A), there is a long exact sequence in homotopy*

$$\cdots \xrightarrow{\delta} \pi_r(A) \longrightarrow \pi_r(X) \longrightarrow \pi_r(X, A) \xrightarrow{\delta} \pi_{r-1}(A) \longrightarrow \cdots$$

Proof. It is clear that the maps $A \to X$ and $(X, *) \to (X, A)$ are maps of (pairs of) spectra, so the induced maps on homotopy groups commute with the colimit, yielding maps $\pi_r(A) \to \pi_r(X)$ and $\pi_r(X) \to \pi_r(X, A)$.

For the map δ, recall that we defined the reduced suspension by $\Sigma X = X \wedge S^1$. Also recall the way δ defined: let I^r be the unit r-cube, and set $J^r = \{0\} \times I^r \cup I \times \partial I^r \subset I^{r+1}$. An element of $\pi_r(X_n, A_n)$ is represented by a map $\alpha : (I^r, \partial I^r, J^{r-1}) \to (X_n, A_n, *)$ and $\delta\alpha \in \pi_{r-1}(A_n)$ is represented by the restriction of α to the face $\{1\} \times I^r$.

It follows quite easily that $\delta\Sigma\alpha = \delta(\alpha \wedge \mathrm{id}_{S^1}) = (\delta\alpha) \wedge \mathrm{id}_{S^1}$, so that δ commutes with the maps in the colimits, thereby defining the δ's in the long exact sequence for relative homotopy groups of spectra. □

4.2 — Generalised cohomology theories

Recall that a reduced (generalised) cohomology theory can be defined as follows:

Definition 4.2.1. [May99] A *reduced cohomology theory* on CW-complexes is given by a set of functors $\tilde{E}^n : \mathrm{hoCW}_*^{\mathrm{op}} \to \mathrm{Ab}$, for $n \in \mathbb{Z}$, such that the following hold:

Exactness For $A \subset X$, $\tilde{E}^n(X/A) \to \tilde{E}^n(X) \to \tilde{E}^n(A)$ is exact.

Suspension There are natural isomorphisms $\tilde{E}^{n+1}(\Sigma X) \cong \tilde{E}^n(X)$.

Additivity If $X = \bigvee_{i \in I} X_i$, the inclusions induce isomorphisms $\tilde{E}^n(X) \xrightarrow{\sim} \prod_{i \in I} \tilde{E}^n(X_i)$.

A morphism of reduced cohomology theories $F : \tilde{E}^* \to \tilde{H}^*$ is a collection of natural transformations $F^n : \tilde{E}^* \to \tilde{H}^*$ such that all squares

$$\begin{array}{ccc} \tilde{E}^n(X) & \xrightarrow{\sim} & \tilde{E}^{n+1}(\Sigma X) \\ \downarrow F^n_X & & \downarrow F^{n+1}_{\Sigma X} \\ \tilde{H}^n(X) & \xrightarrow{\sim} & \tilde{H}^{n+1}(\Sigma X) \end{array}$$

commute. This defines a category GenCoh.

Dually, we have

Definition 4.2.2. [May99] A *reduced homology theory* on CW-complexes is given by a set of functors $\tilde{E}_n : \mathrm{hoCW}_* \to \mathrm{Ab}$, for $n \in \mathbb{Z}$, such that the following hold:

Exactness For $A \subset X$, $\tilde{E}_n(A) \to \tilde{E}_n(X) \to \tilde{E}_n(X/A)$ is exact.

Suspension There are natural isomorphisms $\tilde{E}_{n+1}(\Sigma X) \cong \tilde{E}_n(X)$.

Additivity If $X = \bigvee_{i \in I} X_i$, the inclusions induce isomorphisms $\bigoplus_{i \in I} \tilde{E}_n(X_i) \xrightarrow{\sim} \tilde{E}_n(X)$.

A morphism of reduced homology theories is defined similarly to one of reduced cohomology theories. This defines a category GenH.

Remark 4.2.3. Cohomology theories and reduced cohomology theories are equivalent: for a cohomology theory E^*, define $\tilde{E}^*(X) := E^*(X, *)$. Conversely, for a reduced cohomology theory $\tilde{E}^*$, define $E^*(X, A) := \tilde{E}^*(X/A)$ (this includes $E^*(X) = \tilde{E}^*(X_+)$). Therefore, we will often neglect to write the tildes or the word 'reduced'.

Theorem 4.2.4. *Any cohomology theory E^* is representable by an Ω-spectrum. This gives a (pseudo-)functor* GenCoh $\longrightarrow$ Spec.

This theorem is a consequence of Brown representability[Bro62], already proven in the same paper.

Proof. Given such a cohomology theory E^*, Brown representability[Bro62] implies that there are connected CW-complexes E_n such that $\tilde{E}^n \cong [-, E_n]$ as functors from the homotopy category of connected based CW-complexes. The suspension axiom now show, for all connected X

$$[X, E_n] \cong \tilde{E}^n(X) \cong \tilde{E}^{n+1}(\Sigma X) \cong [\Sigma X, E_{n+1}] \cong [X, \Omega_0 E_{n+1}]$$

By Yoneda, there are weak equivalences $E_n \longrightarrow \Omega_0 E_{n+1}$, showing (E_n) is an Ω_0-spectrum (defined in the obvious way).

Taking now a not necessarily connected X and defining $F_n := \Omega E_{n+1}$,

$$\tilde{E}^n(X) \cong \tilde{E}^{n+1}(\Sigma X) \cong [\Sigma X, E_{n+1}] \cong [X, \Omega E_{n+1}] = [X, F_n]$$

Hence, F_n represents $\tilde{E}^n$. To show this also forms a spectrum we once again apply the same trick:

$$[X, F_n] \cong \tilde{E}^n(X) \cong \tilde{E}^{n+1}(\Sigma X) \cong [\Sigma X, F_{n+1}] \cong [X, \Omega F_{n+1}]$$

Again by Yoneda, we get a weak equivalence $F_n \longrightarrow \Omega F_{n+1}$, so (F_n) is an Ω-spectrum.

Given two cohomology theories, $\tilde{E}^n$ and $\tilde{H}^n$, represented by spectra E and H, respectively, and a morphism $F : \tilde{E}^n \longrightarrow \tilde{H}^n$, Yoneda gives maps $f_n : E_n \longrightarrow H_n$ and the suspension compatibility of F ensures these f_n are compatible with the loop maps of the spectra, hence piecing together to form a map of spectra $f : E \longrightarrow H$. □

It seems intuitively quite clear that on can go in the other direction as well, and this is what the next theorem shows.

Theorem 4.2.5. *Given a spectrum E, define the functors $\tilde{E}^n$:* hoCW $\longrightarrow$ Ab *by*

$$\tilde{E}^n(X) = [\Sigma^\infty X, E[-n]]$$

This defines a generalised cohomology theory, giving a functor quasi-inverse to that of theorem 4.2.4 and hence yields an equivalence of categories between Spec *and* GenCoh.

Proof. Any representable functor preserves limits, proving exactness and additivity. For the suspension axiom, one can calculate:

$$\tilde{E}^{n+1}(\Sigma X) = \big[\Sigma^\infty(\Sigma X), E[-n-1]\big] \cong \big[(\Sigma^\infty X)[-1], E[-n-1]\big] \cong \big[\Sigma^\infty X, E[-n]\big] = \tilde{E}^n(X)$$

Given a morphism of spectra $f : E \longrightarrow H$, there is a commutative diagram

$$\begin{array}{ccccc}
\big[\Sigma^\infty X, E[-n]\big] & \xrightarrow[{[-1]}]{\sim} & \big[(\Sigma^\infty X)[-1], E[-n-1]\big] & \xrightarrow{\sim} & \big[\Sigma^\infty(\Sigma X), E[-n-1]\big] \\
\downarrow {\scriptstyle f_*[-n]} & & \downarrow {\scriptstyle f_*[-n-1]} & & \downarrow {\scriptstyle f_*[-n-1]} \\
\big[\Sigma^\infty X, H[-n]\big] & \xrightarrow[{[-1]}]{\sim} & \big[(\Sigma^\infty X)[-1], H[-n-1]\big] & \xrightarrow{\sim} & \big[\Sigma^\infty(\Sigma X), H[-n-1]\big]
\end{array}$$

So $\{f_*[-n] : [\Sigma^\infty(-), E[-n]] \to [\Sigma^\infty(-), H[-n]]\}$ is a morphism of cohomology theories, making this into a functor.

Let E^* be a cohomology theory, with E its associated spectrum, constructed in theorem 4.2.4. Then $[X, E_n] \cong [\Sigma^r X, E_{n+r}]$ for any $r \geq 0$, so $[X, E_n] \cong [\Sigma^\infty X, E[-n]]$, showing GenCoh $\to$ Spec $\to$ GenCoh is naturally isomorphic to the identity.

On the other hand, representations of functors are unique up to unique isomorphism, so going from spectrum to cohomology theory and back also yields an isomorphic spectrum, proving the equivalence of the two categories. □

Remark 4.2.6. Because of this theorem, spectra and their cohomology theories will often be denoted by the same symbol.

Having a correspondence between spectra and cohomology theories, we would like to identify the associated homology theories as well. They are, as one might expect, constructed in a dual manner.

Theorem 4.2.7. *Let E be a spectrum. Then the collection of functors $\tilde{E}_n : \mathsf{hoCW}_* \to \mathsf{Ab}$ given by*

$$\tilde{E}_n(X) = \pi_n(E \wedge X) = [\mathbb{S}, E \wedge X[n]]$$

where the spectrum $E \wedge X$ is defined by $(E \wedge X)_n = E_n \wedge X$, defines a homology theory.

In the situation of the theorem, the spectrum E is said to corepresent the homology theory $\tilde{E}_*$.

Proof. **Exactness** The map $\pi_{n+r}(E_r \wedge X) \to \pi_{n+r}(E_r \wedge (X/A)) = \pi_{n+r}((E_r \wedge X)/(E_r \wedge A))$ factors through $\pi_{n+r}(E_r \wedge X, E_r \wedge A)$ for all r, in a way compatible with colimits. Using the long exact sequence of relative homotopy, we do indeed get exactness of

$$\pi_{n+r}(E_r \wedge A) \to \pi_{n+r}(E_r \wedge X) \to \pi_{n+r}(E_r \wedge (X/A))$$

which, after taking colimits, shows exactness of $\tilde{E}_*$.

Suspension We have

$$\begin{aligned}\tilde{E}_{n+1}(\Sigma X) &= \left[\mathbb{S}, E \wedge \Sigma X[n+1]\right] \cong \left[\mathbb{S}[-1], E \wedge X \wedge S^1[n]\right] \\ &\cong \left[\mathbb{S} \wedge S^1, E \wedge X \wedge S^1[n]\right] \cong \left[\mathbb{S}, E \wedge X[n]\right] = \tilde{E}_n(X)\end{aligned}$$

Additivity The required statement of additivity is that $\bigoplus_{i\in I} \pi_n(E \wedge X_i) \cong \pi_n(E \wedge X)$ if $X = \bigvee_{i \in I} X_i$. It is clear that $E \wedge X \cong \bigvee_{i\in I} E \wedge X_i$, and we will consider the last space.

Letting $\iota_j : E \wedge X_j \hookrightarrow \bigvee_{i\in I} E \wedge X_j$ be the inclusion, there is a map in one direction given by

$$\sum_{j\in I} (\iota_j)_* : \bigoplus_{j\in I} \pi_n(E \wedge X_j) \to \pi_n\Big(\bigvee_{i\in I} E \wedge X_i\Big)$$

Conversely, let $\alpha \in \pi_n(E \wedge X)$ be represented by a map $a : S^{n+r} \to \bigvee_{i\in I} E_r \wedge X_i$. Then a factors through $S^{n+r}/a^{-1}(*)$, which is homotopy equivalent to a wedge of $(n+r)$-spheres, $\bigvee_j S_j^{n+r}$. The wedge is finite by compactness, and the restriction of a to any of the spheres in the wedge maps into one of the $E_r \wedge X_i$. Define a map $\alpha \mapsto \sum_j [a|_{S_j^{n+r}}] \in \bigoplus_{i\in I} \pi_{n+r}(E_r \wedge X_i)$. This map is a well-defined group homomorphism, and gives an inverse to $\sum_{j\in I}(\iota_j)_*$.

□

The duality between the homology and cohomology theory associated to a spectrum can best be seen in the following form:

$$\begin{aligned}\tilde{E}_n(X) &= \pi_n(E \wedge X) & \qquad \tilde{E}^n(X) &= [\Sigma^\infty X, E[-n]] \\ &= [\mathbb{S}, E \wedge X[n]] & &= [\mathbb{S} \wedge X[n], E]\end{aligned}$$

Example 4.2.8 (Ordinary cohomology)**.** The Eilenberg-MacLane spectrum HA from example 4.1.6 represents ordinary cohomology with coefficients in A, according to theorem A.2.4. Similarly,

$$\pi_n(HA \wedge S^0) = \pi_n(HA) = \operatorname{colim}_r \pi_{n+r}(K(A,r)) = \begin{cases} A & n = 0 \\ 0 & n \neq 0 \end{cases}$$

so HA corepresents ordinary homology as well. This explains the use of the letter H for this spectrum.

Example 4.2.9 (K-theory)**.** Another well-known cohomology theory is complex K-theory. Recall that $K^0(X)$ is the Grothendieck group associated to the monoid of complex vector spaces over X. Equivalently, $\tilde{K}^0(X)$ is the group of stable vector bundles, i.e. vector bundles modulo the relation $E_1 \sim E_2$ if $E_1 \oplus (\mathbb{R}^k \times X) \cong E_2 \oplus (\mathbb{R}^l \times X)$ for some k, l.

By proposition A.1.8, complex vector bundles of dimension n over X are in one-to-one correspondence with $[X, B\mathrm{U}(n)]$, where $\mathrm{U}(n)$ is the unitary group. As we are interested in stable vector bundles, define $\mathrm{U} := \operatorname{colim}_n \mathrm{U}(n)$, where the stabilising maps are $\mathrm{U}(n) \to \mathrm{U}(n+1) : A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We then get $B\mathrm{U} = \operatorname{hocolim}_n B\mathrm{U}(n)$ as well.

The stabilising maps $\mathrm{U}(n) \to \mathrm{U}(n+1)$ are obviously compatible with the notion of stable vector bundles, so

$$\tilde{K}^0(X) \cong \operatorname{colim}_n [X, B\mathrm{U}(n)] \cong [X, \operatorname{hocolim}_n B\mathrm{U}(n)] = [X, B\mathrm{U}]$$

Since $K^0(*) = \tilde{K}^0(S^0) \cong \mathbb{Z}$, given by the difference of dimension over the two points, this shows the representing Ω-spectrum KU of complex K-theory (representing K^*, not $\tilde{K}^*$) starts with $\mathbb{Z} \times B\mathrm{U}$. Therefore, by corollary A.1.7, $KU_{-1} \cong \Omega B\mathrm{U} \cong \mathrm{U}$. By Bott periodicity, $KU_{2n} = \mathbb{Z} \times B\mathrm{U}$ and $KU_{2n+1} = \mathrm{U}$ for all n.

So complex K-theory is represented by the spectrum $KU = (\mathbb{Z} \times B\mathrm{U}, \mathrm{U}, \mathbb{Z} \times B\mathrm{U}, \dots)$.

Theorem 4.2.5 also suggests the following definition of cohomology for spectra:

Definition 4.2.10. Let X be a spectrum and E^* a cohomology theory, represented by a spectrum E. Then the E-cohomology of X is defined as

$$\tilde{E}^n(X) := [X, E[-n]]$$

This way, the $\tilde{E}^n$ define a generalised reduced cohomology theory on Spec.

Remark 4.2.11. One could also extend theorem 4.2.7 to define homology theories on spectra. However, the definition of the smash product on spectra is very involved, so we will not give it here.

For a cohomology theory E^*, its coefficient groups are $E^n(*) = \tilde{E}^n(S^0)$ and similar for homology. If E^* is represented by a spectrum E, lemma 4.1.18 gives

$$E^n(*) = \tilde{E}^n(S^0) = [\mathbb{S} \wedge S^0[n], E] = [\mathbb{S}, E[-n]] = \pi_{-n}(E)$$
$$E_n(*) = \tilde{E}_n(S^0) = [\mathbb{S}, E \wedge S^0[n]] = [\mathbb{S}, E[n]] = \pi_n(E)$$

So, the homotopy groups of a spectrum are the coefficient groups of its cohomology theory.

4.2.1 — Thom spectra and bordisms

Thom spectra are a certain type of spectra that turn out to represent various (co)bordism theories. To define them, one needs the classifying space of the orthogonal group, $\mathrm{O}(n)$ and several related groups. We could just define them as in appendix A.1, but in this case there exists a much more concrete and practical model.

Let $G(n,k)$ be the Grassmannian of n-planes in $\mathbb{R}^k$. There are inclusions $G(n,m) \hookrightarrow G(n,k+1)$ induced by the inclusions $\mathbb{R}^k \cong \mathbb{R}^k \times \{0\} \hookrightarrow \mathbb{R}^{k+1}$. Define $G(n,\infty)$ to be the union of all $G(n,k)$.

For complex vector spaces, the corresponding Grassmannians will be denoted $G_{\mathbb{C}}(n,k)$, for $n \leq k \leq \infty$.

For oriented subspaces, the Grassmannians will be denoted $G^+(n,k)$ and $G^+_{\mathbb{C}}(n,k)$.

Proposition 4.2.12. *The infinite Grassmannian $G(n,\infty)$ is a classifying space for* $\mathrm{O}(n)$.

Proof. It is sufficient to prove that $G(n,\infty)$ is a quotient of a contractible space by a free $\mathrm{O}(n)$-action.

For every $k \geq n$, let $V(n,k)$ be the Stiefel manifold of n orthonormal vectors in $\mathbb{R}^k$. This has a free action of $\mathrm{O}(n)$ by acting on all vectors, with quotient $G(n,k)$. Therefore, $\mathrm{O}(n)$ acts freely on $V(n,\infty)$ with quotient $G(n,\infty)$.

The projection $V(n,k) \longrightarrow S^{k-1} : (e_1,\ldots,e_n) \mapsto e_n$ is a fibration with fiber $V(n-1,k-1)$, so a repeated application of the Serre exact sequence yields $\pi_q(V(n,k)) \cong \pi_q(V(n-1,k-1)) \cong \cdots \cong \pi_q(V(1,k-n+1)) = \pi_q(S^{k-n})$ for $q < k-n+1$, and this goup is trivial for $q < k-n$. Therefore, $\pi_q(V(n,\infty)) = \mathrm{colim}_k\, \pi_q(V(n,k)) = 0$ for all q. By the Whitehead theorem, $V(n,\infty)$ is contractible. □

Corollary 4.2.13. *The Grassmannian $G(n,\infty)$ is also a classifying space for* $\mathrm{GL}(n,\mathbb{R})$.

Proof. Any classifying space $B\mathrm{GL}(n,\mathbb{R})$ classifies principal $\mathrm{GL}(n,\mathbb{R})$-bundles, by proposition A.1.8. These correspond to n-dimensional vector bundles. But as any vector bundle can be given a Riemannian metric, unique up to isomorphism of Riemannian vector bundles, these correspond to principal $\mathrm{O}(n)$-bundles as well, showing that $B\mathrm{GL}(n,\mathbb{R}) \cong B\mathrm{O}(n) \cong G(n,\infty)$. □

Corollary 4.2.14. *The infinite complex Grassmannian $G_{\mathbb{C}}(n,\infty)$, is a classifying space for both* $\mathrm{U}(n)$ *and* $\mathrm{GL}(n,\mathbb{C})$.

Proof. The proof is very similar to that of proposition 4.2.12. The main difference lies in the point that $V_{\mathbb{C}}(n,k)$ fibers over S^{2k-1} in stead of S^{k-1}. □

Corollary 4.2.15. *The infinite oriented Grassmannian $G^+(n,\infty)$ is a classifying space for* $\mathrm{SO}(n)$ *and* $\mathrm{SL}(n,\mathbb{R})$.

The infinite oriented complex Grassmannian $G^+_{\mathbb{C}}(n,\infty)$ is a classifying space for $\mathrm{SU}(n)$ *and* $\mathrm{SL}(n,\mathbb{C})$.

Proof. The proof is again similar to that of proposition 4.2.12, but this time, the Stiefel manifold must be quotiented out by $\mathrm{SO}(n) \subset \mathrm{O}(n)$. This also proves $G^+(n,k) \longrightarrow G(n,k)$ is a two-sheeted covering. □

Definition 4.2.16. For any $0 \leq n \leq k \leq \infty$ with $n < \infty$, the Grassmannian $G(n,k)$ (real or complex) has a *tautological bundle* of dimension n,denoted $\gamma_{n,k}$, whose fibre over a point $p \in G(n,k)$ is the n-space it represents in $\mathbb{R}^k$ or $\mathbb{C}^k$.

If $k = \infty$, it is also called the *universal bundle*, as it classifies n-dimensional vector bundles, via proposition 4.2.12 and its corollaries.

Definition 4.2.17. For a real vector bundle over a paracompact base $\pi : E \longrightarrow B$, its *Thom space* is the space obtained by first taking the fibrewise one-point compactification to get a sphere bundle $S(E) \longrightarrow B$ and then defining $\mathrm{Th}(\pi) = S(E)/B$, where B is included into $S(E)$ via the added points on the fibres.

Equivalently, taking a fibrewise norm on E, $\mathrm{Th}(\pi) = E^{\leq 1}/E^1$, where $E^{\leq 1} = \{e \in E \mid \|e\| \leq 1\}$ and similarly for E^1.

Remark 4.2.18. If the base space B is compact, this is just the one-point compactification of the total space E.

Lemma 4.2.19. *Let $\pi : E \longrightarrow B$ be a real vector bundle and $\varepsilon : \mathbb{R} \times B \longrightarrow B$ the trivial line bundle on B. Then* $\mathrm{Th}(\pi \times \varepsilon) \cong \Sigma\, \mathrm{Th}(\pi)$.

Proof. Define a norm on $E \times \mathbb{R}$ by $\|(e, r)\| = \max\{\|e\|, |r|\}$. Then

$$\begin{aligned}\mathrm{Th}(\pi \times \varepsilon) &\cong (E \times \mathbb{R})^{\leq 1}/(E \times \mathbb{R})^1 \\ &= (E^{\leq 1} \times [-1, 1])/(E^1 \times [-1, 1] \cup E^{\leq 1} \times \{-1, 1\} \\ &\cong \mathrm{Th}(\pi) \times [-1, 1]/(* \times [-1, 1] \cup \mathrm{Th}(\pi) \times \{-1, 1\}) \\ &= \Sigma\,\mathrm{Th}(\pi)\end{aligned}$$

□

We will now use this construction on the infinite Grassmannians defined before to define Thom spectra.

For any finite $n \leq k$, there is an inclusion $G(n, k) \hookrightarrow G(n+1, k+1)$ induced by the inclusion $\mathbb{R}^k \cong \{0\} \times \mathbb{R}^k \hookrightarrow \mathbb{R}^{k+1}$ sending a subspace V to $\mathbb{R} \times V$. These inclusions commute with the inclusions $\mathbb{R}^k \cong \mathbb{R}^k \times \{0\} \hookrightarrow \mathbb{R}^{k+1}$ used to define the infinite Grassmannians, yielding inclusions on the union over k, $G(n, \infty) \hookrightarrow G(n+1, \infty)$.

Under these inclusions, we get restrictions of bundles $\gamma_{n+1,k+1}|_{G(n,k)} \cong \varepsilon \times \gamma_{n,k}$, by definition of the map $V \mapsto \mathbb{R} \times V$, and hence inclusions $\Sigma\,\mathrm{Th}(\gamma_{n,k}) \cong \mathrm{Th}(\varepsilon \times \gamma_{n,k}) \hookrightarrow \mathrm{Th}(\gamma_{n+1,k+1})$. These inclusions also propagate to the colimit, giving inclusions $\Sigma\,\mathrm{Th}(\gamma_{n,\infty}) \hookrightarrow \mathrm{Th}(\gamma_{n+1,\infty})$.

Definition 4.2.20. The *Thom spectrum* of $\mathrm{O}(n)$ is the spectrum $M\mathrm{O}$ having spaces $M\mathrm{O}_n := \mathrm{Th}(\gamma_{n,\infty})$ and inclusions as given above.

For SO, there are compatible inclusions $\mathrm{SO} \hookrightarrow \mathrm{O}(n)$ inducing maps $f_n : B\mathrm{SO}(n) \to B\mathrm{O}(n)$, and the $\gamma_{n,\infty}$ pull back to $B\mathrm{SO}(n)$, still giving inclusions $\Sigma\,\mathrm{Th}(f_n^*\gamma_{n,\infty}) \hookrightarrow \mathrm{Th}(f_{n+1}^*\gamma_{n+1,\infty})$. This defines the Thom spectrum $M\mathrm{SO}$.

For $G = \mathrm{U}, \mathrm{SU}$, things are slightly more subtle, as $G(n) \hookrightarrow \mathrm{O}(2n)$. Therefore, in these cases $MG_{2n} = \mathrm{Th}(f^*\gamma_{2n,\infty})$ and $MG_{2n+1} = \Sigma\,\mathrm{Th}(f^*\gamma_{2n,\infty})$.

In order to define Thom spectra and bordism in full generality, the following definition is in order.

Definition 4.2.21. A *tangential structure* is a set of spaces X_n together fibrations $g_n : X_n \to B\mathrm{O}(n)$ and maps $f_n : X_n \to X_{n+1}$.

Given such a tangential structure X, the universal bundles $\gamma_{n,\infty}$ can be pulled back along g_n. Its *Thom space* MX is then consists of spaces $MX_n = \mathrm{Th}(g_n^*(\gamma_{n,\infty}))$ and the suspension maps are induced by those of $M\mathrm{O}$.

An *X-structure* on an n-dimensional manifold M consists of a pair $(h, \tilde{\nu})$, where $h : M \to \mathbb{R}^{n+m}$ is an embedding, giving a normal bundle $N \to M$ with classifying map $\nu : M \to B\mathrm{O}(m)$ and $\tilde{\nu} : M \to X_m$ is a lift of ν along g.

Two X-structures $(h, \tilde{\nu})$ and $(k, \tilde{\mu})$ are equivalent if for some large m there exists a translation $T : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ such that $k_m = T \circ h_m$ and $\tilde{\nu} \sim \tilde{\mu} : M \to X_m$ are homotopic through liftings of $\nu = \mu : M \to B\mathrm{O}(m)$. An *$X$-manifold* is a manifold with an equivalence class of X-structures.

Definition 4.2.22. Fix a tangential structure X. For a space Y, consider the set of continous maps $f : M \to Y$, where M ranges over all closed n-dimensional X-manifolds. Define two of these maps $f : M \to X$ and $g : N \to X$ to be *X-bordant* if there exists an $(n+1)$-dimensional X-manifold W such that $\partial W = M \sqcup -N$ (as X-manifolds) and a map $F : W \to Y$ such that $F|_M = f$ and $F|_N = g$. The set of equivalence classes of this relation is called the *n-th X-bordism group of* Y, denoted $\Omega_n^X(Y)$. This is an abelian group with addition the disjoint union of manifolds and maps, unit the empty manifold, and inverse of M being $-M$, i.e. opposite orientation: then $M \times I$ gives a bordism from $M \sqcup -M$ to $\emptyset$.

Remark 4.2.23. In the case the tangential structure consists of classifying spaces $BG(n)$ itself, we usually write MG and Ω_*^G for MBG and Ω_*^{BG}

Theorem 4.2.24. *The Thom spectrum MX corepresents the bordism theory Ω_*^X.*

Remark 4.2.25. This theorem was proven by René Thom[Tho54] in the case Y is a point, linking the bordism groups with homotopy groups of the Thom spectra. The general proof is hardly different. The Thom construction and Thom spectra are named in his honour.

Corollary 4.2.26. *Bordism is a generalised homology theory.*

Remark 4.2.27. Important examples of tangential structures are $X = BO$ itself, and $X = BSO$, $X = BU$, BSU, and $X_m = \{*\}$. Their bordism theories are called (unoriented) bordism, oriented bordism, complex bordism, special unitary bordism, and framed bordism, respectively.

Proof of theorem 4.2.24. We will first define a map $F : \Omega_n^X(Y) \to MX_n(Y) = \pi_n(X \wedge Y)$. So consider an n-dimensional X-manifold $(M, h, \tilde{\nu})$ and a map $f : M \to Y$. Then h is represented by an embedding $h : M \hookrightarrow \mathbb{R}^{n+m} \hookrightarrow S^{n+m}$. Let $T(M) \subset S^{n+m}$ be a tubular neighbourhood of $h(M)$. It is isomorphic to the normal bundle of $h(M)$, so $\tilde{\nu}$ defines a map $h(M) \to X_m$, which by definition of MX_m can be extended to a map $(\overline{T(M)}, \partial T(M)) \to (MX_m, *)$.

There is also a map $f \circ \pi : T(M) \to M \to Y$, where $\pi : T(M) \to M$ is the projection. So we get a map $\varphi : T(M) \to MX_m \wedge Y$.

Sending the complement of $T(M) \subset S^{n+m}$ to the basepoint in $MX_m \wedge Y$ then gives a map $\bar{\varphi} : S^{n+m} \to MX_m$, which represents an element $[\varphi] \in [\mathbb{S}, MX \wedge Y[n]] = \pi_n(MX \wedge Y)$.

The class $[\varphi]$ is independent of the choice of $\tilde{\nu}$ in its equivalence class, as $\tilde{\nu}$ varies by homotopy, and $[\varphi]$ is a limit of homotopy classes. It does not depend on the choice of h, as it is a colimit, and equivalent h agree up to homotopy after a finite amount of steps in the colimit diagram.

Finally, bordant X-manifolds M, N define the same class in $\pi_n(MX \wedge Y)$, because the embedding of the bordism W into some higher $R^{n+m'} \subset S^{n+m'-1} \times \mathbb{R}$ yields a homotopy between the maps $\varphi, \psi : S^{n+m'-1} \to MX_{m'} \wedge Y$ defined by M and N. Hence, the assignment $F : \Omega_n^X(Y) \to MX_n(Y) : (M, h, \tilde{\nu}, f) \mapsto [\varphi]$ is well-defined.

The map F is a homomorphism, as for a disjoint union of X-manifolds, one can choose their embeddings in R^{n+m} to lie on opposite sides of a hyperplane, and which shows that their image under F factors through the pinching of that hyperplane, seen as an equator of S^{n+m}. Hence, the disjoint union of X-manifolds maps to the sum of homotopy classes.

To prove surjectivity, consider a class $[\varphi] \in \pi_n(MX \wedge Y)$. Choose a representative $\varphi : S^{n+m} \to MX_m \wedge Y$ such that $p \circ \varphi : S^{n+m} \to MX_m \wedge Y \to MX_m = \mathrm{Th}(g^*\gamma_{m,\infty})$ is transversal to the zero section. Then the inverse image of the zero section is an n-dimensional submanifold M of S^{n+m} which is disjoint from the point at infinity, so it is a submanifold of $\mathbb{R}^{n+m}$ via a map h. As it is the inverse image of the zero section of $\mathrm{Th}(g^*\gamma_{m,\infty})$, it has a canonical map $\tilde{\nu} : M \to (\mathrm{Th}(g^*\gamma_{m,\infty}))_0 \cong X_m$ lifting the classification map of the normal bundle. Finally, φ restricted to M yields a morphism $M \to MX_m \wedge Y$, which by definition is $\tilde{\nu} \wedge f$ for some f. Hence, $[\varphi] = F(M, h, \tilde{\nu}, f)$.

Finally, to prove injectivity, consider an X-manifold with map into Y, $(M, h, \tilde{\nu}, f) \in \Omega_n^X(Y)$ such that $F(M, h, \tilde{\nu}, f) = 0 \in \pi_n(MX \wedge Y)$. Then we may assume that for some large m, h maps into $\mathbb{R}^{n+m}$ and the image in $MX_m \wedge Y$ is contractible. Letting this contraction homotopy be given by $H : \Sigma^{m+n} \times I \to MX_m \wedge Y$, then by the surjectivity argument above, we can construct a $W \subset \mathbb{R}^{m+n} \times I$ as the inverse image of the zero section, and make it an X-manifold with a map to Y. Then by construction $W \cap (\mathbb{R}^{n+m} \times \{0\}) = M$ and $W \cap (\mathbb{R}^{n+m} \times \{1\}) = \emptyset$, giving a bordism between M and the empty manifold. □

Because of this result, there is a natural definition for cobordisms, which is given below.

Definition 4.2.28. For a tangential structure X, define the *X-cobordism theory* to be the cohomology theory Ω_X^* represented by the spectrum MX.

Remark 4.2.29. The words 'bordism' and 'cobordism' are often used interchangeably for what is called 'bordism' in this text. 'Cobordism' is the older term, derived from French, where 'cobordant' means 'bounding together'. However, in the context of bordism and cobordism theory as given here, the 'co' was considered confusing, and is therefore often dropped.

The original proof of the Madsen-Weiss theorem was inspired by and modeled on bordism theory: they define a space $\Omega^\infty \mathbf{hV}$, homotopy equivalent to $\Omega^\infty \mathbb{CP}^\infty_{-1}$. This space classifies concordance classes of pairs $(q, \delta q)$, where q is a proper morphisms of a manifold M^{n+2} into a fixed space X^n, and δq a stable bundle epimorphism $TM \longrightarrow q^*TX$ with oriented kernel. Here concordance classes are an similar notion to bordism classes in the bordism theories.

This is related to submersions, the case $\delta q = dq$. These last maps are classified by $\coprod_g \mathrm{Diff}(S_g)$. Combining these two notions yields a map of classifying spaces, which is the first step in the original proof, cf. [MW07].

4.3 — Infinite loop spaces

One of the main reasons to consider spectra is the adjointness between the suspension functor and the loop space functor, which connects spectra to homotopy groups. This adjunction, $\Sigma \dashv \Omega$, induces inductions on all higher powers $\Sigma^n \dashv \Omega^n$, by repeated application. So far, we have defined an infinite version of Σ, the suspension spectrum functor $\Sigma^\infty : \mathrm{CW}_* \longrightarrow \mathrm{Spec}$, but there is no functor Ω^∞ defined yet to be its right adjoint. We will define one now.

Definition 4.3.1. Define the *infinite loop space functor* $\Omega^\infty : \mathrm{Spec} \longrightarrow \mathrm{CW}_*$ by choosing, for any spectrum X, $\Omega^\infty X$ to be the zeroeth space of a homotopy-equivalent Ω-spectrum X'. For a morphism of spectra $f : X \longrightarrow Y$, define $\Omega^\infty f$ to be the zeroeth map in the induced morphism of spectra $f' : X' \longrightarrow Y'$.

Remark 4.3.2. A priori, this does not really seem to be a real definition at all: we do not even know whether every spectrum is homotopy equivalent to an Ω-spectrum, nor does a morphism of spectra have a well-defined component at degree zero.

However, there is in fact a canonical choice of such Ω-spectra, up to unique homotopy equivalence: by theorem 4.2.5, every spectrum represents a generalised cohomology theory, and by theorem 4.2.4, any generalised cohomology theory is represented by an Ω-spectrum. Taking the composition of these two functors yields Ω-spectra in a functorial way.

As we now have Ω-spectra, any morphism of spectra f can be defined on degree zero by repeatedly applying the loop space functor to larger and larger parts of a cofinal subspectrum on which f is defined.

Another way to obtain an Ω-spectrum from an arbitrary spectrum is given in the following lemma.

Lemma 4.3.3. *Any spectrum $X = (X_n)_{n\in\mathbb{Z}}$ is homotopy equivalent to the Ω-spectrum Y defined by $Y_n = \mathrm{colim}_{m\to\infty} \Omega^m X_{n+m}$.*

Proof. The morphism $X \longrightarrow Y$ is given by the maps $X_n = \Omega^0 X_{n+0} \longrightarrow \mathrm{colim}_m \Omega^m X_{n+m}$. It is an homotopy equivalence, because it induces isomorphisms on all homotopy groups:

$$\pi_n(Y) = \mathrm{colim}_m \, \pi_{n+m}(\mathrm{colim}_p(X_{m+p})) = \mathrm{colim}_m \, \pi_{n+m}(X_m) = \pi_n(X)$$

As the definition of Y_{n-1} can equivalently be written $Y_{n-1} = \mathrm{colim}_m \Omega^{m+1} X_{n+m}$, it is clear that Y is an Ω-spectrum. □

Lemma 4.3.4. *The infinite loop space functor is right adjoint to the suspension spectrum functor.*

Proof. By definition of the cohomology theory associated to a spectrum, and using the model of Ω^∞ in the remark, we get for any pointed CW-complex X and spectrum Y

$$[\Sigma^\infty X, Y] =: \tilde{Y}^0(X) =: [X, \Omega^\infty Y]$$

where $\tilde{Y}^0$ is the zeroeth reduced cohomology associated to Y. □

Definition 4.3.5. An *infinite loop space* is any space homotopy equivalent to a CW-complex in the image of the infinite loop space functor Ω^∞.

Infinite loop spaces should be considered as the 'right' analogue of commutative groups in homotopy theory in some sense. For instance, any commutative topological group is an infinite loop space by example 4.1.6. To capture this idea more precisely, one needs to introduce the notion of operads, following Boardman and Vogt[BV68]. The current exposition is mainly based on May[May72].

Definition 4.3.6. An *operad* $\mathcal{O}$ consists of the following data:

- a set of spaces $\mathcal{O}_n \in \mathbb{C}$ for $n \geq 0$, such that $\mathcal{O}_0 = \{*\}$;
- For any k-tuple $(j_1, \ldots, j_k)$ with $\sum_{s=1}^k j_s = j$ a composition morphism $\gamma : \mathcal{O}_k \times (\mathcal{O}_{j_1} \times \cdots \mathcal{O}_{j_k}) \to \mathcal{O}_j$ which is associative in the sense that

  $$\gamma(\gamma(c; d_1, \ldots, d_k); e_1, \ldots, e_l) = \gamma\big(c; \gamma(d_1; e_1, \ldots, e_{l_1}), \ldots, \gamma(d_k; e_{l-l_k+1}, \ldots, e_l)\big)$$

  where $c \in \mathcal{O}_k$, $d_i \in \mathcal{O}_{l_i}$, and e_j in any $\mathcal{O}_n$.
- an identity $1 \in \mathcal{O}_1$, such that $\gamma(1; d) = d$ and $\gamma(c; 1, \ldots, 1) = c$ for any d, c.
- a right action of the symmetric group Σ_n on $\mathcal{O}_n$ such that for any $c \in \mathcal{O}_k$, $d_i \in \mathcal{O}_{j_i}$, $\sigma \in \Sigma_k$, and $\tau_i \in \Sigma_{j_i}$, we get

  $$\begin{aligned} \gamma(c\sigma; d_1, \ldots, d_k) &= \gamma(c; d_{\sigma^{-1}(1)}, \ldots, d_{\sigma^{-1}(k)})\sigma(j_1, \ldots j_k) \\ \gamma(c; d_1\tau_1, \ldots d_k\tau_k) &= \gamma(c; d_1, \ldots d_k)(\tau_1 \oplus \cdots \oplus \tau_k) \end{aligned}$$

  where $\sigma(j_1, \ldots, j_k) \in \Sigma_j$ permutes the blocks of sizes j_i according to σ and $\tau_1 \oplus \ldots \oplus \tau_k \in \Sigma_j$ acts as τ_1 on the fist j_1 coordinates, as τ_2 on the next j_2 coordinates, &c.

Remark 4.3.7. One should consider elements of $\mathcal{O}_n$ as n-ary product operators. Then definition 4.3.6 expresses that all compositions of products are associative and definition 4.3.6 defines some commutativity property, depending on the action of the symmetric group.

Definition 4.3.8. A *morphism of operads* $\psi : \mathcal{O} \to \mathcal{P}$ is a sequence of Σ_n-equivariant maps $\psi_n : \mathcal{O}_n \to \mathcal{P}_n$ that commute with the compositions in the obvious way.

This forms a category of operads, Op.

An operad $\mathcal{O}$ is Σ-*free* if the action of Σ_n on $\mathcal{O}_n$ is free for all n.

Definition 4.3.9. For a based space X, the *endomorphism operad* $\mathcal{E}_X$ is the operad where $(\mathcal{E}_X)_n$ is the space of (based) maps $X^n \to X$, where

$$\begin{aligned} \gamma(f; g_1, \ldots, g_k) &= f \circ (g_1 \times \cdots g_k) \\ 1 &= \mathrm{id}_X \\ f\sigma &: (x_1, \ldots, x_k) \mapsto f(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(k)}) \end{aligned}$$

Definition 4.3.10. An *operation* of an operad $\mathcal{O}$ on a space X is a morphism $\vartheta : \mathcal{O} \to \mathcal{E}_X$. Such a pair (X, ϑ) is called an $\mathcal{O}$-*space*.

There is an obvious notion of morphism of $\mathcal{O}$-spaces, forming a category of $\mathcal{O}$-spaces, $\mathcal{O}[\mathrm{Sp}_*]$.

Definition 4.3.11. Define the operad $\mathcal{N}$ by $\mathcal{N}_n = \{f_n\}$, a single point, and defining the compositions, identity, and symmetric group action in the only possible way.

Proposition 4.3.12. *The category of commutative monoids is isomorphic to the category* $\mathcal{N}[\mathrm{Sp}_*]$ *of* $\mathcal{N}$*-spaces.*

Proof. Given a commutative topological monoid M, define $\vartheta : \mathcal{N} \to \mathcal{E}_M$ by defining $\vartheta_n(f_n)$ to be the n-fold product on M. This is a morphism of operads, because the γ's in $\mathcal{N}$ correspond to associative composition of products, $f_1 = 1$ gets mapped to the one-fold product, which is the identity, and the trivial action of the symmetric group on $\mathcal{N}$ corresponds to the commutativity of the product on M.

Conversely, for an $\mathcal{N}$-space (X, ϑ), declare $\vartheta(f_2)$ to be a (binary) product on X. By the converse reasoning to the previous paragraph, this product has an identity and is associative and commutative.

Both of these constructions clearly define functors and are inverse (not just quasi-inverse!) to each other. □

Definition 4.3.13. An E_∞*-operad* is a Σ-free operad $\mathcal{O}$ such that all $\mathcal{O}_n$ are contractible. Equivalently, the unique map (called *augmentation*) $\varepsilon : \mathcal{O} \to \mathcal{N}$ is a *local equivalence* in the sense that $\varepsilon_n : \mathcal{O}_n \to \mathcal{N}_n = \{f_n\}$ is a homotopy equivalence for all n.

An E_∞*-space* or *homotopy everything space* is an $\mathcal{O}$-space for some E_∞-operad $\mathcal{O}$.

Remark 4.3.14. Historically, the terms 'E_∞' and 'homotopy everything' come from analogy to an older notion, A_n and A_∞ or 'associativity everything'. An A_n-space is an H-space with unit such that all ways of placing parenthetheses in an n-fold product are homotopic (thus the space is 'homotopy associative up to stage n'), and an A_∞-space has all this property for all n. In contrast, an E_∞-space has all products homotopic, where entries can be permuted as well. This leads to a space where all algebraic operations are homotopic, hence 'homotopy everything'.

The main theorem of [May72] is then the following:

Theorem 4.3.15 ([May72])**.** *A connected space X is an infinite loop space if and only if it is a homotopy everything space.*

This theorem confirms the statement that infinite loop spaces are the right commutative group analogue in homotopy theory: they are exactly those spaces having a product which is homotopy associative and homotopy commutative in all iterations.

The theorem will not be proven, as the proof is rather lengthy and would take us quite far away from the idea of this thesis. However, this result is worth mentioning, as the first proof that $B\Gamma_\infty^+$ is an infinite loop space was given by Tillmann[Til97] using this theorem.

Chapter 5 — The Madsen-Weiss theorem

We have now finally done enough preliminary work to state and prove the Madsen-Weiss theorem. After having done this, we will use the theorem to calculate the rational cohomology of $B\Gamma_\infty$, confirming Mumford's conjecture.

In this chapter we will assume $g > 2$ if not doing so would make the argument more involved. Since we ultimately consider the stable mapping class group, this will not pose a problem.

5.1 — Statement and proof of the Madsen-Weiss theorem

The Madsen-Weiss theorem states $B\Gamma_\infty^+$ is homotopy equivalent to some infinite loop space. So before we can state it, we first need to show the plus construction can be applied to $B\Gamma_\infty$ and define the required loop space. We will attack these problems first.

Proposition 5.1.1. *Let $g \geq 2$, r arbitrary. Then the mapping class group $\Gamma_{g,r}$ is perfect. In other words, $H_1(\Gamma_{g,r}) = 0$.*

Proof. We will prove this for $g \geq 3$. The case $g = 2$ then follows from Harer stability. The proof is due to [Har83].

Let S be a four-holed sphere, with boundary circles $C_0, \dots, C_3$. Let c_i be a circle in the interior of S isotopic to C_i, and let c_{ij} be the circles around the two holes i and j. Writing τ_i and τ_{ij} for the Dehn twists around c_i and c_{ij}, we then have the lantern relation $\tau_0\tau_1\tau_2\tau_3 = \tau_{12}\tau_{13}\tau_{23}$ in $\Gamma(S)$.

We will first show $\Gamma_{g,r}$ is generated by Dehn twists on non-separating curves. If a curve γ is isotopic to a boundary component $\partial_0 S_{g,r}$ with $g \geq 2$, choose an embedding i of S into $S_{g,r}$ mapping C_0 to $\partial_0 S_{g,r}$ such that $S_{g,r} \setminus i(C_1 \cup C_2)$ is connected. Then $\gamma \sim c_0$ and the images of all other c_i, c_{ij} are non-separating, so τ_γ is a product of Dehn twists around non-separating curves.

If γ is a general separating curve in $S_{g,r}$, $g \geq 3$, split $S_{g,r}$ along γ and use the previous argument on the resulting component with genus ≥ 2.

Letting γ, δ be two non-separating curves in $S_{g,r}$ there exists an $h \in \Gamma_{g,r}$ such that $h(\gamma) \sim \delta$, so $h\tau_\gamma h^{-1} = \tau_\delta$ in $\Gamma_{g,r}$. Hence, $H_1(\Gamma_{g,r})$ is cyclic with generator τ. Taking an embedding of S in $S_{g,r}$ with all c_i and c_{ij} non-separating, we see $\tau^3 = \tau^4$ in $H_1(\Gamma_{g,r})$, so $H_1(\Gamma_{g,r}) = 0$. □

Hence, $\pi_1(B\Gamma_{g,r})$, which is $\Gamma_{g,r}$ by definition, is perfect, so $B\Gamma_{g,r}^+ := (B\Gamma_{g,r})^+$ is well-defined.

To define the infinite loop space, consider the anticanonical bundle on $G^+(2,n)$: the $(n-2)$-dimensional bundle $\gamma_{2,n}^\perp \to G^+(2,n)$ whose fibre over a point $V \in G^+(2,n)$ is the orthogonal complement of V as a plane in $\mathbb{R}^n$ It is the orthogonal complement to $\gamma_{2,n}$ in the trivial bundle $\mathbb{R}^n \times G^+(2,n)$ with standard metric. There are natural inclusions $G^+(2,n) \hookrightarrow G^+(2,n+1)$ and $\gamma_{2,n+1}^\perp|_{G^+(2,n)} = \gamma_{2,n}^\perp \times \varepsilon$, where ε is the trivial line bundle. As seen before, on the level of Thom spaces, this means that $\mathrm{Th}(\gamma_{2,n+1}^\perp|_{G^+(2,n)}) = \mathrm{Th}(\gamma_{2,n}^\perp \times \varepsilon) = \Sigma\,\mathrm{Th}(\gamma_{2,n}^\perp)$. Using this observation, we can define a spectrum in a somewhat similar manner to the Thom spectra of section 4.2.1. However, in stead of taking infinite Grasmannians and varying the dimension of the subspaces, we fix the dimension of the subspaces and vary the dimension of the ambient space.

Definition 5.1.2. Define the spectrum $\mathbb{CP}_{-1}^\infty = (\mathrm{Th}(\gamma_{2,n}^\perp))_{n=0}^\infty$ with the suspension maps given by $\Sigma\,\mathrm{Th}(\gamma_{2,n}^\perp) \to \mathrm{Th}(\gamma_{2,n+1}^\perp)$ induced by the isomorphisms above.

Now we are finally ready to state the Madsen-Weiss theorem in full.

Theorem 5.1.3 (Madsen-Weiss[MW07]). *There exists a homotopy equivalence* $B\Gamma_\infty^+ \xrightarrow{\simeq} \Omega_0^\infty \mathbb{CP}_{-1}^\infty$.

Remark 5.1.4. Note that, although $B\Gamma_\infty$ is only a homology type, $B\Gamma_\infty^+$ is simply connected as well, so by the relative Hurewicz theorem, it is a homotopy type. As it is a CW-complex, $B\Gamma_\infty^+$ is a well-defined space up to homotopy equivalence.

For the remainder of this chapter, we will use the model of infinite loop space of a spectrum as given in lemma 4.3.3.

Our plan of attacking this proof is derived from [Hat14], who adapted it from a more general argument involving Thom spectra in [GR10]. However, we prove the existence of an actual homotopy equivalence, in stead of an abstract isomorphism on integral cohomology.

First we will construct an explicit model for $B\Gamma_{g,r}$. Consider $\mathcal{E}(S_{g,r}, \mathbb{R}^n)$, the space of smooth embeddings $S_{g,r} \to \mathbb{R}^n$, and let $\mathcal{E}(S_{g,r}, \mathbb{R}^\infty)$ be the direct limit over these spaces via the inclusions $\mathbb{R}^n \to \mathbb{R}^{n+1} : x \mapsto (x, 0)$. This has a natural action of the diffeomorphism group $\mathrm{Diff}(S_{g,r})$, acting on the surface before the embedding, with quotient $C(S_{g,r}, \mathbb{R}^\infty)$, the space of oriented surfaces in $\mathbb{R}^\infty$ (abstractly) homeomorphic to $S_{g,r}$ (the C stands for "configurations" of $S_{g,r}$ in $\mathbb{R}^\infty$).

Lemma 5.1.5. *The embedding space* $\mathcal{E}(S_{g,r}, \mathbb{R}^\infty)$ *is contractible and*

$$\begin{array}{ccc} \mathrm{Diff}(S_{g,r}) & \rightarrowtail & \mathcal{E}(S_{g,r}, \mathbb{R}^\infty) \\ & & \downarrow \\ & & C(S_{g,r}, \mathbb{R}^\infty) \end{array}$$

is a fibre bundle. Hence, $C(S_{g,r}, \mathbb{R}^\infty) \simeq B\,\mathrm{Diff}(S_{g,r}) \simeq B\Gamma_{g,r}$.

Proof. For the first statement, any embedding $S_{g,r} \to \mathbb{R}^\infty$ can be composed with a linear endomorphism of $\mathbb{R}^\infty$ pushing all of $\mathbb{R}^\infty$ into the odd coordinates. Then a linear isotopy to a fixed embedding of $S_{g,r}$ in the even coordinates gives a contraction.

For the second statement, fix a $P \subset \mathbb{R}^n$ homeomorphic to $S_{g,r}$ and take a tubular neighbourhood of it, viewed as the normal bundle. Then local sections of that bundle form a neighbourhood of $P \in C(S_{g,r}, \mathbb{R}^n)$, and projection onto the zero section gives a local product structure, which is compatible with the limit over n. □

For a large part of the proof we will consider several different embedding spaces, defined below.

Definition 5.1.6. Define C^n to be the space of smooth oriented surfaces (of any genus) in $\mathbb{R}^n$ that are properly embedded, including the empty surface as a base point. A basis of the topology is given by sets $C(r, V)$ with $r \in \mathbb{R}_+$, V an open set in the C^∞ topology of properly embedded surfaces in the closed ball $B_r := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 \leq r\}$, and $C(r, V)$ all surfaces S meeting ∂B_r transversely such that $B_r \cap S \in V$.

Lemma 5.1.7. *For* $n > 2$, C^n *is path-connected.*

Proof. For any surface $S \subset \mathbb{R}^n$, we can move it by translation such that $0 \notin S$. Choosing an increasing homeomorphism $\lambda : [0, 1) \to [1, \infty)$, we can define a path from S to the empty surface by radial expansion of $\mathbb{R}^n$ by factors $\lambda(t)$ from the origin:

$$\alpha : [0,1] \to C^n : t \mapsto \begin{cases} \lambda(t)S = \{\lambda(t)x \mid x \in S\} & t < 1 \\ \emptyset & t = 1 \end{cases}$$

As a basis of the empty surface is given by the open sets $C(r, \emptyset)$, and the radial expansion ensures that for any r there exists a $t \in [0, 1)$ such that $\alpha(t) \cap B_r = \emptyset$, i.e. $\alpha(t) \in C(r, \emptyset)$, this path is continuous at 1. As α is clearly continuous on $[0, 1)$, this proves it is a path from S to the empty manifold. □

Definition 5.1.8. Define an ascending filtration on C^n by defining $C^{n,k}$ to be the subspace of C^n consisting of surfaces in $\mathbb{R}^k \times (0,1)^{n-k}$.

Lemma 5.1.9. *There is a loop map $C^{n,k} \longrightarrow \Omega C^{n,k+1}$, given by sending a surface to the loop translating that surface from $-\infty$ to ∞ in the $(k+1)$-st coordinate, defining a loop based at the empty surface. Composing these maps yields a map $C^{n,0} \longrightarrow \Omega^n C^n$.*

Proof. The construction of such a loop α and its continuity at the endpoints is proven similarly to the proof of lemma 5.1.7. The main difference is that this map is defined by translation in stead of dilatation, and that it goes to infinity at both ends. □

One reason to look at these spaces is the following proposition.

Proposition 5.1.10. *The space C^n is homotopy equivalent to the Thom space of the anticanonical bundle over $G(2,n)$, $\mathrm{Th}(\gamma_{2,n}^{\perp})$.*

Proof. The total space of $\gamma_{2,n}^{\perp}$ is homeomorphic to the space of all oriented affine planes in $\mathbb{R}^n$, i.e. not neccessarily through the origin, by associating to a pair $(P,v) \in G^+(2,n) \times P^{\perp}$ the oriented affine plane $v + P = \{x \in \mathbb{R}^n | x - v \in P\}$. As v can be recovered from an affine plane by taking the unique point on that plane so that the line through it and the origin in orthogonal, this is a bijection. The Thom space of this vector bundle is then the one-point compactification, which means adding the empty subset, the 'affine plane at infinity'.

Now, given an oriented surface $S \in C^n$, take a tubular neighbourhood $p : W \subset \mathbb{R}^n \longrightarrow S$ of it (varying continuously with S). If $0 \notin W$, we scale $\mathbb{R}^n$ radially from 0 by a factor increasing from 1 to infinity. If $0 \in W$, we still scale the same way in tangential directions, but in normal directions we scale from 1 to $\lambda(S)$, for a continuous $\lambda : C \longrightarrow [1,\infty)$, such that $\lambda(S) = 1$ if $0 \in S$ and $\lambda(S) \longrightarrow \infty$ if 0 gets closer to the boundary of W.

This is a retraction from C^n to the set of oriented affine planes (plus empty surface) and this set gets mapped to itself, so it is a homotopy equivalence. □

Hence, if we can show all maps $C^{n,k} \longrightarrow \Omega C^{n,k+1}$ are well-behaved with respect to homotopy groups, this would greatly help in proving the Madsen-Weiss theorem, as $C^{n,0}$ is clearly related to $C(S_{g,r}, \mathbb{R}^\infty) \simeq B\Gamma_{g,r}$ in some way.

For $k < n$, the space $C^{n,k}$ is an H-space, with multiplication defined by juxtaposition of surfaces in the $(k+1)$-st coordinates and compressing the interval $(0,2)$ to $(0,1)$, just as in the definition of homotopy groups. Clearly, there is a homotopy-equivalent Moore-like strict monoid $M^{n,k}$, of pairs $(S,a) \in C^n \times [0,\infty)$ with $S \subset \mathbb{R}^k \times (0,a) \times (0,1)^{n-k-1}$. Here multiplication is juxtaposition without compressing. We have the following proposition:

Proposition 5.1.11 (First delooping)**.** *For $0 < k < n$, $\pi_0 M^{n,k}$ is a group and there is a weak homotopy equivalence from the classifying space $BM^{n,k}$ to $C_0^{n,k+1}$, the path-connected component of the empty surface in $C^{n,k+1}$. Hence the loop map $C^{n,k} \longrightarrow \Omega C^{n,k+1}$ is a weak homotopy equivalence.*

However, this proposition does not hold for $k = 0$, as $\pi_0 M^{n,0}$ is manifestly not a group. We will therefore define a new monoid M^n.

Definition 5.1.12. Let $Z = \mathbb{R} \times C(0, \frac{1}{2}) \times \{0\} \subset \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^{n-3} = \mathbb{R}^n$ be a fixed cylinder, where $C(0, \frac{1}{2})$ is the circle based at 0 with radius $\frac{1}{2}$. Let the monoid $M^n \subset C^{n,1} \times [0,\infty)$ be given by pairs (S,a) with S a compact, connected, oriented surface in $[0,a] \times (-1,1)^{n-1}$ such that:

a) $\partial S = Z \cap (\{0,a\} \times (-1,1)^{n-1}$ consists of exactly two circles, and S is tangent to infinite order along Z at the boundary. Also, $\mathrm{int}(S) \subset (0,a) \times (-1,1)^{n-1}$;
b) $S \cap ([0,a] \times (-1,0] \times (-1,-1)^{n-2}) = Z \cap ([0,a] \times (-1,0] \times (-1,-1)^{n-2})$. I.e., 'the lower half of the surface is fixed'.

The monoid structure on M^n is given by juxtaposition in the first coordinate.

Define the 'infinity component' of the monoid as $M^\infty = \bigcup_n M^n \simeq \operatorname{hocolim}(M^n)$.

Remark 5.1.13. The infinity component M^∞ is defined in a manner consistent with the Group Completion Theorem. This is of course no coincidence, as we will use the Group Completion Theorem for this monoid in the proof of the Madsen-Weiss theorem.

We now get the following proposition, which is an analogue of proposition 5.1.11 for the case $k = 0$.

Proposition 5.1.14 (Second delooping)**.** *The classifying space of M^∞ is weakly homotopy equivalent to $C^{\infty,1}$: there is a weak homotopy equivalence $\sigma : BM^\infty \xrightarrow{\sim}_w C^{\infty,1}$*

Assuming these two delooping propositions for the moment, the proof of the Madsen-Weiss theorem is not that hard. We give it here and then prove the two delooping propositions 5.1.11 and 5.1.14 in section 5.1.1.

Proof of theorem 5.1.3 (assuming propositions 5.1.11 and 5.1.14). The path components of M^∞, which are denoted M_g^∞, consist of all surfaces in M^∞ of genus g. They have two boundary components. It is quite clear that M_g^∞ is homotopy equivalent to the configuration space $C(S_{g,2}, \mathbb{R}^\infty)$, as any subsurface homeomorphic to $S_{g,2}$ (and hence compact) in $\mathbb{R}^\infty$ can be isotoped to one in M^∞, in a way that is continuous over all surfaces. Assembling all the definitions and propositions, we then have the following chain of maps, of which most are equalities or (weak) homotopy equivalences (with the weak equivalences in the right direction, so they compose to one map):

$$\begin{aligned}
B\Gamma_\infty &:= \operatorname{hocolim}_g B\Gamma_{g,2} && \text{by definition 2.5} \\
&= \operatorname{hocolim}_g C(S_{g,2}, \mathbb{R}^\infty) && \text{by lemma 5.1.5} \\
&\simeq \operatorname{hocolim}_g M_g^\infty && \text{by above} \\
&\longrightarrow \Omega_0 BM^\infty && \text{by corollary 3.2.4} \\
&\simeq_w \Omega_0 C^{\infty,1} && \text{by proposition 5.1.14} \\
&= \operatorname{colim}_n \Omega_0 C^{n,1} && \text{by definition} \\
&\simeq_w \operatorname{colim}_n \Omega_0^n C^n && \text{by proposition 5.1.11} \\
&\simeq \operatorname{colim}_n \Omega_0^n \operatorname{Th}(\gamma_{2,n}^\perp) && \text{by proposition 5.1.10} \\
&=: \Omega_0^\infty \mathbb{CP}_{-1}^\infty
\end{aligned}$$

where the arrow is a homology equivalence.

So now we have a map $B\Gamma_\infty \to \Omega_0^\infty \mathbb{CP}_{-1}^\infty$ which is a homology equivalence from a space with perfect fundamental group (by proposition 5.1.1) to a simply connected space, as $H_1(\Omega_0^\infty \mathbb{CP}_{-1}^\infty) \cong H_1(\Gamma_\infty) = 0$ and the fundamental group of a loop space is abelian. Hence, this map factors through $B\Gamma_\infty^+$ and the map $B\Gamma_\infty^+ \to \Omega_0 \mathbb{CP}_{-1}^\infty$ is a homology equivalence of simply-connected spaces. By the relative Hurewicz theorem, it must therefore be a weak homotopy equivalence. As both spaces are (homotopy equivalent to) connected CW-complexes, by the Whitehead theorem this is an actual homotopy equivalence (this last argument can be found in e.g. [Mad07]). □

5.1.1 — Proof of the delooping propositions

In this subsection we will prove the two delooping propositions 5.1.11 and 5.1.14, thereby finishing the proof of the Madsen-Weiss theorem. We will give the proof of proposition 5.1.11 first, starting with the following lemma:

Lemma 5.1.15. *If $k > 2$, $\pi_0 M^{n,k} = 0$. If $k = 1, 2$, $\pi_0 M^{n,k} \cong \Omega_{2-k}^{SO}(\mathbb{R}^{n-k})$.*

Proof. Let $S \in M^{n,k}$ be a surface in $\mathbb{R}^k \times (0,a) \times (0,1)^{n-k-1}$. Then the projection $p : S \longrightarrow \mathbb{R}^k$ is proper, and we can perturb S slightly so that p is transverse to $0 \in \mathbb{R}^k$.

If $k > 2$, this means $p^{-1}(0) = \emptyset$, so we can blow up the first k coordinates of $\mathbb{R}^n$ from 1 to ∞, defininig a path from S to the empty surface. This shows $M^{n,k}$ is path-connected, i.e. $\pi_0 M^{n,k} = 0$ for $k > 2$.

If $k = 1, 2$, we have $p^{-1}(0) = S_0 \subset S$ is a closed submanifold of dimension $2-k$. It obtains an orientation from that of S and the pullback of the orientation of $\mathbb{R}^k$ to the normal bundle of S_0 in S. Also, it is a submanifold of $\pi^{-1}(0) = \mathbb{R}^{n-k}$, so it defines an element $[S_0] \in \Omega^{\mathrm{SO}}_{2-k}(\mathbb{R}^{n-k})$. The assignment $S \mapsto [S_0]$ is a homomorphism as the monoid operation is disjoint union at both sides.

Given an $[S_0] \in \Omega^{\mathrm{SO}}_{2-k}(\mathbb{R}^{n-k})$, which we may perturb to lie inside $(0,1)^{n-k}$, the manifold $\mathbb{R}^k \times S_0 \in C^{n,k}$ maps to $[S_0]$, showing surjectivity.

For injectivity, let $S \in M^{n,k}$ be such that S_0 is cobordant to $\emptyset$ in $\mathbb{R}^{n-k}$, i.e. $S_0 \times \{0\} = \partial M$ for some $M \subset (0,a) \times (0,1)^{n-k-1} \times I$. By radial expansion in the $\mathbb{R}^k$ direction and transversality, we get a path from S to $\mathbb{R}^k \times S_0$. This can be composed by a path

$$I \longrightarrow M^{n,k} : t \mapsto \mathbb{R}^{k-1} \times (-\infty, \lambda_t) \times S_0 \cup_{\{0\} \times \{\lambda_t\} \times S_0} \mathbb{R}^{k-1} \times (M + \lambda_t)$$

where λ is a fixed map scaling I to $\mathbb{R} \cup \{-\infty, \infty\}$ and $M + \lambda_t$ is the manifold M, embedded in $[\lambda_t, \lambda_{t+1}] \times (0,a) \times (0,1)^{n-k-1}$. This second path connects $\mathbb{R}^k \times S_0$ to the empty surface, by translating a one-side infinite capped cylinder, so $[S_0] = [\emptyset] \in \Omega^{\mathrm{SO}}_{2-k}(\mathbb{R}^{n-k})$ implies $[S] = [\mathbb{R}^k \times S_0] = [\emptyset] \in \pi_0 M^{n,k}$, showing injectivity. □

Remark 5.1.16. Although we do not need it here, it is clear that $\Omega^{\mathrm{SO}}_1(\mathbb{R}^{n-1}) = 0$, as a one-manifold is a disjoint union of circles and circles bound discs, and $\Omega^{\mathrm{SO}}_0(\mathbb{R}^{n-2}) \cong \mathbb{Z}$, generated by an oriented (i.e. signed) point, with its inverse having the opposite orientation.

Proof of proposition 5.1.11. The first statement, that $\pi_0 M^{n,k}$ is a group, has been proven above, in lemma 5.1.15.

The second statement is that there is a weak homotopy equivalence from $BM^{n,k}$ to $C^{n,k+1}_0$. We will start by defining this map $\sigma : BM^{n,k} \longrightarrow C^{n,k+1}_0$. So take a point in $BM^{n,k}$. It is given by a p-tuple of surfaces $S_i \subset \mathbb{R}^k \times (0, a_i) \times (0,1)^{n-k-1}$ and a point $(w_0, \ldots, w_p) \in \Delta^p$. The product $S_1 \cdots S_p$ is a surface in $\mathbb{R}^k \times (0, \sum a_i) \times (0,1)^{n-k-1}$, which seems like a good candidate for $\sigma((S_i),(w_j))$. When we move one of the w_i, $1 \le i \le p-1$ to zero, this works well, as we replace the two surfaces S_i and S_{i+1} by their disjoint union, which does not change the sum. However, when either w_0 or w_p goes to zero, suddenly either S_1 or S_p vanishes, and the image 'jumps' by a distance a_1 as well in the case $w_0 \longrightarrow 0$. Therefore, this naive definition of σ does not work.

To remedy the jumping, we can translate the manifold as follows. Let S_i lie between c_{i-1} and c_i in the disjoint union (so $c_i = c_{i-1} + a_i$). We then translate such that the barycentre $\sum_{i=0}^p w_i c_i$ is equal to zero.

To correct the sudden disappearance of S_0 or S_p, define $c_i^+ = \max\{c_i, 0\}$ and $c_i^- = \min\{c_i, 0\}$ and set $b^+ = \sum_{i=0}^p w_i c_i^+$ and $b^- = \sum_{i=0}^p w_i c_i^-$. Taking a fixed oriented homeomorphism $L : \mathbb{R} \xrightarrow{\sim} (0,1)$, and $\varphi_b : (0,1) \longrightarrow (b^-, b^+)$ the unique affine map, define

$$\sigma\big((S_i),(w_j)\big) := \big(1_{\mathbb{R}^k} \times (\varphi_b \circ L) \times 1_{\mathbb{R}^{n-k-1}}\big)^{-1}(S_1 \cdots S_p)$$

This is a continuous map, as the $b^{\pm}$ vary continously and are well-behaved when one of the w_i goes to zero. Since $BM^{n,k}$ is path-connected and the base point of $BM^{n,k}$ is sent to the empty surface in $C^{n,k+1}$, the image of σ does lie in $C^{n,k+1}_0$.

To show σ is surjective on the qth homotopy group, consider a map $f : D^q \longrightarrow C^{n,k+1}_0$ which is the empty manifold on ∂D^q. We want to lift it, up to homotopy, to a map $g : D^q \longrightarrow BM^{n,k}$. First, consider a fixed $t \in D^q$, and let $S_t := f(t)$. It is a properly embedded surface in $\mathbb{R}^{k+1} \times (0,1)^{n-k-1}$, so the projection $p : S_t \to \mathbb{R}^{k+1}$ is proper. Hence, its set of regular values is open and dense. Picking such a regular value x, $S_x := p^{-1}(x)$ is a manifold of dimension $2-k-1$, so it is either empty or a discrete set.

If $k \geq 2$, S_x is empty, so we can choose a closed ball $B \subset \mathbb{R}^k$ and a closed interval $J \subset \mathbb{R}$ such that $B \times J \cap p(S_t) = \emptyset$. We can then radially expand the B with a factor depending continuously on J such that the factor is 0 near ∂J and is infinite on a subinterval $J' \subset J$. The deformed S_t is then disjoint from $\mathbb{R}^k \times J' \times (0,1)^{n-k-1}$.

If $k = 1$, we can perturb S_t to look like $\mathbb{R}^2 \times S_x$ over a similar neighbourhood $B \times J$ of x. Using the same radial expansion, we deform S_t to agree with $\mathbb{R}^2 \times S_x$ in $\mathbb{R} \times J' \times (0,1)^{n-2}$. Now, $[S_x] = 0 \in \Omega^{SO}_{0,n-2}$ as $S \in C^{n,2}_0$, so using the same 'translation of capped cylinder' argument as in the proof of lemma 5.1.15, we can again choose S_t to be disjoint from $\mathbb{R}^1 \times J' \times (0,1)^{n-2}$.

When varying t, we can take the same choice of $B \times J$ for a neighbourhood. By compactness of D^q, we can take a finite cover $\{U_i\}$ with corresponding $B_i \times J_i$, where we can shrink the J_i to be disjoint. Choosing another cover $\{V_i\}$ (with same index set) of D^q such that $\bar{V}_i \subset U_i$, we can deform the S_t simultaneously over each V_i according to $B_i \times J_i$, damping out the deformation of $B_i \times J_i$ to zero over $U_i \setminus \bar{V}_i$. This gives a new map $f : (D^q, \partial D^q) \to (C^{n,k+1}, \{\emptyset\})$, homotopic to the old one.

Choosing $a_i \in J'_i$ and a partition of unity $\{\rho_i\}$ subordinate to $\{U_i\}$, taking $a_0 = 0$ and $\rho_0 = 1$ in a neighbourhood U_0 of S^{q-1}, we can now define a map $D^q \to BM^{n,k}$ by

$$g : D^q \to BM^{n,k} : t \mapsto \big((S_t|_{\mathbb{R}^k \times (a_{i-1}, a_i) \times (0,1)^{n-k-1}}), (\rho_i(t)) \big)$$

i.e. restricting the surface to the pieces between the a_i, and giving weights $\rho_i(t)$ to the corresponding vertices.

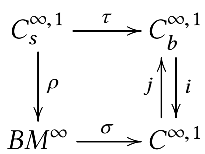
Then the composition σg is homotopic to f by expanding the (b_t^-, b_t^+) to $\mathbb{R}$ for all $g(t)$, proving surjectivity.

To prove injectivity, take a map $h : S^q \to BM^{n,k}$ and an extension $f : D^{q+1} \to C^{n,k+1}_0$ of σg. We can deform f in the same way as before, deforming h with it, as the empty slices of the disjoint union are preserved. We then obtain a map g as above, which, however, does not agree with h on S^q yet, as we still have the expansion of (b_t^-, b_t^+) to $\mathbb{R}$ to take care of. So, taking this expansion at S^q and damping it out to zero in a neighbourhood of the centre of D^{q+1}, we get a map $g' : D^{q+1} \to BM^{n,k}$ which agrees with h on S^q and is homotopic to g, proving injectivity. □

The only part left to prove the Madsen-Weiss theorem is the proof of proposition 5.1.14. This proof is the most intricate part of the Madsen-Weiss theorem. Whereas all other steps can be generalised from surfaces to manifolds of arbitrary dimension, this one will only give a weaker statement: that $C^{\infty,1}$ is weakly homotopy equivalent to the classifying space of a cobordism category. In higher dimensions, this category is not a monoid, so the Group Completion Theorem does not apply.

Proof of proposition 5.1.14. In this proof, surfaces of $C^{\infty,1}$ will be embedded in $\mathbb{R} \times (-1,1)^\infty$ in stead of $\mathbb{R} \times (0,1)^\infty$ for notational convenience. We also introduce notation for *slices* $S(a) := \{a\} \times (-1,1)^\infty$ and *slabs* $S[a,b] := [a,b] \times (-1,1)^\infty$.

Define the map σ to be the same map as in proposition 5.1.11, but using connected sums in stead of disjoint union. Then we will construct a diagram, commutative up to homotopy,



with i and j being homotopy inverse to each other, and ρ and τ weak homotopy equivalences, showing σ is a weak homotopy equivalence as well.

The space $C_s^{\infty,1}$ consists of surfaces $S \in C^{\infty,1}$ together with numbers $a_0 \leq \cdots \leq a_p$ and weights $w_0, \ldots w_p \geq 0$ such that $\sum_{i=0}^p w_i = 1$, where a_i can be deleted if $w_i = 0$, satisfying a couple of conditions:

a) S is based: $S \cap (\mathbb{R} \times (-1,0] \times (-1,1)^\infty) = Z \cap (\mathbb{R} \times (-1,0] \times (-1,1)^\infty =: B$, the *base*, as oriented spaces;
b) $S \cap S(a_i) = Z \cap S(a_i)$ and S and Z are tangent to order infinity at $S(a_i)$ for all i;
c) $S \cap S[a_i, a_{i+1}]$ is connected for all i.

Similarly, $C_b^{\infty,1}$ is the subspace of $C^{\infty,1}$ of surfaces satisfying the first condition.

The map i is the inclusion, and j is defined as follows: for a surface S, compress the the second coordinate from $(-1,1)$ to $(\frac{2}{3},1)$ to obtain S', and take the disjoint union with Z. There is a homotopy from the identity on $C^{\infty,1}$ to the composition ij by first compressing from $(-1,1)$ to $(\frac{2}{3},1)$ by isotopy and then translating a one-side infinite capped cylinder which agrees with Z on the infinite side from $-\infty$ to ∞, similar to the path in the proof of lemma 5.1.15.

A homotopy from ji to the identity is given on a surface S by moving the zipper in figure 5.1 from $-\infty$ to ∞ and adjusting $S \setminus B$, so i and j are homotopy inverses.

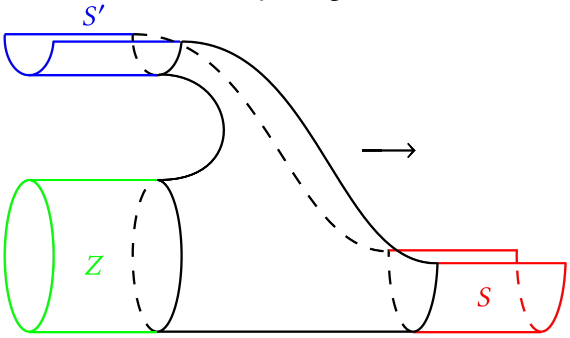


Figure 5.1: The path from $S' \cup Z$ to S.

The map ρ is defined by restricting a surface S to the slabs $S[a_i, a_{i+1}]$. As both its source and its target are path-connected, to show it is a weak homotopy equivalence, it is sufficient to show that for each map $f : D^q \longrightarrow BM^\infty$ with a partial lift $g : \partial D^q \longrightarrow C_s^{\infty,1}$, there is a map $\tilde{g} : D^q \longrightarrow C_s^{\infty,1}$ such that $\rho\tilde{g} \sim f$ and $\tilde{g}|_{\partial D^q} \sim g$, with the homotopies corresponding under σ, for all $q > 0$.

So pick such an f and g, writing $S_t := f(t)$ and considering it as a family of surfaces in $\mathbb{R}^\infty$ with slices $S(a_i(t))$ and weights $w_i(t)$. We will assume $\sum_{i=0}^p a_i(t)w_i(t) = 0$ for all $t \in D^q$. Then g gives an extension of the surfaces to infinity for $t \in \partial D^q$ and we want to complete this to an extension for all $t \in D^q$. First we change f and g slightly so that all slices have a neighbourhood $S[a_i(t) - w_i(t), a_i(t) + w_i(t)]$ on which S_t agrees with Z, by 'growing' these slabs out of the slices and moving the surfaces in between.

Fixing a t for the moment, choose a $a^-(t)$ and $a^+(t)$ among the $a_i(t)$ such that $a^-(t) \leq 0 \leq a^+(t)$, and choose $\alpha^-(t) \in (a^-(t) - w^-(t), a^-(t))$ and $\alpha^+(t) \in (a^+(t), a^+(t) + w^+(t))$. Varying t again, we can choose the same values of $a^\pm(t)$ and $\alpha^\pm(t)$ on a neighbourhood and extend these maps by letting them be zero outside a slightly smaller neighbourhood and interpolating. Taking a finite cover of these neighbourhoods, we can define $[c^-(t), c^+(t)] = \bigcup_j [\alpha_j^-(t), \alpha_j^+(t)]$, where j is an index for the cover.

Also taking a partition of unity $\eta_j(t)$ subordinate to this cover, define $\tilde{f} : D^q \longrightarrow BM^\infty$ to associate to the surface S_t the slices $S(a_j^\pm(t))$ and weights $\eta_j(t)$. It is homotopic to f by linearly moving the weights from w_i to η_j in the simplex of all slices $S(a_i(t))$ and it lifts to a homotopy $g \sim \tilde{g}$. Now, because $[c^-(t), c^+(t)]$ always contains all the $a^\pm(t)$ by construction, we can extend the interval $(c^-(t), c^+(t))$ to $\mathbb{R}$ to obtain a homotopy $\tilde{f} \sim \hat{f}$ lifting to $\tilde{g} \sim \hat{g}$ over all of D^q, as the surfaces do not have to be extended to infinity anymore. This proves ρ is a weak homotopy equivalence.

The map τ forgets the slices $S(a_i)$ and the weights w_i. To show it is a weak equivalence, we will again consider a map $f : D^q \to C_b^{\infty,1}$ and a lift on the boundary $g : \partial D^q \to C_s^{\infty,1}$. Again, we will denote by S_t the surface associated to a point $t \in D^q$. To extend the lift, we need to find slices $S(a_i(t))$ such that $S_t \cap S(a_i(t))$ are circles and the parts of S_t between the slices are connected. But first we will ensure the given slices on ∂D^q all lie within $[-1, 1]$.

To do this, it would be convenient to use compactness of ∂D^q to find a uniform bound for the values of $a_i(t)$. However, the $a_i(t)$ a priori only vary continuously over the set where $w_i(t)$ are non-zero. Hence, we modify the weights by an affine map from $[0, \max\{w_i(t)\}$ to $[-1, \max\{w_i(t)\}$, discarding all negative weights, and rescaling so that the sum is equal to 1 once more. Now the values of $a_i(t)$ are bounded by compactness, so we can translate so that the barycentre $\sum w_i(t)a_i(t)$ is zero for all $t \in \partial D^q$, and rescale the first coordinate of $\mathbb{R}^\infty$ so that all $a_i(t)$ lie in $[-1, 1]$. Damping off the translation and scaling towards the centre of D^q, these operations can be extended to deform f as well as g.

Next, we will make sure that, for any $t \in D^q$, $S_t \cap S[-1, 1]$ lies in the path component of the base B in S_t (not neccessarily via paths in $S_t \cap S[-1, 1]$). We will first fix t and choose $a \leq -1$, $b \geq 1$ such that both $S(a)$ and $S(b)$ are transverse to S_t. For any path component C of $S_t \cap S[a, b]$ not in the base component of S_t, we can choose one small disc in C and one near the base B, and deform S_t by first pulling both discs to $-\infty$ in the first coordinate and then dragging them back joined together as a tube such that the orientations match. Since our embedding space is $\mathbb{R}^\infty$, this can be done for all components C, ensuring $S_t \cap S[-1, 1] \subset S_t \cap S[a, b]$ is in the base component.

To do this for all $t \in D^q$, use compactness of D^q to choose a finite open cover $\{U_i\}_{i=1}^n$ of D^q with corresponding $a_i \leq -1$ and $b_i \geq 1$ such that $S(a_i)$ and $S(b_i)$ are transverse to S_t for $t \in U_i$. We will inductively choose small discs to connect via the procedure above. The discs close to B can be chosen without any problem, as the surfaces hardly change near B. For the other discs, call them D_{ij}, we can assume by induction we have chosen all D_{kl} for $k < i$. As $S_t \cap S[a_i, b_i]$ varies by isotopy over $\bar{U}_i$, we can assume all D_{kl} for $k < i$ are constant over $\bigcap_{j \leq i} \bar{U}_j$. Hence the D_{ij} can be chosen disjointly over this set, and can be extended to $\bar{U}_i$, again by the isotopy argument.

Having chosen the disjoint discs, we can create the tubes as before, damping the tube-forming deformations to zero on the boundary of the U_i and letting them go through entirely on slightly smaller subsets of U_i that still cover D^q.

On a neighbourhood of ∂D^q, the a_i and b_i may be chosen to be -1 and 1, respectively, and no tubes are necessary here, by existence of the lift g. Choosing the deformation this way, the lift of the deformation of f is still g.

Finally, we want to make each S_t intersect at least one slice $S(a)$ in $S[-1, 1]$ transversely in a single circle. Again, we first do this for fixed $t \in D^q$. Choose a slice $S(a)$ in $S[-1, 1]$ that meets S_t transversely. If the intersection $S_t \cap S(a)$ is a single circle, we are done. If not, denote by C_j the circles in $S_t \cap S(a)$ disjoint from B. For every j, pick points $p_j \in C_j$ and q_j near B, and choose paths α_j connecting them (these paths exists by the previous step). We can assume the p_j, q_j, and α_j are all disjoint. Now, extend the α_j to $-\infty$, parallel to, but not in B from q_j and choose paths β_j from $+\infty$ to $-\infty$ parallel to B and disjoint from each other and from all α_j. Then S can be deformed by moving a thin tube from minus infinity with one end on α_j and the other on β_j, such that their first coordinates are the same at each point of the deformation, see figure 5.2. We can do this for all j at once, without the tubes intersecting, as we are working in $\mathbb{R}^\infty$. This makes $S_t \cap S(a)$ connected. We can make S_t coincide with Z around S_a by an isotopy supported near S_a fixing B.

Varying t in a small neighbourhood, the curves C_j vary by isotopies, and we can choose the α_j and β_j to vary by isotopies as well. Also, we can choose the α_j and β_j to be disjoint for all t in the neighbourhood. Again, by compactness of D^q, we can choose a finite cover $\{U_i\}$ of D^q such that for each i, we have arcs $\alpha_{ij}(t)$ and $\beta_{ij}(t)$ on S_t as before, defined over U_i and disjoint for fixed i and t.

We still need the $\alpha_{ij}(t)$ and $\beta_{ij}(t)$ to be disjoint for varying i as well on intersections of the U_i.

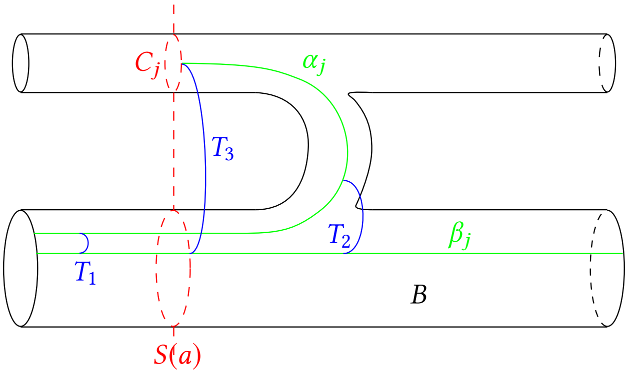


Figure 5.2: Moving a tube (represented in the picture by the blue arcs) along α_j and β_j, via T_1 and T_2 to T_3.

For the $\beta_{ij}(t)$ this is easy, as all of them are parallel to B, so choosing them close to B makes them disjoint from each other, and the $\alpha_{ij}(t)$ as well. The $\alpha_{ij}(t)$ will be made disjoint by induction on i, so assume the $\alpha_{kl}(t)$ are disjoint for $k < i$. If a certain $\alpha_{ij}(t)$ intersects a $\alpha_{kl}(t)$ for all t on which they are both defined, we can pick such a t and push $\alpha_{ij}(t)$ off $\alpha_{kl}(t)$ by moving the intersection towards the end of $\alpha_{kl}(t)$, damping this pushing off in a small neighbourhood of t. Hence, we can assume $\alpha_{ij}(t)$ is disjoint from all $\alpha_{jk}(t)$ for a certain t. If $\alpha_{ij}(t')$ starts intersecting some $\alpha_{kl}(t')$ near t, this can happen in three ways:

- a loop of α_{ij} is pulled over the interior of α_{kl}, making two intersection points. This can be pushed off again quite easily by a deformation of α_{ij};
- the endpoint of α_{kl} intersects α_{ij}. This can be avoided by looping α_{ij} around that point;
- the endpoint of α_{ij} intersects the interior of α_{kl}. This time, α_{kl} can be looped around the point, making sure no new intersections occur.

Iterating this procudure and using induction, we can choose all $\alpha_{ij}(t)$ to be disjoint.

Having constructed the $\alpha_{ij}(t)$ and $\beta_{ij}(t)$, we can use them to introduce the tubes T_{ij} over U_i, as above. As before, we damp the deformations introducing the tubes near the boundaries of the U_i. However, this damping may introduce new problems, as moving a certain tube T_{ij} through a slice $S(a_k)$ of another U_k may make the intersection $S_t \cap S(a_k)$ disconnected. To remedy this, let the tubes grow thinner in the damping off part, where they move to $-\infty$, than in the part where they are stationary around $S(a_i)$. As a tube T_{ij} moves along the slice $S(a_k)$ it will then never completely cover any of the tubes T_{kl}, so we can replace the slice $S(a_k)$ by two new slices, which will never be covered simultaneously, see figure 5.3.

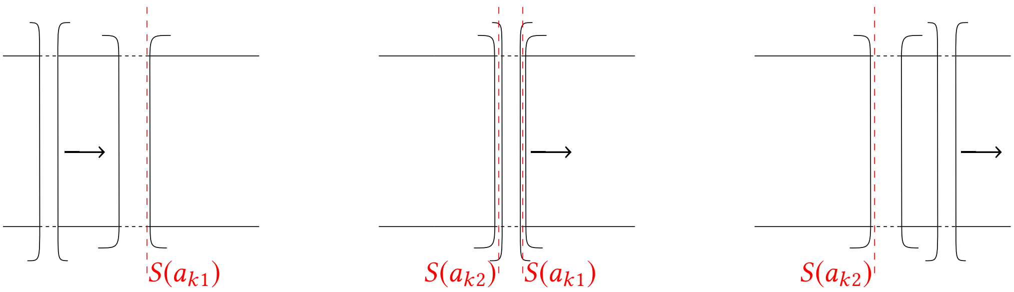


Figure 5.3: Moving a tube in front of another. The weight of the slice $S(a_k)$ is divided by partition of unity into weights for $S(a_{k1})$ and $S(a_{k2})$, such that the weights are zero when the tube moves through the slice.

Having dealt with this last obstruction, we have now arranged for the surfaces S_t to meet certain slices in single circles with given weights assigned by a partition of unity. The slabs between the slices are connected, since inserting the extra tubes of the last step does not destroy the connectedness created before. This gives a homotopy from $f : D^q \to C_b^{\infty,1}$ to $\tilde{f}$ which has a lift $\tilde{g} : D^q \to C_s^{\infty,1}$, extending the induced homotopy of $g : \partial D^q \to C_s^{\infty,1}$, proving τ is a weak homotopy equivalence.

Finally, there is a homotopy $\sigma\rho \sim i\tau$ by expanding (b^-, b^+) to $\mathbb{R}$ (in the definition of σ), proving the lemma. □

5.2 — The calculation of the rational cohomology algebra

Now we have proven the Madsen-Weiss theorem, we can use this to calculate the rational cohomology of the stable mapping class group Γ_∞: it is the same as that of the infinite loop space $\Omega_0^\infty \mathbb{CP}_{-1}^\infty$. This cohomology is calculated in the following theorem, proven along the outline given in appendix C of [Hat14].

Theorem 5.2.1. *The rational cohomology ring $H^*(\Omega_0^\infty \mathbb{CP}_{-1}^\infty; \mathbb{Q})$ is a polynomial algebra on infinitely many generators, $\mathbb{Q}[c_1, c_2, c_3, \dots]$, where each c_i has degree $2i$.*

This theorem hence gives a positive answer to Mumford's conjecture, mentioned in [Mum83], which stated that the stable rational cohomology of $\mathcal{M}_g$, which is equal to that of Γ_g by corollary 1.3.9, is given by the same ring.

Proof. As $\gamma_{2,n}^\perp$ is an oriented vector bundle over $G^+(2,n)$, there are, for all i, Thom isomorphisms $\tilde{H}^i(\mathrm{Th}(\gamma_{2,n}^\perp)) \cong H^{i-(n-2)}(G^+(2,n))$.

Letting $V^+(2,n)$ be the oriented Stiefel manifold, there is a fibration $S^{n-2} \to V^+(2,n) \to S^{n-1}$, so $V^+(2,n)$ is $(n-3)$-connected. By the fibration $S^1 \to V^+(2,n) \to G^+(2,n)$, this shows that $G^+(2,n)$ has homotopy $\mathbb{Z}$ in degree 2 and 0 for $i < n-3$ otherwise. So it is a '$K(\mathbb{Z},2)$ up to degree $n-3$'. By the same argument, $V^+(2,\infty)$ is weakly contractible, and $G^+(2,\infty)$ is an actual $K(\mathbb{Z},2)$. The inclusion $G^+(2,n) \hookrightarrow G^+(2,\infty)$ is therefore a homology isomorphism up to degree $n-3$. As $H^*(G^+(2,\infty)) = \mathbb{Z}[e]$, where $e \in H^2(G^+(2,\infty))$ is the Euler class, the powers of e restrict to an additive basis of $H^*(G^+(2,n)$ up to degree $n-3$. By theorem A.2.4 and the Thom isomorphisms, these powers give a map

$$f : \mathrm{Th}(\gamma_{2,n}^\perp) \to K(\mathbb{Z}, n-2) \times K(\mathbb{Z}, n) \times \cdots \times K\big(\mathbb{Z}, n + 2\lfloor \frac{n-5}{2} \rfloor\big)$$

Because $H^*(K(\mathbb{Z},2k);\mathbb{Q}) \cong \mathbb{Q}(x_{2k})$ and $H^*(K(\mathbb{Z},2k+1)) \cong \Lambda_{\mathbb{Q}}[x_{2k+1}]$ with x_j in degree j by proposition B.2.16, f is an isomorphism on rational cohomology in dimensions up to at least $2n-5$, as the first non-trivial product on the right can occur in degree $2n-4$.

Because both spaces are simply connected, we get a map $f_{\mathbb{Q}}$ on $\mathbb{Q}$-localisations of the spaces, cf. [Sul70], which is an isomorphism on integral homology up to degree $2n-5$, as integral and rational homology coincide for $\mathbb{Q}$-localised spaces. By the relative Hurewicz theorem, $f_{\mathbb{Q}}$ is also an isomorphism on homotopy up to degree $2n-5$. As these homotopy groups are exactly the rational homotopy groups of the source and target of f, it follows that f is an isomorphism on rational homotopy up to degree $2n-5$.

Applying the loop space functor n times, the factor $K(\mathbb{Z}, n-2)$ disappears. Restricting to the pointed component, the factor $K(\mathbb{Z}, n)$ disappears as well, and we get a map

$$\Omega_0^n f : \Omega_0^n \,\mathrm{Th}(\gamma_{2,n}^\perp) \to K(\mathbb{Z},2) \times K(\mathbb{Z},4) \times \cdots \times K\big(\mathbb{Z}, 2\lfloor \frac{n-5}{2} \rfloor\big)$$

which is an isomorphism on rational homotopy up to degree $n-5$.

By the same $\mathbb{Q}$-localisation argument as before, using that the left space is an H-space, and therefore has abelian fundamental group with trivial action on higher homotopy so that $\mathbb{Q}$-localisation exists, we get that $\Omega_0^n f$ is an isomorphism on rational (co)homology up to degree $n-5$ as well. By the Künneth formula, $H^*(\Omega_0^n \operatorname{Th}(\gamma_{2,n}^\perp); \mathbb{Q}) \cong \mathbb{Q}[c_1, c_2, \dots]$ up to degree $n-5$.

To prove the theorem, all that is left to prove is that this isomorphism behaves well with respect to the suspension maps of the spectrum $\mathbb{CP}_{-1}^\infty$. They will turn out to stabilise in a wide enough range of dimensions.

First consider the map $\operatorname{Th}(\gamma_{2,n}^\perp) \to \Omega \operatorname{Th}(\gamma_{2,n+1}^\perp)$. It is the transpose of the map $S \operatorname{Th}(\gamma_{2,n}^\perp) \to \operatorname{Th}(\gamma_{2,n+1}^\perp)$ under the adjunction $\Sigma \dashv \Omega$ and therefore is given as a composition

$$\operatorname{Th}(\gamma_{2,n}^\perp) \to \Omega\Sigma \operatorname{Th}(\gamma_{2,n}^\perp) \to \Omega \operatorname{Th}(\gamma_{2,n+1}^\perp)$$

By the first part of this proof, $\operatorname{Th}(\gamma_{2,n}^\perp)$ is $n-1$-connected, so by the Freudenthal suspension theorem and the shifting of homotopy under Ω, the first map is an isomorphism in homotopy up to degree $2n-2$.

For the second map, using that the pair $(G^+(2, n+1), G^+(2, n))$ is $(n-3)$-connected via their common inclusion in $G^+(2, \infty)$, the Thom isomorphism shows

$$\begin{aligned} \tilde{H}^i(\Sigma \operatorname{Th}(\gamma_{2,n}^\perp)) &\cong \tilde{H}^i(\operatorname{Th}(\gamma_{2,n}^\perp \times \varepsilon) \\ &\cong H^{i-(n-1)}(G^+(2,n)) \\ &\cong H^{i-(n-1)}(G^+(2,n+1)) && \text{for } i-(n-1) \leq n-3 \\ &\cong \tilde{H}^i(\operatorname{Th}(\gamma_{2,n+1}^\perp)) \end{aligned}$$

so for $i \leq 2n-4$. As both spaces are finite CW-complexes, this is an isomorphism for homology and homotopy as well. Looping loses one degree of the range, so $\Omega\Sigma \operatorname{Th}(\gamma_{2,n}^\perp) \to \Omega \operatorname{Th}(\gamma_{2,n+1}^\perp)$ is an isomorphism in homotopy up to degee $2n-5$.

Combining the maps, $\operatorname{Th}(\gamma_{2,n}^\perp) \to \Omega \operatorname{Th}(\gamma_{2,n+1}^\perp)$ is an isomorphism in homotopy up to degree $2n-5$, so $\Omega_0^n \operatorname{Th}(\gamma_{2,n}^\perp) \to \Omega_0^{n+1} \operatorname{Th}(\gamma_{2,n+1}^\perp)$ is an isomorphism in homotopy, and hence in (co)homology, up to degree $n-5$, proving the theorem. □

Corollary 5.2.2. *The rational cohomology of Γ_∞ is a polynomial algebra $\mathbb{Q}[c_1, c_2, \dots]$, with $c_i \in H^{2i}(\Gamma_\infty; \mathbb{Q})$*

Proof. This follows from the previous theorem with the Madsen-Weiss theorem. □

The final positive answer to Mumford's conjecture is given in the next corollary.

Corollary 5.2.3. *The rational cohomology of the moduli space $\mathcal{M}_{g,r}$, $H^*(\mathcal{M}_{g,r}; \mathbb{Q})$ is, in low degrees, a polynomial algebra on one generator in each even degree. The range in which this holds is $* \leq \frac{2g-2}{3}$.*

Proof. This follows from the previous corollary with Harer stability, theorems 2.3 and 2.4 and corollary 1.3.9. □

Appendix A — Classifying spaces and group (co)homology

In this appendix, we will give some basic results on classifying spaces of topological monoids and, in particular, topological groups. The results are well-known, at can be found in e.g. [May99].

A.1 — Classifying spaces

We will start by giving some basic definitions, according to the Eilenberg-MacLane construction.

Definition A.1.1. Let $C = (O, M, (1 : O \to M), (s, t : M \to O))$ be a topological category. Define its *nerve* B_*C to be the simplicial space with $B_nC = M \;{}_s\!\times_t M \;{}_s\!\times_t \cdots {}_s\!\times_t M$ (n-fold fibre product), and face and degeneracies given by

$$\partial_n^i : B_nC \to B_{n-1}C : (f_1, \dots, f_n) \mapsto \begin{cases} (f_2, \dots, f_n) & i = 0 \\ (f_1, \dots, f_i \circ f_{i+1}, \dots, f_n) & 1 \le i \le n-1 \\ (f_1, \dots, f_{n-1}) & i = n \end{cases}$$
$$s_n^i : B_nC \to B_{n+1}C : (f_1, \dots, f_n) \mapsto (f_1, \dots, f_i, 1_{s(f_i)}, f_{i+1}, \dots, f_n)$$

The *classifying space* of C is then defined as its geometric realisation, $BC = |B_*C|$.

The nerve and classifying space constructions extend to functors $B_* : \mathrm{Sp(Cat)} \to [\Delta^{\mathrm{op}}, \mathrm{Sp}]$ and $B : \mathrm{Sp(Cat)} \to \mathrm{Sp}$ by defining morphisms componentwise.

Remark A.1.2. This definition is mostly used in the cases that C is a monoid, i.e. a one-object category, or in the case C is the poset of inclusions of finite intersections of a good open cover of a space X, where good means that all finite intersections must be contractible. In this last case, BC is known to be homotopy equivalent to X.

Definition A.1.3. Given a monoid M and a right action of M on a space X, we get a topological category with object space X, morphism space $X \times M$, identity $1 : x \mapsto (x, e)$, and source and target maps $s : (x, m) \mapsto x$ and $t : (x, m) \mapsto xm$. Define the *Borel construction* E_MX to be the classifying space of this category. Its face and degeneracy maps are given as follows:

$$\partial_n^i : E_n(M, X) \to E_{n-1}(M, X) : (x, m_1, \dots, m_n) \mapsto \begin{cases} (xm_1, m_2, \dots, m_n) & i = 0 \\ (x, m_1, \dots, m_i m_{i+1}, \dots, m_n) & 1 \le i \le n-1 \\ (x, m_1, \dots, m_{n-1}) & i = n \end{cases}$$
$$s_n^i : E_n(M, X) \to E_{n+1}(M, X) : (x, m_1, \dots, m_n) \mapsto (x, m_1, \dots, m_i, e, m_{i+1}, \dots, m_n)$$

If $X = M$, write $EM := E_MM$.

The assignment $E : M \mapsto EM$ extends to a functor $\mathrm{Sp(Mon)} \to [\Delta^{\mathrm{op}}, \mathrm{Sp}] \to \mathrm{Sp}$ by defining $E(f)$ componentwise, for $f \in \mathrm{Mor(Sp(Mon))}$.

Lemma A.1.4. *The nerve functor preserves finite products. Hence, the classifying space functor does too.*

Proof. Given two topological categories $\mathcal{C} = (M, O)$ and $\mathcal{D} = (N, P)$, there are obvious shuffle homeomorphisms

$$B_n(\mathcal{C} \times \mathcal{D}) = (M \times N) \ {}_s\!\times_t \cdots {}_s\!\times_t \ (M \times N) \cong (M \ {}_s\!\times_t \cdots {}_s\!\times_t \ M) \times (N \ {}_s\!\times_t \cdots {}_s\!\times_t \ N) = B_n\mathcal{C} \times B_n\mathcal{D}$$

As composition in $\mathcal{C} \times \mathcal{D}$ is defined componentwise, these homeomorphisms clearly commute with the faces and degeneracies, showing the nerve preserves products. As geometric realisation preserves products, so does B. □

Lemma A.1.5. *For any topological monoid, the space EM is contractible.*

Proof. A contraction to the vertex e is given, for any p-simplex $(m, m_1, \ldots, m_n) \times \Delta^p$, by contracting linearly through the $(p+1)$-simplex $(e, m, m_1, \ldots, m_n) \times \Delta^{p+1}$. □

Proposition A.1.6. *If G is a topological group, there is a principal fibre bundle $G \longrightarrow EG \longrightarrow BG$.*

Proof. The natural action of G on EG on the left, given by $G \times EG \longrightarrow EG : (g', (g, g_1, \ldots, g_n)) \mapsto (g'g, g_1, \ldots, g_n)$ is free and the quotient EG/G is homeomorphic to BG by projecting away from the first coordinate. □

Corollary A.1.7. *For a topological group G, there is a weak homotopy equivalence $G \longrightarrow \Omega BG$.*

Proof. There is a map of fibrations

$$\begin{array}{ccc} EG & \longrightarrow & PBG \\ & \searrow \quad \swarrow & \\ & BG & \end{array}$$

sending a point in EG to the projection of the path traced out by the point under the contraction of the proof of lemma A.1.5. By the Serre exact sequence and the Five Lemma, $\pi_i(G) \cong \pi_i(\Omega BG)$ for all i. □

Proposition A.1.8. *The fibre bundle $p_G : EG \longrightarrow BG$ is universal for principal G-bundles, i.e. there is a one-to-one correspondence between isomorphism classes of principal G-bundles over a space X and $[X, BG]$.*

Proof. The idea of this proof is derived from [Seg68].

Given a map $f : X \longrightarrow BG$, we get the pullback square

$$\begin{array}{ccc} f^*EG & \longrightarrow & EG \\ \downarrow p & \lrcorner & \downarrow p_G \\ X & \xrightarrow[f]{} & BG \end{array}$$

As pullbacks of fiber bundles along homotopic are isomorphic, this gives a well-defined map from $[X, BG]$ to principal G-bundles over X.

Conversely, let $E \longrightarrow X$ be a principal G-bundle. Take a good open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ of X, closed under finite intersections, over which the bundle is trivial. As in the remark under definition A.1.1, this is a category. Considering G as a one-object category, the choice of trivialisations $E|_{U_\alpha} \cong U_\alpha \times G$ induces a functor $F : \mathfrak{U} \longrightarrow G$ by sending an inclusion $U_\alpha \hookrightarrow U_\beta$ to the associated transition function (which is an element of G). Then the required map $X \cong B\mathfrak{U} \longrightarrow BG$ is BF.

These two constructions are clearly inverse to each other. □

Remark A.1.9. Because of the above propositions, any space which is the quotient of a contractible space by a free action of a group G is called a classifying space for G, and similarly for any space classifying principal G-bundles in the sense of proposition A.1.8. All these possible spaces are homotopy equivalent.

A.2 — Eilenberg-Maclane spaces

Definition A.2.1. An *Eilenberg-MacLane space* is a space with exactly one non-trival homotopy group. If this group is G and it occurs as the n-th homotopy group, the space is denoted $K(G, n)$.

Proposition A.2.2. *For a discrete group G, BG is a $K(G, 1)$.*

Proof. The Serre exact sequence for the fibration $G \longrightarrow EG \longrightarrow BG$ of proposition A.1.6 together with lemma A.1.5 yields $\pi_i BG \cong \pi_{i-1} G$. As G is discrete, $\pi_0 G = G$ and $\pi_i G = 0$ for $i > 0$. □

Corollary A.2.3. *If G is abelian, the group structure of G induces an abelian group structure on BG. Hence, $B^n G = B \cdots BG$ is a $K(G, n)$ for abelian discrete G and $n \geq 2$.*

Proof. Using lemma A.1.4, there is an isomorphism $BG \times BG \cong B(G \times G)$. As G is abelian, multiplication $m : G \times G \longrightarrow$ is a group homomorphism and hence induces $Bm : BG \times BG \cong B(G \times G) \longrightarrow BG$ with obvious identity $*$, the unique vertex. Again by commutativity, the inverse map $i : G \longrightarrow G$ is a group homomorphism and induces $Bi : BG \longrightarrow BG$. As equalities are preserved by any functor, $(BG, *, Bm, Bi)$ is again an abelian group.

Using the same argument as in proposition A.2.2, $\pi_i(B^n G) \cong \pi_{i-n}(G)$, so $B^n G$ is a $K(G, n)$ for discrete abelian G. □

Theorem A.2.4. *Eilenberg-MacLane spaces represent reduced cohomology in* hoCW_*, *the homotopy category of CW-complexes pointed by a vertex. In other words, for every n, there are isomorphisms $\tilde{H}^n(X; G) \cong [X, K(G, n)]$, natural in X and G. The abelian group structure is the one induced by that of $K(G, n)$.*

Proof. This will be proven by checking the Eilenberg-Steenrod axioms for reduced cohomology (definition 4.2.1, cf. [May99]) for the functors $X \longrightarrow [X, K(G, n)]$.

Dimension By definition, $[S^0, K(G, n)] =: \pi_0(K(G, n))$, which is G if $n = 0$ and $\{0\}$ otherwise.

Exactness For a pointed pair (X, A), the sequence

$$[X/A, K(G, n)] \longrightarrow [X, K(G, n)] \longrightarrow [A, K(G, n)]$$

is exact by definition of X/A.

Suspension Combining corollaries A.2.3 and A.1.7, we get

$$[X, K(G, n)] \cong [X, \Omega BK(G, n)] \cong [X, \Omega K(G, n+1)] \cong [\Sigma X, K(G, n+1)]$$

Additivity The statement of additivity is $[\bigvee_i X_i, K(G, n)] \cong \prod_i [X_i, K(G, n)]$, which is obvious as any pointed map from a wedge is defined by maps from the components.

□

Corollary A.2.5. *There are natural isomorphisms $H^n(X; G) \cong [X_+, K(G, n)]$.*

Proof. This is the equivalence between the Eilenberg-Steenrod axioms for cohomology on a topological category and their reduced analogues for the pointed category, which were proven in theorem A.2.4. See e.g. [May99]. □

A.3 — Group (co)homology

For a topological group, we would like to have a notion of (co)homology that incorporates both the group structure and the topology. An appropriate definition is given as follows:

Definition A.3.1. Given a topological group G and a topological G-module M, its *group homology* $H_*(G;M)$ is defined to be the homology of its classifying space $H_*(BG;M)$ with coefficients in the coefficient system induced by the module M. Dually, its *group cohomology* is the cohomology of BG.

In the case G is discrete, this definition seems somewhat awkward: we would like to have a purely algebraic description of group (co)homology, as these are algebraic objects associated to other algebraic objects. Such a description is given in the next propositions. For discrete group (co)homology, a reference would be [Wei94].

Lemma A.3.2 (Bar resolution)**.** *For a discrete group G, the simplicial chain complex $(C_\bullet(E_*G), \partial)$ (sometimes just written $E_\bullet G$) is a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$ and is given by*

$$\cdots \xrightarrow{\partial} \mathbb{Z}G^{\otimes 3} \xrightarrow{\partial} \mathbb{Z}G^{\otimes 2} \xrightarrow{\partial} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z}$$

Here the tensor products are over $\mathbb{Z}$. Identifying $\mathbb{Z}G^{\otimes k} \cong (\mathbb{Z}G)\cdot G^{k-1}$, the free $\mathbb{Z}G$-module on basis G^{k-1}, as left-$\mathbb{Z}G$-modules and writing $\partial = \sum_{i=0}^{k}(-1)^i\partial^i$ and the maps are given on generators by

$$\varepsilon : \mathbb{Z}G \to \mathbb{Z} : g \mapsto 1$$

$$\partial^i : (\mathbb{Z}G)\cdot G^k \to (\mathbb{Z}G)\cdot G^{k-1} : (g_1,\ldots,g_k) \mapsto \begin{cases} g_1(g_2,\ldots,g_k) & i=0 \\ (g_1,\ldots,g_ig_{i+1},\ldots,g_k) & 1\le i\le k-1 \\ (g_1,\ldots,g_{k-1}) & i=k \end{cases}$$

Proof. That this is indeed the complex is direct from definition A.1.3. It is a resolution because its homology is zero by lemma A.1.5. □

Proposition A.3.3. *For a discrete group G and a G-module M, there are natural isomorphisms*

$$H_*(G;M) \cong \mathrm{Tor}_*^{\mathbb{Z}G}(\mathbb{Z},M) \qquad \text{and} \qquad H^*(G;M) \cong \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z},M)$$

Proof. By lemma A.3.2, $(C_\bullet(E_*G),\partial)$ is a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$. Also, as can be seen from the definition, $C_\bullet(E_*G)\otimes_{\mathbb{Z}G}\mathbb{Z} = C_\bullet(B_*G)$. Combining this yields

$$\begin{aligned} H_*(G;M) &:= H_*(BG;M) = H_*(C_\bullet(B_*G)\otimes_{\mathbb{Z}} M) \\ &= H_*(C_\bullet(E_*G)\otimes_{\mathbb{Z}G}\mathbb{Z}\otimes_{\mathbb{Z}} M) \\ &= H_*(C_\bullet(E_*G)\otimes_{\mathbb{Z}G} M) =: \mathrm{Tor}_*^{\mathbb{Z}G}(\mathbb{Z},M) \end{aligned}$$

The proof that $H^*(G;M)\cong \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z},M)$ is completely dual. □

Remark A.3.4. Besides the theoretical result that $H_*(G;M)\cong \mathrm{Tor}_*^{\mathbb{Z}G}(\mathbb{Z},M)$, the Bar resolution is also of great import for explicit calculations.

Definition A.3.5. For a group G and a G-module M, define the *invariant subgroup* to be $M^G = \{m \in M \mid gm = m\ \forall g\in G\}$.

Define the coinvariant quotient to be $M_G = M/\langle gm - m \mid g\in G, m\in M\rangle$.

Proposition A.3.6. *If G is a discrete group and M a G-module, group homology computes the left derived functors of the coinvariant quotient functor: $L_*(-_G)(M) = H_*(G;M)$. Similarly, group cohomology computes the right derived functors of the invariant subgroup functor $R^*(-^G)(M) = H^*(G;M)$.*

Proof. The coinvariant assignment $(-)_G$ is a functor $\mathbb{Z}G$-Mod $\to$ $\mathbb{Z}$-Mod $=$ Ab, left adjoint to the trivial module functor T, sending an abelian group A to the $\mathbb{Z}G$-module on which each $g \in G$ acts trivially:

$$\mathrm{Ab}(M_G, A) \cong \mathbb{Z}G\text{-Mod}(M, TA)$$

because all $\mathbb{Z}G$-morphisms $\varphi : M \to TA$ must factor through M_G. But since $\mathbb{Z}$ represents the forgetful functor Ab $\to$ Set, T is given by $\mathrm{Hom}_{\mathbb{Z}}(T\mathbb{Z}, -)$. So we have

$$(-)_G \dashv T = \mathrm{Hom}_{\mathbb{Z}}(T\mathbb{Z}, -) \vdash \mathbb{Z} \otimes_{\mathbb{Z}G} (-)$$

proving that $(-)_G \cong \mathbb{Z} \otimes_{\mathbb{Z}G} (-)$, where $\mathbb{Z} = T\mathbb{Z}$ is the trivial $\mathbb{Z}G$-module. Taking derived functors and using proposition A.3.3 shows $L_*(-_G)(M) \cong L_*(\mathbb{Z} \otimes_{\mathbb{Z}G} (-))(M) = \mathrm{Tor}_*^{\mathbb{Z}G}(\mathbb{Z}, M) \cong H_*(G; M)$.

The invariant subgroup assignment $(-)^G$ is a functor $\mathbb{Z}G$-Mod $\to$ Ab. It is right adjoint to T:

$$\mathrm{Ab}(A, M^G) \cong \mathbb{Z}G\text{-Mod}(TA, M)$$

because the image of TA under any $\mathbb{Z}G$-morphism $\varphi : TA \to M$ must be G-invariant, hence in M^G. So $M^G \cong \mathrm{Ab}(\mathbb{Z}, M^G) \cong \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, M)$, where T is again suppressed. Taking derived functors and using proposition A.3.3 again, we get $R^*(-^G)(M) \cong R^*(\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, -))(M) = \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, M) \cong H^*(G; M)$. □

A.4 — Shapiro's lemma

Shapiro's lemma is a result in group homology, relating homology of groups to that of their subgroups. It is often a very convenient tool in computations using spectral sequence. We use it as such as well.

Lemma A.4.1 (Shapiro's lemma). *Let $H \subset G$ be groups and M a $\mathbb{Z}H$-module. Then*

$$H_*(G; \mathbb{Z}G \otimes_{\mathbb{Z}H} M) \cong H_*(H; M)$$

Proof. As $\mathbb{Z}G$ is a free $\mathbb{Z}H$-module, a projective $\mathbb{Z}G$-resolution $P_\bullet \to \mathbb{Z}$ is also projective as a $\mathbb{Z}H$-resolution. Taking the homology of the complex

$$P_\bullet \otimes_{\mathbb{Z}G} (\mathbb{Z}G \otimes_{\mathbb{Z}H} M) \cong P_\bullet \otimes_{\mathbb{Z}H} M$$

we get

$$\begin{array}{ccc} \mathrm{Tor}_*^{\mathbb{Z}G}(\mathbb{Z}, \mathbb{Z}G \otimes_{\mathbb{Z}H} M) & \xrightarrow{\cong} & \mathrm{Tor}_*^{\mathbb{Z}H}(\mathbb{Z}, M) \\ \updownarrow \cong & & \updownarrow \cong \\ H_*(G; \mathbb{Z}G \otimes_{\mathbb{Z}H} M) & & H_*(H; M) \end{array}$$

□

Corollary A.4.2. *Let G be a group acting transitively on a set X. Then*

$$H_*(X \times_G EG) = H_*(\mathbb{Z}X \otimes_G E_\bullet G) \cong H_*(\mathrm{Stab}(x))$$

Here, $\mathbb{Z}X$ is the free abelian group on elements of X, $E_\bullet G$ is the complex of simplicial chains of EG, which is a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$ (see lemma A.3.2), and $\mathrm{Stab}(x)$ is the stabiliser in G of any $x \in X$.

Proof. By transitivity, $\mathbb{Z}X \cong \mathbb{Z}G \otimes_{\mathrm{Stab}(x)} \mathbb{Z}$. Hence, using Shapiro's lemma A.4.1,

$$H_*(\mathrm{Stab}(x); \mathbb{Z}) \cong H_*(G; \mathbb{Z}G \otimes_{\mathrm{Stab}(x)} \mathbb{Z}) \cong H_*(G; \mathbb{Z}X) := H_q(\mathbb{Z}X \otimes_G E_\bullet G)$$

□

Corollary A.4.3. *Suppose G and H are groups acting transitively on sets X and Y, respectively. Let $\varphi : G \to H$ be a homomorphism and $f : X \to Y$ a map such that $f(gx) = \varphi(g)f(x)$. Then, with the same notation as above,*

$$H_*(\mathbb{Z}Y \otimes_H E_\bullet H, \mathbb{Z}X \otimes_G E_\bullet G) \cong H_*(\mathrm{Stab}(f(x)), \mathrm{Stab}(x))$$

Proof. This is just the relative version of corollary A.4.2. □

Corollary A.4.4. *Let G be a topological group acting transitively on a space X. Then, for any $x \in X$,*

$$H_*(X \times_G EG) \cong H_*(\mathrm{Stab}(x))$$

Proof. This is just the continuous version of corollary A.4.2. □

Appendix B — Spectral sequences

A spectral sequence is a certain tool for computing homology and cohomology. It is ubiquitous in homology theory, and comes in many different guises. But in essence all spectral sequences have the same idea: they approximate the homology of a certain object (be it a space, a group, a functor, or something else) by dividing that object into parts, computing the homologies of the separate parts, and glueing them together using more homologies.

Notable examples are the spectral sequence of a filtered complex, computing its total homology by first computing all homologies of the graded complex and then using their interrelation, or the Grothendieck spectral sequence, computing the derived functor of a composition of functors from the derived functors of the composed functors.

This Grothendieck spectral sequence is perhaps the most general and powerful spectral sequence of all, and many others are special cases of this one. However, to prove it, we first need to define spectral sequences and construct some other examples.

Spectral sequences were developed first by Leray as a prisoner of war during World War II, and first published in [Ler46], for computing sheaf cohomology. Koszul[Kos47] generalised the notion to an algebraic setting.

Most of the material of this section comes from Weibel's book on homological algebra[Wei94].

B.1 — Definitions

A spectral sequence consists of a lot of data, and it is hard to graps the full power (or use, for that matter) of the subject entirely. However, to see this use, one needs to go through the basic definitions first. These are given here.

Definition B.1.1. A *homology spectral sequence* in an abelian category $\mathcal{A}$ consists of the following data:

- a non-negative integer a;
- for each $p, q, r \in \mathbb{Z}$ with $r \geq a$, an object E^r_{pq} of $\mathcal{A}$;
- differentials $d^r_{pq} : E^r_{pq} \longrightarrow E^r_{p-r,q+r-1}$ such that $d^r \circ d^r = 0$;
- isomorphisms $\alpha^r_{pq} : \ker(d^r_{pq})/\operatorname{Im}(d^r_{p+r,q-r+1}) \xrightarrow{\sim} E^{r+1}_{pq}$.

The collections $E^r_{\bullet\bullet}$ are called *sheets* or *pages* and are usually represented in a lattice with p and q the horizontal and vertical coordinates, respectively. The differentials divide the sheets into complexes of slope $-\frac{r}{r+1}$.

The *total degree* of E^r_{pq} is $n = p + q$. All differentials are of total degree -1.

A *morphism of spectral sequences* $f : E \longrightarrow E'$ is a collection of morphisms $f^r_{pq} : E^r_{pq} \longrightarrow E'^r_{pq}$ for $r \geq a'' \geq \max\{a, a'\}$, natural with respect to the differentials and the isomorpisms α and α'.

Remark B.1.2. There exists a dual notion of *cohomology spectral sequence* $(a, E^{pq}_r, d^{pq}_r, \alpha^{pq}_r)$, defined by letting the differentials go the other way. As the entire theory is the same up to re-indexing, we will mostly consider homology spectral sequences here.

Lemma B.1.3 (Mapping lemma). *Let $f : E \to E'$ be a morphism of spectral sequences such that for a certain r, all maps $f^r_{pq} : E^r_{pq} \to E'^r_{pq}$ are isomorphisms. Then all maps f^s_{pq} for $s \geq r$ are isomorphisms as well.*

Proof. Using the Five Lemma, this is immediate. □

Definition B.1.4. A spectral sequence $\{E^r_{pq}\}$ *collapses* at sheet r if at that sheet only one row or column contains non-zero objects.

It *degenerates* at sheet r if all d^r are zero. This implies all later sheets are equal.

As with usual sequences, spectral sequences have a notion of convergence. However, this is not at all straightforward, and there are many different kinds of convergence. Terminology is not uniform, so we will stick to that of [Wei94].

Definition B.1.5. A spectral sequence E is *bounded* if at each sheet there are only finitely many non-zero objects of each total degree.

It is *bounded below* if for each n there is a $p(n)$ such that for $p < p(n)$, $E^a_{p,n-p} = 0$.

A *first quadrant spectral sequence* has non-zero objects only for $p, q \geq 0$.

A spectral sequence is *regular* if for all p, q there exists an $r(p, q)$ such that $d^r_{pq} = 0$ for all $r \geq r(p, q)$.

By definition, any E^{r+1}_{pq} is (isomorphic to) a subquotient of E^r_{pq}, and hence by induction a subquotient of E^a_{pq}. Therefore, there is an infinite sequence of inclusions

$$0 = B^a_{pq} \subseteq B^{a+1}_{pq} \subseteq \cdots \subseteq B^r_{pq} \subseteq \cdots \subseteq Z^r_{pq} \subseteq \cdots \subseteq Z^{a+1}_{pq} \subseteq Z^a_{pq} = E^a_{pq}$$

such that $E^r_{pq} \cong Z^r_{pq}/B^r_{pq}$.

Definition B.1.6. Define $B^\infty_{pq} := \bigcup_r B^r_{pq}$ and $Z^\infty_{pq} := \bigcap_r Z^r_{pq}$, if they exist—which is always the case in a category of modules over a ring. Clearly, $B^\infty_{pq} \subseteq Z^\infty_{pq}$.

The ∞-*sheet* is then given by the terms $E^\infty_{pq} := Z^\infty_{pq}/B^\infty_{pq}$.

Definition B.1.7. A spectral sequence *converges weakly* to a series of objects in $\mathcal{A}$, $H_* = (H_n)_{n\in\mathbb{Z}}$, if each H_n is filtered and there are isomorphisms $\beta_{pq} : E^\infty_{pq} \xrightarrow{\sim} F_pH_{p+q}/F_{p-1}H_{p+q}$.

It *approaches* H_* if also $H_n = \bigcup F_pH_n$ and $0 = \bigcap F_pH_n$.

If a spectral sequence is regular, approaches H_*, and $H_n = \varprojlim H_n/F_pH_n$, it is said to *converge*.

If E and E' converge weakly to H_* and H'_*, respectively, and $f : E \to E'$ is a morphism of spectral sequences, a morphism $g : H_* \to H'_*$ is *compatible* with f if it is a collection of morphisms of filtered objects $g : F_pH_n \to F_pH'_n$ and the associated maps of graded objects correspond to f^∞ under β.

The object H_n should be thought of as occupying the entire antidiagonal $p + q = n$.

Remark B.1.8. In the case of a bounded spectral sequence E, the filtration on each H_n is actually finite. Bounded convergence is usually denoted

$$E^a_{pq} \Rightarrow H_{p+q}$$

Remark B.1.9. In particular, if E is a first quadrant spectral sequence, H_n has a filtration of length $n+1$. In this case also, E^∞_{0n} is a subobject of E^a_{0n} and E^∞_{n0} a quotient of E^a_{n0}. Hence, we get maps, called *edge homomophisms*, $H_n \twoheadrightarrow E^\infty_{n0} \hookrightarrow E^a_{n0}$ and $E^a_{0n} \twoheadrightarrow E^\infty_{0n} \hookrightarrow H_n$.

The terms E^r_{0n} and E^r_{n0} are called *fibre* and *base* terms, respectively. See example B.2.15 for the origin of this notation.

Theorem B.1.10 (Comparison theorem). *Let E and E' be spectral sequences converging to H_* and H'_* respectively. Let $f : E \to E'$ be a morphism and $g : H_* \to H'_*$ a map compatible with f. If there is an r such that f^r_{pq} is an isomorphism, then g is an isomorphism.*

Proof. By the mapping lemma, B.1.3, $f^\infty_{pq} : E^\infty_{pq} \to E'^\infty_{pq}$ is an isomorphism.

For all n, p and s, weak convergence gives maps of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_{p-1}H_n/F_sH_n & \longrightarrow & F_pH_n/F_sH_n & \longrightarrow & E^\infty_{p,n-p} & \longrightarrow & 0 \\
 & & \downarrow \bar{g} & & \downarrow \bar{g} & & \cong \downarrow \bar{f} & & \\
0 & \longrightarrow & F_{p-1}H'_n/F_sH'_n & \longrightarrow & F_pH'_n/F_sH'_n & \longrightarrow & E'^\infty_{p,n-p} & \longrightarrow & 0
\end{array}$$

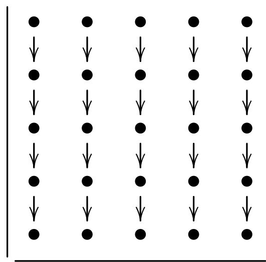
By induction on p and using the Five Lemma, all $\bar{g} : F_pH_n/F_sH_n \to F_pH'_n/F_sH'_n$ are isomorphisms. Taking the direct limit over p, using $H_n = \bigcup_p F_pH_n$, we get isomorphisms $\bar{g} : H_n/F_sH_n \to H'_n/F_sH'_n$. Passing through to the inverse limit over s yields $g : H_n \xrightarrow{\sim} H'_n$ □

B.2 — Examples

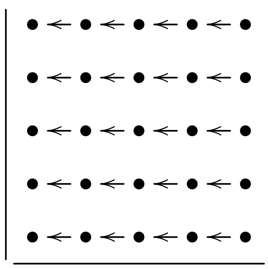
We will give some visualisation of spectral sequences and a large range of examples. Several of these examples occur in the main part of this thesis as well.

B.2.1 — Visualisation of spectral sequences

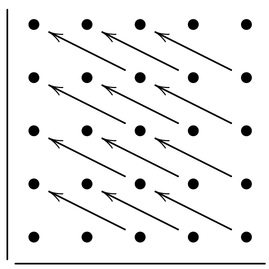
Example B.2.1 (Visualisation). As stated before, the sheets of a spectral sequence should be visualised as two-dimensional lattices. The conventional way to draw these sheets is given in this example. The zeroeth sheet looks like:



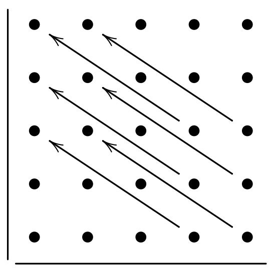
Here the bullets denote the terms and the arrows are the differentials. The next three sheets then have the form:



E^1_{pq}



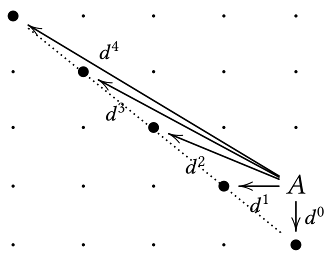
E^2_{pq}



E^3_{pq}

Usually, arguments involving spectral sequences do not explicitly involve sheets higher than the third or fourth.

The sequence of arrows starting at a certain object A can be viewed as follows:



B.2.2 — Filtered and double complexes

Example B.2.2 (Filtered complex)**.** A filtered complex is a complex C with filtrations $\ldots \subset F_pC_q \subset F_{p+1}C_q \subset \ldots$ such that $dF_pC_q \subset F_pC_{q-1}$.

To any filtered complex there is an associated natural spectral sequence, converging to the homology of C under favourable circumstances. We will first define some of these nice conditions on the filtration.

Definition B.2.3. A filtration on C is *bounded* if for all n there are s and t such that $F_sC_n = 0$ and $F_tC_n = C_n$. It is *bounded below* or *bounded above* if only such s or t exists, respectively.

If in the above case $s = -1$ and $t = n$, the filtration is *canonically bounded.*

A filtration is *exhaustive* if $C = \bigcup_p F_pC$. It is *Hausdorff* if $\bigcap_p F_pC = 0$. It is complete if $C = \varprojlim C/F_pC$.

The *completion* of a filtered complex is $\hat{C} = \varprojlim C/F_pC$.

As the construction of the spectral sequence will only use 'finite' filtered parts of the complex, the spectral sequences of C and $\widehat{\bigcup_p F_pC}$ are equal. Hence we will assume the filtration to be exhaustive and complete.

Set $E^0_{pq} = F_pC_{p+q}/F_{p-1}C_{p+q}$. We will construct a tower of subobjects

$$0 = B^0_{pq} \subset \cdots \subset B^r_{pq} \subset B^\infty_{pq} \subset Z^\infty_{pq} \subset \ldots \subset Z^r_{pq} \subset \ldots \subset Z^0_{pq} = E^0_{pq}$$

and show that $E^r_{pq} \coloneqq Z^r_{pq}/B^r_{pq}$ with induced differentials form a spectral sequence.

The idea is to approximate $H_{p+q}(C)$ by letting Z be the inverse image of lower and lower degrees of the filtrations under d and letting B contain more and more of the image of d. To keep notation manageable, we will suppress q and the position in C (q can be retrieved by recalling $E^0_{pq} = F_pC_{p+q}$ and $d : C_n \to C_{n-1}$). So define

$$\begin{aligned}
A^r_p &\coloneqq F_pC \cap d^{-1}F_{p-r}C \\
Z^r_p &\coloneqq A^r_p/(F_{p-1}C \cap A^r_p) = A^r_p/A^{r-1}_{p-1} \\
B^r_p &\coloneqq (A^{r-1}_{p-1} + dA^{r-1}_{p+r-1})/(F_{p-1}C \cap (A^{r-1}_{p-1} + dA^{r-1}_{p+r-1})) \\
&\qquad = (A^{r-1}_{p-1} + dA^{r-1}_{p+r-1})/(A^{r-1}_{p-1} + dA^{r}_{p+r-1}) \cong dA^{r-1}_{p+r-1}/dA^r_{p+r-1} \\
E^r_p &= A^r_p/(A^{r-1}_{p-1} + dA^{r-1}_{p+r-1})
\end{aligned}$$

The Z^r_p is defined as the subobject that goes down r steps in the filtration under d and the B^r_p is the subobject coming from $r-1$ steps up under d (the term A^{r-1}_{p-1} is only so that the quotient is well-defined).

To prove d^r has E^{r+1}_{pq} as its homology, we need the following lemma:

Lemma B.2.4. *The map* $d : Z^r_p/Z^{r+1}_p \to B^{r+1}_{p-r}/B^r_{p-r}$ *is an isomorpism.*

Proof. Writing out the notation, using examples B.2.2 and B.2.2, we get

$$Z_p^r/Z_p^{r+1} = (A_p^r/A_{p-1}^{r-1})\big/(A_p^{r+1}/A_{p-1}^r) \cong A_p^r\big/(A_{p-1}^{r-1} + A_p^{r+1})$$
$$B_{p-r}^{r+1}/B_{p-r}^r \cong (dA_p^r/dA_p^{r+1})\big/(dA_{p-1}^{r-1}/dA_{p-1}^r) \cong dA_p^r\big/(dA_p^{r+1} + dA_{p-1}^{r-1})$$

As clearly $\ker(d : A_p^r \to F_{p-r}C) \subset A_p^{r+1}$, the lemma follows. □

Now,

$$\ker d_p^r = \frac{\{z \in A_p^r \mid dz \in A_{p-r-1}^{r-1} + dA_{p-1}^{r-1}\}}{(A_{p-1}^{r-1} + dA_{p+r-1}^{r-1})} = \frac{A_{p-1}^{r-1} + A_p^{r+1}}{(A_{p-1}^{r-1} + dA_{p+r-1}^{r-1})} = \frac{Z_p^{r+1}}{B_p^r}$$

and d_p^r factors as

$$d_p^r : Z_p^r/B_p^r \twoheadrightarrow Z_p^r/Z_p^{r+1} \xrightarrow{\sim} B_{p-r}^{r+1}/B_{p-r}^r \hookrightarrow Z_{p-r}^r/B_{p-r}^r$$

So $\operatorname{Im} d_{p+r}^r = B_p^{r+1}/B_p^r$ and $\ker d_p^r/\operatorname{Im} d_{p-r}^r = Z_p^{r+1}/B_p^{r+1} =: E_{pq}^{r+1}$, proving that we do indeed have a spectral sequence.

Having constructed the spectral sequence, we would like to have some convergence properties. The following theorem gives the basic case:

Theorem B.2.5 (Classical Convergence Theorem)**.** *Suppose the filtration of a complex C is exhaustive and bounded below. Then the spectral sequence is bounded below and converges to $H_*(C)$. If the filtration is bounded, so is the spectral sequence and*

$$E_{pq}^0 = F_pC_{p+q}/F_{p-1}C_{p+q} \Rightarrow H_{p+q}(C)$$

The convergence is natural: given $f : (C, F) \to (C', F')$, the induced maps on spectral sequences and homology are compatible.

Proof. The statements about boundedness are obvious. As the filtration is bounded below, A_p^r stabilises for r large with limit value A_p^∞:

$$\begin{aligned}
A_p^\infty &:= F_pC \cap d^{-1}\Big(\bigcap_r F_{p-r}C\Big) = F_pC \cap d^{-1}(0) = \ker d : F_pC \to F_pC \\
Z_p^\infty &= A_p^\infty/A_{p-1}^\infty \\
B_p^\infty &= \Big(A_{p-1}^\infty + d\Big(\bigcup_r F_{p+r-1}C\Big) \cap F_pC\Big)\Big/A_{p-1}^\infty = \big(A_{p-1}^\infty + dC \cap F_pC\big)\big/A_{p-1}^\infty
\end{aligned}$$

using exhaustiveness for B. Hence we get

$$\begin{aligned}
F_pH_{p+q}(C)/F_{p-1}H_{p+q}(C) &= \frac{\operatorname{Im}(H_{p+q}(F_pC) \to H_{p+q}(C))}{\operatorname{Im}(H_{p+q}(F_{p-1}C) \to H_{p+q}(C))} \\
&= \frac{\ker(d_p : F_pC \to F_pC)}{\operatorname{Im}(d : C \to C) \cap F_pC + \ker(d_{p-1} : F_{p-1}C \to F_{p-1}C)} \\
&= \frac{A_p^\infty}{dC \cap F_pC + A_{p-1}^\infty} = \frac{A_p^\infty/A_{p-1}^\infty}{(A_{p-1}^\infty + dC \cap F_pC)/A_{p-1}^\infty} \\
&= \frac{Z_{pq}^\infty}{B_{pq}^\infty} = E_{pq}^\infty
\end{aligned}$$

which proves weak convergence. But the filtration on $H_*(C)$ is exhaustive and bounded below because C is, and the spectral sequence is regular because it is bounded below. Hence, it converges to $H_*(C)$. □

Remark B.2.6. In general, one only needs to assume the filtration on C is complete and exhaustive and the spectral sequence is regular to prove it converges weakly to $H_*(C)$ and if the spectral sequence is in addition bounded above it converges to it. For a proof, see theorem 5.5.10 of [Wei94].

Example B.2.7 (Double complex)**.** Let (C_{pq}, d, δ) be a double complex, with d the vertical and δ the horizontal differential (which anticommute by assumption). Both the vertical and horizontal grading induce filtrations on the total complex, which induce spectral sequences ${}^vE^r_{pq}$ and ${}^hE^r_{pq}$ according to the previous example. They both start at sheet 0 and converge to the homology of the total complex under some mild conditions.

Let us first consider ${}^vE^r_{pq}$, induced by the filtration $F_p \operatorname{Tot}^{\oplus}_n = \bigoplus_{k=-\infty}^{p} C_{k,n-k}$. The zeroeth sheet is then $C_{\bullet\bullet}$ itself, together with the vertical differential: ${}^vE^0_{pq} = C_{pq}$.

As the total differential is given by $D = d + \delta$, the induced differential d^1 on ${}^vE^1_{pq} = H^d_q(C_{p,\bullet})$ is the horizontal one (d is clearly zero).

The terms of the second sheet are given by ${}^vE^2_{pq} = H^\delta_p H^d_q(C_{\bullet\bullet})$. Now, the (p,q)th term of this second sheet is mapped to zero under δ in the first sheet, which is the image of the $(p-1,q+1)$th term under d in the zeroeth sheet. Hence, the induced differential is $d^2 = \delta \circ d^{-1} \circ \delta$, where d^{-1} is any lift along d. This is well-defined, because different lifts differ by homology $H^d_{q+2}(C_{p-1,\bullet})$, which is mapped to zero in $H^\delta_{p-2}H^d_{q+2}(C_{\bullet\bullet})$.

This process, of inverting d and defining the higher derivatives by $d^r := d \circ (d^{-1} \circ \delta)^r$, yields the spectral sequence ${}^vE^r_{pq}$. Similarly, ${}^hE^r_{pq}$ is defined by setting ${}^hE^0_{pq} = C_{q,p}$ with the horizontal differential and working analogously. Note the switch of indices!

In an abelian category with elements, an a_0 in a class $\alpha \in {}^vE^r_{pq}$ can be extended to a tuple $(a_0, \dots, a_{r-1})$ with $a_i \in C_{p-i,q+i}$ such that $da_0 = 0$ and $\delta a_i = -da_{i+1}$ for $0 \le i \le r-2$, unique up to elements $(db_0, \delta b_0 + db_1, \dots, \delta b_{r-2})$. Then $d^r(a_0, \dots, a_{r-1}) = \delta a_{r-1}$.

In a bounded below double complex (bounded below in p), ${}^vE^r_{pq}$ converges to $H_n(\operatorname{Tot}^{\oplus}_\bullet(C_{\bullet\bullet}))$, using theorem B.2.5, because the filtration is clearly exhaustive.

Similarly, in a bounded above double complex (bounded below in q), we get that ${}^hE^r_{pq}$ converges to $H_n(\operatorname{Tot}^{\oplus}_\bullet(C_{\bullet\bullet}))$ with filtration induced by $F'_q \operatorname{Tot}^{\oplus}_n(C_{\bullet\bullet}) = \bigoplus_{k=-\infty}^{q} C_{n-k,k}$. Hence, if the double complex is bounded, both converge to the same limit.

Most applications of this example are concerned with first quadrant double complexes and there use one of the spectral sequences to deduce information about the other.

B.2.3 — The Grothendiek spectral sequence and specialisations

Proposition-definition B.2.8. Let $A_\bullet$ be a chain complex in an abelian category $\mathcal{A}$ with enough projectives. Then there exists an upper half-plane double complex $P_{\bullet\bullet}$ with a chain map $\varepsilon : P_{\bullet 0} \to A_\bullet$, the *augmentation map*, such that $P_{p\bullet} = 0$ if $A_p = 0$ and the induced maps $B_p(\varepsilon) : B_p(P, d^h) \to B_p(A)$ and $H_p(\varepsilon) : H_p(P, d^h) \to H_p(A)$ from horizontal boundaries and cohomology are projective resolutions.

This double complex is called the *Cartan-Eilenberg resolution* of A_*.

Proof. Choose projective resolutions $P^B_{p\bullet} \to B_p(A)$ and $P^H_{p\bullet} \to H_p(H)$. Using the short exact sequences $0 \to B_p(A) \to Z_p(A) \to H_p(A) \to 0$ and $0 \to Z_p(A) \to A_p \to B_{p-1}(A) \to 0$, we get projective resolutions defined (as objects) as $P^Z_{p\bullet} := P^B_{p\bullet} \oplus P^H_{p\bullet} \to Z_p(A)$ and $P_{p\bullet} := P^Z_{p\bullet} \oplus P^B_{p-1,\bullet} \to A_p$, using projectivity and induction to define the differentials.

The horizontal differential is

$$
d^h_{p\bullet} : \quad
\begin{array}{ccccccc}
P_{p\bullet} & \longrightarrow & P^B_{p-1,\bullet} & \longrightarrow & P^Z_{p-1,\bullet} & \longrightarrow & P_{p-1,\bullet} \\
\| & & \| & & \| & & \| \\
P^B_{p\bullet} \oplus^H_{p\bullet} \oplus P^B_{p-1,\bullet} & \xrightarrow{\pi_3} & P^B_{p-1,\bullet} & \xhookrightarrow{\iota_1} & P^B_{p-1,\bullet} \oplus P^H_{p-1,\bullet} & \xhookrightarrow{\iota_1 \oplus \iota_2} & P^B_{p-1,\bullet} \oplus P^H_{p-1,\bullet} \oplus P^B_{p-2,\bullet}
\end{array}
$$

and the vertical differential must be multiplied by $(-1)^p$ in order for the differentials to anticommute.

□

Lemma B.2.9. *Chain homotopic maps of complexes $f \sim g : A_\bullet \to B_\bullet$ induce chain homotopic maps of Cartan-Eilenberg resolutions $\tilde{f} \sim \tilde{g} : P^A_{\bullet\bullet} \to P^B_{\bullet\bullet}$. In particular, Cartan-Eilenberg resolutions are unique up to homotopy equivalence.*

Proof. A standard computation in homological algebra. □

Definition B.2.10. Let $F : \mathcal{A} \to \mathcal{B}$ be a right-exact functor and assume $\mathcal{A}$ has enough projectives. Then define the *left hyper-derived functors*

$$\mathbb{L}_i F : \mathsf{Kom}^+(\mathcal{A}) \to \mathcal{B} : \begin{cases} A_\bullet & \mapsto H_i(\mathrm{Tot}^\oplus_\bullet(FP_{\bullet\bullet})) \\ f & \mapsto H_i(\mathrm{Tot}^\oplus_\bullet(F\tilde{f})) \end{cases}$$

where $P_{\bullet\bullet}$ is a Cartan-Eilenberg resolution of $A_\bullet$ and $\tilde{f}$ is an extension of f on Cartan-Eilenberg resolutions. If $\mathcal{B}$ has all direct sums, this definition can be extended to $\mathsf{Kom}(\mathcal{A})$.

Lemma B.2.11. *There is a convergent spectral sequence*

$$(L_p F)(H_q(A_\bullet)) \Rightarrow \mathbb{L}_{p+q} F(A_\bullet)$$

If A is bounded below, there is also a convergent spectral sequence

$$H_p((L_q F)(A_\bullet)) \Rightarrow \mathbb{L}_{p+q} F(A_\bullet)$$

Proof. These are the spectral sequences associated to the double complex $F(P_{\bullet\bullet})$, see example B.2.7. □

Example B.2.12 (Grothendieck spectral sequence)**.** Let $\mathcal{A}$, $\mathcal{B}$ and C be abelian categories with $\mathcal{A}$ and $\mathcal{B}$ having enough projectives. If $G : \mathcal{A} \to \mathcal{B}$ and $F : \mathcal{B} \to C$ are right-exact functors such that G sends $\mathcal{A}$-projectives to F-acyclic objects, then there is a first quadrant convergent spectral sequence

$$E^2_{pq} = (L_p F)(L_q G)(A) \Rightarrow L_{p+q}(FG)(A)$$

for each $A \in \mathcal{A}$.

The construction uses lemma B.2.11. Take a projective resolution $P_\bullet \to A$ and a Cartan-Eilenberg resolution $Q_{\bullet\bullet} \to GP_\bullet$. Then the lemma yields two spectral sequences for $GP_\bullet$. First consider lemma B.2.11: $H_p((L_q F)(GP_\bullet)) \Rightarrow \mathbb{L}_{p+q}(GP_\bullet)$. Because GP_p is acyclic by assumption,

$$H_p((L_q F)(GP_\bullet)) = H_p(H_q(FQ_{\bullet\bullet})) = \begin{cases} H_p(FGP_\bullet) = L_p(FG)(A) & \text{if } q = 0 \\ 0 & \text{else} \end{cases}$$

Hence, the spectral sequence collapses and $(\mathbb{L}_p F)(GP_\bullet) = L_p(FG)(A)$. Inserting this in the first spectral sequence lemma B.2.11, we get

$$(L_p F)(H_q(GP_\bullet)) = (L_p F)(L_q G(A)) \Rightarrow L_{p+q}(FG)(A)$$

Example B.2.13 (Lyndon-Hochschield-Serre spectral sequence)**.** Let $H \triangleleft G$ be finite groups, and M a G-module. Then there is a composition of functors $(-)_G = (-)_{G/H} \circ (-)_H$. Taking derived functors using proposition A.3.6 we get an instance of the Grothendieck spectral sequence (example B.2.12)

$$E^2_{pq} = H_p(G/H; H_q(H; M)) \Rightarrow H_n(G; M)$$

It is called the *Lyndon-Serre-Hochschield spectral sequence.*

Example B.2.14 (Leray spectral sequence)**.** Given a map $f : X \to Y$ and a sheaf $\mathcal{F}$ on X, there is a *Leray spectral sequence* (in cohomology)

$$E_2^{pq} = H^p(Y, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F})$$

This can be deduced from the Grothendieck spectral sequence, example B.2.12, by taking the functors $f_* : \mathsf{Sh}(X) \to \mathsf{Sh}(Y)$, $\Gamma_Y : \mathsf{Sh}(Y) \to \mathsf{Ab}$ and $\Gamma_X = \Gamma_Y \circ f_* : \mathsf{Sh}(X) \to \mathsf{Ab}$. Indeed, $\mathsf{Sh}(X)$ and $\mathsf{Sh}(Y)$ have enough injectives and f_* sends $\mathsf{Sh}(X)$-injectives to Γ_Y-acyclic objects. Historically, this was the first spectral sequence to be considered.

Example B.2.15 (Leray-Serre spectral sequence)**.** Let $F \to E \xrightarrow{\pi} B$ be a Serre fibration with B path-connected, and let A be a ring. Then $H_q(F; A)$ is a local coefficient system on B and there is a *Leray-Serre spectral sequence*

$$E_2^{pq} = H^p(B; H^q(F; A)) \Rightarrow H^{p+q}(E; A)$$

This is a special case of the Leray spectral sequence, using $f = \pi$. Indeed, if we use Čech cohomology on B using a good cover, the only thing left to prove is that $R^q \pi_*(\underline{A}_E)$ computes the cohomology of F, with $\underline{A}_E$ the constant sheaf on E. But on a contractible $U \subset B$, $\pi_* \underline{A}_E(U) = \underline{A}_E(\pi^{-1}(U)) \cong \Gamma(\pi^{-1}(U), A) \cong \Gamma(F, A)$, so this is in fact true. The twisting of the coefficients is obvious by the Čech construction.

As an application of the Leray-Serre spectral sequence, we will calculate the rational cohomology of the Eilenberg-MacLane spaces $K(\mathbb{Z}, n)$.

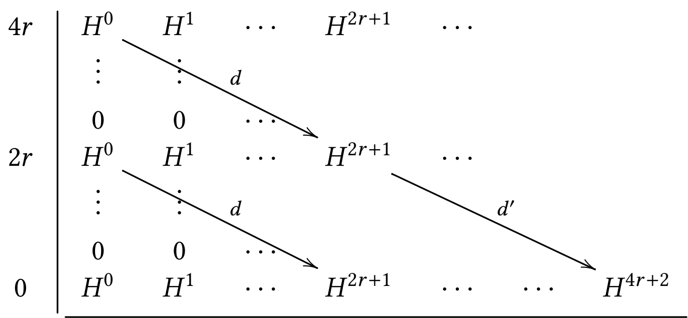
Proposition B.2.16. *The rational cohomology of the Eilenberg-MacLane space $K(\mathbb{Z}, n)$ is a graded polynomial algebra on one generator in degree n. Hence, if n is even, $H^*(K(\mathbb{Z}, n); \mathbb{Q}) = \mathbb{Q}[x_n]$ and if n is odd, $H^*(K(\mathbb{Z}, n); \mathbb{Q}) = \mathbb{Q} \cdot x_n$.*

Proof. This is certainly true for $K(\mathbb{Z}, 1) = S^1$ and $K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$. For the rest, we will use induction, using a fibration $K(\mathbb{Z}, n-1) \to P \to K(\mathbb{Z}, n)$ with P weakly contractible. There then is an associated Leray-Serre spectral sequence

$$H^p\big(K(\mathbb{Z}, n); H^q(K(\mathbb{Z}, n-1))\big) \Rightarrow \begin{cases} \mathbb{Q} & p + q = 0 \\ 0 & \text{else} \end{cases}$$

First consider the step from $k - 1 = 2r - 1$ to $k = 2r$. As $H^q(K(\mathbb{Z}, 2r-1); \mathbb{Q}) = \mathbb{Q}$ for $q = 0, 2r - 1$ and 0 else, the second sheet of the spectral sequence has only two non-zero rows at those degrees, and is $H^*(K(\mathbb{Z}, 2r); \mathbb{Q})$ there. Therefore, the only possibly non-zero differentials are $d_{2r} : H^p(K(\mathbb{Z}, 2r); \mathbb{Q}) \to H^{p+2r}(K(\mathbb{Z}, 2r); \mathbb{Q})$ and these must all be isomorphisms, except for $d_{2r}^{-2r, 2r-1}$, to eliminate all positive-degree cohomology. This shows $H^{2kr}(K(\mathbb{Z}, 2r); \mathbb{Q}) = \mathbb{Q}$ and the rest is zero. As the differentials are readily seen to be the cup product with d of a generator of $H^{2r-1}(K(\mathbb{Z}, 2r-1); \mathbb{Q})$ (which is the Euler class), the induction step holds.

Now consider the step from $k - 1 = 2r$ to $k = 2r + 1$. Now the non-zero rows are at $q = 2jr$ and they are again all $H^* := H^*(K(\mathbb{Z}, 2r+1); \mathbb{Q})$:



The pictured differentials are the first non-zero ones, at sheet $2r+1$. As again only E^{00} survives to infinity, the lower d in the diagram must be an isomorphism. As both d's are the same map up to a non-zero factor, $d' = 0$. Since all other differentials related to H^{4k+2} are zero as well, $H^{4k+2} = 0$. Inductively, moving to higher and higher lines of slope $-\frac{2r}{2r+1}$, all higher cohomology groups must be zero, proving the second induction step, and thereby the proposition. □

Example B.2.17 (Segal spectral sequence)**.** For a semi-simplicial space Y_*, define a filtration on $Y := |Y_*|$ by setting $F_p Y := \mathrm{Im}(\Delta^p \times Y_p \to Y)$. This induces a bounded filtration on singular chains $\tilde{C}_*(Y)$ by $F_p\tilde{C}_q(Y) = \tilde{C}_q(F_p Y)$. By theorem B.2.5, there is an associated spectral sequence

$$E^0_{pq}Y_* = F_p\tilde{C}_{p+q}(Y)/F_{p-1}\tilde{C}_{p+q}(Y) = \tilde{C}_{p+q}(F_pY, F_{p-1}Y) \Rightarrow H_{p+q}(Y)$$

As $(F_pY, F_{p-1}Y)$ is the p-fold suspension of (Y_p, Y_p^d), where Y_p^d is the degenerate part of Y_p, we get $E^1_{pq}Y_* = H_{p+q}(F_pY, F_{p-1}Y) = H_q(Y_p, Y_p^d)$. Hence, $E^2_{pq}Y_* = H_p^{\mathrm{simp}}H_q^{\mathrm{sing}}(Y_\bullet, Y_\bullet^d) = H_p^{\mathrm{simp}}H_q^{\mathrm{sing}}(Y_\bullet)$, as the degenerate part consists of boundaries by definition. This yields the *Segal spectral sequence*[Seg68]

$$E^2_{pq}Y_* = H_p^{\mathrm{simp}}H_q^{\mathrm{sing}}(Y_\bullet) \Rightarrow H_{p+q}(Y)$$

We can clearly also take $E^1_{pq}Y_* = H_q(Y_p)$.

Bibliography

[Ada78] John F. Adams. *Infinite loop spaces.* Herman Weyl lectures 90. Princeton University Press, 1978.

[Ada95] John F. Adams. *Stable homotopy and generalised homology.* Chicago lectures in mathematics. University of Chicago press, 1995.

[Bol12] Søren K. Boldsen. "Improved homological stability for the mapping class group with integral or twisted coefficients". In: *Mathematische Zeitschrift* 270.1-2 (2012), pp. 297–329.

[BP72] Michael Barratt and Stewart Priddy. "On the homology of non-connected monoids and their associated groups". In: *Commentarii Mathematici Helvetici* 47.1 (1972), pp. 1–14.

[Bro62] Edgar H. Brown Jr. "Cohomology Theories". In: *Annals of Mathematics.* Second Series 75.3 (1962), pp. 467–484.

[Bus92] Peter Buser. *Geometry and Spectra of Compact Riemann Surfaces.* 1st. Vol. 106. Progress in Mathematics. Birkhäuser, 1992.

[BV68] John Michael Boardman and Rainer M Vogt. "Homotopy-everything H-spaces". In: *Bulletin of the American mathematical society* 74.6 (1968), pp. 1117–1122.

[EE69] Clifford J. Earle and James Eells. "A fibre bundle description of Teichmüller theory". In: *Journal for differential geometry* 3 (1969), pp. 19–43.

[ES70] Clifford J. Earle and Alfred Schatz. "Teichmüller theory for surfaces with boundary". In: *Journal for differential geometry* 4 (1970), pp. 169–185.

[GR10] Søren Galatius and Oscar Randal-Williams. "Monoids of moduli spaces of manifolds". In: *Geometry & Topology* 14.3 (2010), pp. 1243–1302.

[Gra73] André Gramain. "Le type d'homotopie du groupe des diffeomorphismes d'une surface compacte". In: *Annales scientifiques de l'École Normale Supérieure* 6.1 (1973), pp. 53–66.

[Har83] John L. Harer. "The second homology group of the mapping class group of an orientable surface". In: *Inventiones mathematicae* 72.2 (1983), pp. 221–239.

[Har85] John L. Harer. "Stability of the homology of the mapping class groups of orientable surfaces". In: *Annals of Mathematics* 121.2 (1985), pp. 215–249.

[Har93] John L. Harer. "Improved stability for the homology of the mapping class groups of surfaces". `http://www.math.duke.edu/preprints/1993.html`. 1993.

[Hat14] Allen Hatcher. "A short exposition of the Madsen-Weiss theorem". In: *arXiv* preprint arXiv:1103.5223v2 (2014).

[Hur92] Adolf Hurwitz. "Über algebraische Gebilde mit eindeutigen Transformationen in sich". In: *Mathematische Annalen* 41.3 (1892), pp. 403–442.

[Iva89] Nikolai V. Ivanov. "Stabilization of the homology of Teichmüller modular groups". In: *Algebra i Analiz* 1.3 (1989), pp. 110–126.

[Iva93] Nikolai V. Ivanov. "On the homology stability for Teichmüller modular groups: closed surfaces and twisted coefficients". In: *Contemporary Mathematics* 150 (1993), pp. 149–194.

[Jar89] J. F. Jardine. “The homotopical foundations of algebraic K-theory”. In: *Algebraic K-theory and algebraic number theory* 83 (1989), pp. 57–82.

[Kos47] Jean-Louis Koszul. “Sur les opérateurs de dérivation dans un anneau”. In: *Comptes Rendus de l'Académie des Sciences* 225 (1947), pp. 217–219.

[Ler46] Jean Leray. “Structure de l'anneau d'homologie d'une représentation”. In: *Comptes Rendus de l'Académie des Sciences* 222 (1946), pp. 1419–1422.

[Lod76] Jean-Louis Loday. “K-théorie algébrique et représentations de groupes”. In: *Annales scientifiques de l'É.N.S. 4e série* 9.3 (1976), pp. 309–377.

[Mad07] Ib Madsen. “Moduli spaces from a topological viewpoint”. In: *Proceedings of the International Congress of Mathematicians: Madrid, August 22-30, 2006: invited lectures.* 2007, pp. 385–411.

[May72] J. Peter May. *The geometry of iterated loop spaces.* 1st. Vol. 271. Lecture Notes in Mathematics. Springer, 1972.

[May75] J. Peter May. *Classifying spaces and fibrations.* American Mathematical Society, 1975.

[May99] J. Peter May. *A concise course in algebraic topology.* Chicago Lecture Series in Mathematics. University of Chicago Press, 1999.

[Mil56] John W. Milnor. “Construction of Universal Bundles II”. In: *Annals of Mathematics* 63.3 (1956).

[MS76] Dusa McDuff and Graeme Segal. “Homology fibrations and the "group-completion" theorem”. In: *Inventiones mathematicae* 31.1 (1976), pp. 279–284.

[MT01] Ib Madsen and Ulrike Tillman. “The stable mapping class group and $Q(\mathbb{C}P^\infty_+)$”. In: *Inventiones mathematicae* 145.1 (2001), pp. 509–544.

[Mum83] David Mumford. “Towards an enumerative geometry of the moduli space of curves”. In: *Arithmetic and geometry.* Springer, 1983, pp. 271–328.

[MW07] Ib Madsen and Michael Weiss. “The stable moduli space of Riemann surfaces: Mumford's conjecture”. In: *Annals of mathematics* 165.3 (2007), pp. 843–941.

[Qui70] Daniel Quillen. “Cohomology of Groups”. In: *Actes, Congrès international Mathematique* 2 (1970), pp. 47–51.

[Qui71] Daniel Quillen. *On the group completion of a simplicial monoid.* preprint. 1971.

[Qui73] Daniel Quillen. “Higher algebraic K-theory: I”. In: *Lecture notes in mathematics* 341 (1973), pp. 85–147.

[Ran14] Oscar Randal-Williams. “Resolutions of moduli spaces and homological stability”. In: *arXiv* 0909.4278 (2014). Version 4.

[Seg68] Graeme Segal. “Classifying spaces and spectral sequences”. In: *Publications Mathématiques de l'IHÉS* 34.1 (1968), pp. 105–112.

[Ste51] Norman E. Steenrod. *The topology of fibre bundles.* Vol. 14. Princeton University Press, 1951.

[Sul70] Dennis Sullivan. *Geometric Topology: Localisation, Periodicity, and Galois Symmetry.* Massachusetts Institute of Technology, 1970.

[Tho54] René Thom. “Quelques propriétés globales des variétés différentiables”. In: *Commentarii Mathematici Helvetici* 28.1 (1954), pp. 17–86.

[Til97] Ulrike Tillmann. “On the homotopy of the stable mapping class group”. In: *Inventiones Mathematicae* 130.1 (1997), pp. 257–275.

[Wah12] Nathalie Wahl. "Homological stability for mapping class groups of surfaces". In: *Handbook of Moduli, Vol. III* 26.1 (2012), pp. 547–583.

[Wal60] C. T. C. Wall. "Generators and relations for the Steenrod algebra". In: *Annals of Mathematics.* Second series 72.3 (1960), pp. 429–444.

[Wei94] Charles A. Weibel. *An introduction to homological algebra.* Cambridge studies in advanced mathematics 38. Cambridge University Press, 1994.