Some formulae of tensor calculus and differential geometry

1 General notation

We use Greek letters $\alpha, \beta \ldots = 0, 1, 2, 3$ for components of 4-vectors,tensors, etc and Roman letters $i, j, k \ldots = 1, 2, 3$ for their spatial components. Ordinary partial derivative with respect to coordinate x^α is often denoted by comma

$$\frac{\partial}{\partial x^\alpha} A\left(x^\alpha\right) \equiv A_{,\alpha} \tag{1}$$

Up or down position of the index after comma is generally important.

We often differentiate with respect to vectors (note, coordinates x^α themselves do not constitute vectors) or even tensors, for example, the Lagrangian density

$$\frac{\partial \mathcal{L}}{\partial u^\alpha} = \left(\frac{\partial \mathcal{L}}{\partial u^0}, \frac{\partial \mathcal{L}}{\partial u^1}, \frac{\partial \mathcal{L}}{\partial u^2}, \frac{\partial \mathcal{L}}{\partial u^3}\right) \tag{2}$$

Note, that it matters, whether one differentiate with respect to covariant or contravariant components, i.e

$$\frac{\partial \mathcal{L}}{\partial u_\alpha} = \left(\frac{\partial \mathcal{L}}{\partial u_0}, \frac{\partial \mathcal{L}}{\partial u_1}, \frac{\partial \mathcal{L}}{\partial u_2}, \frac{\partial \mathcal{L}}{\partial u_3}\right) \tag{3}$$

is a different object. We always write the derivatives with respect to vectors explicitly. One needs some care with notation when the vector that we differentiate with respect to is itself the gradient of a scalar function. Then we get the notation like $\frac{\partial \mathcal{L}}{\partial \phi_{,\alpha}}$ which means $\frac{\partial \mathcal{L}}{\partial (\phi_{,\alpha})}$.

2 Coordinate transformations

Coordinate transformation is given by the set of functions $x' = x'(x)$. It is often convenient to add prime to the index, i.e write $x^{\alpha'}\left(x^\alpha\right)$. This way α' not only means that the vector is considered in other frame, but also α' being different index than α we save on Greek letters. Otherwise we would have to write $x'^\beta\left(x^\alpha\right)$ to designate the dependence of β primed coordinate on α original one. This is especially useful for differentiation, so we can write $A^\alpha{}_{,\beta'}$ to say the α's component of vector A in original frame is differentiated wrt to β coordinate in *another*, primed, system.

We consider coordinate transformation invertible, i.e., there exist the inverse functions $x^\alpha\left(x^{\alpha'}\right)$. In this case, using chain differentiation rule

$$\frac{\partial x^\alpha}{\partial x^{\alpha'}} \frac{\partial x^{\alpha'}}{\partial x^\beta} = \frac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha{}_\beta \tag{4}$$

the latter step following from condition that coordinates in the same system are independent on each other. Similarly,

$$\frac{\partial x^{\alpha'}}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial x^{\beta'}} = \delta^{\alpha'}{}_{\beta'} \tag{5}$$

$\frac{\partial x^\alpha}{\partial x^{\alpha'}}$ can be viewed as square matrices, but they are not tensors, since they are not objects defined in one particular coordinate system (actually they mix two coordinate systems). Coordinates x^α's themselves do not constitute a vector, and when they are viewed as functions of another coordinate set, the transformation matrix is build from usual derivatives of these functions wrt their arguments. Quite similar with what you have to deal with when, for example, you change variables in a multidimensional integral.

3 Tensor transformation rules

Tensors are defined by their transformation properties under coordinate change. One distinguishes *covariant* and *contravariant* indexes. Number of indexes is tensor's *rank*, scalar and vector quantities are particular case of tensors of rank zero and one.

Consider coordinate change $x^\alpha = x^\alpha(x^{\alpha'})$. Transformation rules are

Scalar

$$S = S' \; - \; scalar\ (tensor\ of\ 0\ rank)\ is\ invariant\ under\ transformations \tag{6}$$

Vector

$$V^\alpha = V^{\alpha'}\frac{\partial x^\alpha}{\partial x^{\alpha'}} \; - \; contravariant\ vector\ (tensor\ of\ rank\ 1) \tag{7}$$

$$V_\alpha = V_{\alpha'}\frac{\partial x^{\alpha'}}{\partial x^\alpha} \; - \; covariant\ vector \tag{8}$$

Tensor

$$T^{\alpha\dots}{}_{\beta\dots} = T^{\alpha'\dots}{}_{\beta'\dots}\frac{\partial x^\alpha}{\partial x^{\alpha'}}\frac{\partial x^{\beta'}}{\partial x^\beta}\cdots \; - \; tensor\ of\ higher\ rank\ with\ mixed\ indexes \tag{9}$$

In general, the position of the indexes matters. Above case where all covariant indexes are at the end is a special case.

Contraction Contraction is a summation over a pair of one covariant and one contravariant indexes. It creates a tensor of rank less than original by two. We use shorthand that when two indexes of different type are labeled by the same latter it implies a summation over them.

$$S = V_\alpha V^\alpha, \qquad V^\alpha = T^{\alpha\beta}{}_\beta \tag{10}$$

4 Special invariant tensors

There are two special tensors, which components are invariant under arbitrary coordinate transformations.

The first one is rank-2 unit tensor that is represented by the unit matrix

$$\delta^\alpha{}_\beta \equiv 1\ if\ \alpha = \beta, \quad 0,\ if \alpha \neq \beta \tag{11}$$

which is often called *Kronecker* symbol. Note that it has mixed components. One may encounter $\delta_{\alpha\beta}$ which in some coordinate system is represented by unit matrix, but such tensor will have its components changed in another frame (check !). The effect of contraction of the Kronecker symbol with another vector is the replacement of the component index

$$\delta^\alpha{}_\beta A^\beta = A^\alpha, \qquad \delta^\alpha{}_\beta A^\alpha = A^\beta \tag{12}$$

The second special tensor has rank equal to dimensionality N of the space, and is defined as

$$\varepsilon^{\alpha\beta\gamma\dots} = \pm 1 \quad if\ and\ only\ if \quad \alpha \neq \beta \neq \gamma \dots \tag{13}$$

with $e^{012\dots} = 1$ and changing sign with each permutation of a pair of indexes. Thus, $e^{102\dots} = -1$, etc. This tensor is known as *fully antisymmetric tensor of rank N* or *Levi-Civita* symbol. With its help one can define a dual tensor to any fully antisymmetric tensor of rank r less than N. Such dual tensor will have rank $N - r$. For example in 4D if $F_{\gamma\delta}$ is antisymmetric, its dual is

$$(F^*)^{\alpha\beta} = \varepsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta} \tag{14}$$

while for a tensor of rank 3, the dual will be a vector

$$(A^*)^\alpha = \varepsilon^{\alpha\beta\gamma\delta}A_{\beta\gamma\delta} \tag{15}$$

5 The metric tensor

Definition The metric tensor $g_{\alpha\beta}$ specifies the invariant interval (distance) between two neighbouring points (events)

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \tag{16}$$

Lowering of indexes

$$A_\alpha = g_{\alpha\beta} A^\beta, \quad T_{\alpha\beta} = g_{\alpha\gamma} g_{\beta\sigma} T^{\gamma\sigma} \tag{17}$$

Defining $g^{\alpha\beta}$

$$g_{\alpha\beta} \equiv g_{\alpha\gamma} g^\gamma_\beta \Rightarrow g^\gamma_\beta = \delta^\gamma_\beta, \ldots \Rightarrow \ldots g_{\beta\sigma} g^{\gamma\sigma} (\equiv g^\gamma_\beta) = \delta^\gamma_\beta \tag{18}$$

Rising of indexes

$$A^\alpha = g^{\alpha\beta} A_\beta, \quad T^{\alpha\beta} = g^{\alpha\gamma} g^{\beta\sigma} T_{\gamma\sigma} \tag{19}$$

6 Affine connection (Christoffel symbols)

Affine connection $\Gamma^\alpha_{\beta\gamma}$ describes relation between vectors at two neighbouring points.

$$\delta V^\alpha = -\Gamma^\alpha_{\beta\gamma} V^\beta dx^\gamma \tag{20}$$

Covariant derivatives We denote ordinary derivatives with comma and the covariant ones with semi-colon

$$\begin{aligned} S_{;\mu} &= S_{,\mu} \quad - \quad for\ scalars\ derivatives\ are\ equal. && (21)\\ V^\alpha{}_{;\mu} &= V^\alpha{}_{,\mu} + \Gamma^\alpha_{\mu\gamma} V^\gamma && (22)\\ V_{\alpha;\mu} &= V_{\alpha,\mu} - \Gamma^\gamma_{\mu\alpha} V_\gamma && (23)\\ T^\beta{}_{\alpha;\mu} &= T^\beta{}_{\alpha,\mu} - \Gamma^\gamma_{\mu\alpha} T^\beta{}_\gamma + \Gamma^\beta_{\mu\gamma} T^\gamma{}_\alpha && (24) \end{aligned}$$

Relation between $\Gamma^\alpha_{\beta\gamma}$ and $g_{\alpha\beta}$: In GR we use affine connection which is related to the first derivatives of the metric tensor by the requirement that $g_{\alpha\beta;\mu} = 0$ and restriction that connection is symmetric $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$. Then

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\sigma} \left(g_{\sigma\beta,\gamma} + g_{\sigma\gamma,\beta} - g_{\beta\gamma,\sigma} \right) \tag{25}$$

7 Curvature, Riemann and Ricci tensors, Ricci scalar

Covariant derivative in general is not commutative, $V^\alpha{}_{;\nu;\mu} - V^\alpha{}_{;\mu;\nu} \neq 0$. Namely

$$V^\alpha{}_{;\nu;\mu} - V^\alpha{}_{;\mu;\nu} \equiv R^\alpha{}_{\gamma\mu\nu} V^\gamma \tag{26}$$

defines **Riemann tensor** $R^\alpha{}_{\gamma\mu\nu}$ which gives invariant measure of the curvature of the space. The space is flat if $R^\alpha{}_{\gamma\mu\nu} = 0$.

Riemann tensor $R^\alpha{}_{\gamma\mu\nu}$

$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\beta\mu,\nu} + \Gamma^\alpha_{\gamma\mu} \Gamma^\gamma_{\beta\nu} - \Gamma^\alpha_{\gamma\nu} \Gamma^\gamma_{\beta\mu} \tag{27}$$

Ricci tensor $R_{\alpha\beta}$ is the contraction of the Riemann tensor

$$R_{\alpha\beta} \equiv R^\gamma{}_{\alpha\gamma\beta} \tag{28}$$

Ricci scalar R is the further contraction

$$R \equiv g^{\alpha\beta} R_{\alpha\beta} = R^\alpha{}_\alpha \tag{29}$$

8 Useful Computational Formulae

$$
\begin{aligned}
R_{\beta\nu} &= \Gamma^{\mu}_{\beta\nu,\mu} - \Gamma^{\mu}_{\beta\mu,\nu} + \Gamma^{\mu}_{\gamma\mu}\Gamma^{\gamma}_{\beta\nu} - \Gamma^{\mu}_{\gamma\nu}\Gamma^{\gamma}_{\beta\mu} && (30)\\
\Gamma^{\gamma}_{\alpha\gamma} &= \left[\ln\left(\sqrt{-g}\right)\right]_{,\alpha} \quad , \qquad g = |g_{\alpha\beta}| && (31)
\end{aligned}
$$