FORMULA SHEET FOR ASSIGNMENT 2

1 Tensor transformation rules

Tensors are defined by their transformation properties under coordinate change. One distinguishes *covariant* and *contravariant* indexes. Number of indexes is tensor's *rank*, scalar and vector quantities are particular case of tensors of rank zero and one.

Consider coordinate change $x^\alpha = x^\alpha(x'^\alpha)$. Transformation rules are

Scalar

$$S = S' \ - \ scalar \ (tensor \ of \ 0 \ rank) \ is \ invariant \ under \ transformations \tag{1}$$

Vector

$$V^\alpha = V^{\alpha'} \frac{\partial x^\alpha}{\partial x^{\alpha'}} \ - \ contravariant \ vector \ (tensor \ of \ rank \ 1) \tag{2}$$

$$V_\alpha = V_{\alpha'} \frac{\partial x^{\alpha'}}{\partial x^\alpha} \ - \ covariant \ vector \tag{3}$$

Tensor

$$T^{\alpha\dots}_{\beta\dots} = T^{\alpha'\dots}_{\beta'\dots} \frac{\partial x^\alpha}{\partial x^{\alpha'}} \frac{\partial x^{\beta'}}{\partial x^\beta} \cdots \ - \ tensor \ of \ higher \ rank \ with \ mixed \ indexes \tag{4}$$

Contraction Contraction is a summation over a pair of one covariant and one contravariant indexes. It creates a tensor of rank less than original by two. We use shorthand that when two inderxes of different type are labeled by the same latter it implies a summation over them.

$$S = V_\alpha V^\alpha, \qquad V^\alpha = T^{\alpha\beta}_{\ \ \ \beta} \tag{5}$$

2 The metric tensor

Definition The metric tensor $g_{\alpha\beta}$ specifies the invariant interval (distance) between two neighbouring points (events)

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \tag{6}$$

Lowering of indexes

$$A_\alpha = g_{\alpha\beta} A^\beta, \quad T_{\alpha\beta} = g_{\alpha\gamma} g_{\beta\sigma} T^{\gamma\sigma} \tag{7}$$

Defining $g^{\alpha\beta}$

$$g_{\alpha\beta} \equiv g_{\alpha\gamma} g^\gamma_\beta \Rightarrow g^\gamma_\beta = \delta^\gamma_\beta, \dots \Rightarrow \dots g_{\beta\sigma} g^{\gamma\sigma} (\equiv g^\gamma_\beta) = \delta^\gamma_\beta \tag{8}$$

Rising of indexes

$$A^\alpha = g^{\alpha\beta} A_\beta, \quad T^{\alpha\beta} = g^{\alpha\gamma} g^{\beta\sigma} T_{\gamma\sigma} \tag{9}$$

3 Affine connection (Christoffel symbols)

Affine connection $\Gamma^\alpha_{\beta\gamma}$ describes relation between vectors at two neighbouring points.

$$\delta V^\alpha = -\Gamma^\alpha_{\beta\gamma} V^\beta dx^\gamma \tag{10}$$

Covariant derivatives We denote ordinary derivatives with comma and the covariant ones with semicolon

$$
\begin{aligned}
S_{;\mu} &= S_{,\mu} \quad - \quad \mathit{for\ scalars\ derivatives\ are\ equal.} && (11)\\
V^{\alpha}_{;\mu} &= V^{\alpha}_{,\mu} + \Gamma^{\alpha}_{\mu\gamma} V^{\gamma} && (12)\\
V_{\alpha;\mu} &= V_{\alpha,\mu} - \Gamma^{\gamma}_{\mu\alpha} V_{\gamma} && (13)\\
T^{\beta}_{\alpha;\mu} &= T^{\beta}_{\alpha,\mu} - \Gamma^{\gamma}_{\mu\alpha} T^{\beta}_{\gamma} + \Gamma^{\beta}_{\mu\gamma} T^{\gamma}_{\alpha} && (14)
\end{aligned}
$$

Relation between $\Gamma^{\alpha}_{\beta\gamma}$ and $g_{\alpha\beta}$: In GR we use affine connection which is related to the first derivatives of the metric tensor by the requirement that $g_{\alpha\beta;\mu} = 0$ and restriction that connection is symmetric $\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta}$. Then

$$
\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\sigma} \left(g_{\sigma\beta,\gamma} + g_{\sigma\gamma,\beta} - g_{\beta\gamma,\sigma} \right) \tag{15}
$$

4 Curvature, Riemann and Ricci tensors, Ricci scalar

Covariant derivative in general is not commutative, ${V^{\alpha}}_{;\nu;\mu} - {V^{\alpha}}_{;\mu;\nu} \neq 0$. Namely

$$
{V^{\alpha}}_{;\nu;\mu} - {V^{\alpha}}_{;\mu;\nu} \equiv {R^{\alpha}}_{\gamma\mu\nu} V^{\gamma} \tag{16}
$$

defines **Riemann tensor** ${R^{\alpha}}_{\gamma\mu\nu}$ which gives invariant measure of the curvature of the space. The space is flat if ${R^{\alpha}}_{\gamma\mu\nu} = 0$.

Riemann tensor ${R^{\alpha}}_{\gamma\mu\nu}$

$$
{R^{\alpha}}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\alpha}_{\gamma\mu} \Gamma^{\gamma}_{\beta\nu} - \Gamma^{\alpha}_{\gamma\nu} \Gamma^{\gamma}_{\beta\mu} \tag{17}
$$

Ricci tensor $R_{\alpha\beta}$ is the contraction of the Riemann tensor

$$
R_{\alpha\beta} \equiv {R^{\gamma}}_{\alpha\gamma\beta} \tag{18}
$$

Ricci scalar R is the further contraction

$$
R \equiv g^{\alpha\beta} R_{\alpha\beta} = {R^{\alpha}}_{\alpha} \tag{19}
$$

5 Useful Computational Formulae

$$
\begin{aligned}
R_{\beta\nu} &= \Gamma^{\mu}_{\beta\nu,\mu} - \Gamma^{\mu}_{\beta\mu,\nu} + \Gamma^{\mu}_{\gamma\mu} \Gamma^{\gamma}_{\beta\nu} - \Gamma^{\mu}_{\gamma\nu} \Gamma^{\gamma}_{\beta\mu} && (20)\\
\Gamma^{\gamma}_{\alpha\gamma} &= \left[\ln \left(\sqrt{-g} \right) \right]_{,\alpha} \quad , \qquad g = |g_{\alpha\beta}| && (21)
\end{aligned}
$$