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ADVANCED SCALING TECHNIQUES FOR THE MODELING OF MATERIALS PROCESSING

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Abstract

This paper presents a formalism to automate portions of the traditional scaling procedure. This formalism, called here Order of Magnitude Scaling (OMS), captures essential components of the governing equations into a set of matrices and arrays amenable to computer implementation using standard linear algebra tools. This approach is valid and specially useful for large systems of multicoupled equations. The matrix approach proposed enables one to automate operations such as normalization, dominant balance, obtention of characteristic values, and check for self-consistency. Also, automation permits an exhaustive exploration of all possible balances and the discovery of unsuspected scaling laws. As an example, OMS is applied to the analysis of the viscous boundary layer with no external pressure gradient. The obtained scaling laws for boundary layer thickness and transverse velocity are the same as those obtained with traditional scaling, and in addition, a meaningful estimation for pressure variations within the boundary layer is obtained; this last estimation is typically not addressed in the manual analysis.

Introduction

The focus of this work is on "scaling," which provides solutions for a whole family of systems instead of solutions for a particular embodiment. Scaling a particular system provides a simple closed-form solution applicable to any system of a family that falls within a range of parameters. Scaling provides many benefits for the modeling of materials processes. It helps to determine the relative importance of the multiple driving forces acting on a system, to determine which terms can be simplified in a given problem, to generate simpler equations which still contain all the relevant phenomena, to obtain characteristic values for the unknowns, and to generate a set of dimensionless group based only on the problem parameters.

In scaling, the estimated characteristic values typically have the form of power-laws. Power laws consist of the multiplication of parameters raised to an exponent. For example, the following estimated pressure has a power-law expression: $\widehat{p_c} = \frac{1}{2}\rho V^2$, where ρ and V are the parameters with exponents 1 and 2 respectively. Power-laws can have a numerical coefficient such as the $\frac{1}{2}$ in the example. Because the estimation of characteristic values almost invariably yields power-laws, regardless of the procedure used to obtain them, some authors use the term "scaling" for the obtention of power laws by some other procedures too. For example, the scaling of interest in this work is different than the "scaling" of interest to Barenblatt [1, 2], which is obtained through advanced dimensional analysis procedures, but not a explicit self-consistent approach. Mendez and Ordóñez [3] obtain scaling laws from

a combination of linear regressions and dimensional analysis. And Kokar [4] and Washio and Motoda [5] propose artificial intelligence approaches to the discovery of scaling laws of physical processes.

The goal of this work is to extend traditional scaling techniques to complex multicoupled transport problems. Scaling analysis involves iterations to identify the proper dominant terms. Performing these tasks manually becomes impractical or impossible for systems described by many equations involving many coupled driving forces. This paper introduces a framework to perform these iterations automatically. Before addressing this topic, basic concepts of scaling need to be introduced.

Definition of Scaling Within the scope of this work, scaling means a procedure to obtain characteristic values for the unknown variables in the governing equations [6]. The procedure is based on self-consistent simplifications to determine which terms in the equations can be neglected [7]. Self consistency means that the test of whether the neglected terms are indeed small is made using the estimations which already assume those terms are negligible [8]. A characteristic value is typically considered to be the maximum order of magnitude of a variable over the volume and time span considered [6]. This definition can be meaningless in systems involving singularities, such as the case of the viscous boundary layer. A better definition of characteristic value is part of work in progress, and in this paper finite values of relevant parts of the domain are chosen as characteristic values, even if there are higher values elsewhere in the domain.

Typical Scaling Procedure All scaling approaches have the same conceptual core in their procedure:

1. Write the governing equations including boundary conditions.
2. Scale the dependent, independent variables, and differential expressions. Some characteristic values may be unknown.
3. Replace scaled expressions into governing equations.
4. Normalize governing equations using the term expected to be dominant.
5. Solve for the unknown characteristic values by choosing terms where they are present and making their coefficients equal to 1.
6. Verify that the terms not chosen are not larger than one.
7. If any term is larger than one, then normalize equations again and/or pick different terms to solve for the unknowns.

Some authors state these items as a clear set of steps to be performed, for example Denn [9] and Sides [10], while other authors list them as useful heuristics, for example Astarita [11]. In all cases, to have some qualitative knowledge of the problem is a prerequisite. Some general idea about the shape of the functions is necessary to establish the correct characteristic values and to properly normalize the differential expressions.

The iterative procedure described above can be very tedious to perform manually for large systems of equations (of the order of four or more), especially if they are coupled. In many

cases, it is impossible to exhaust all possible iterations by hand, and the scaling might contain inconsistencies or ambiguities.

State of the Art

The original and simplest approach to obtaining scaling laws from differential equations is inspectional analysis, which involves the construction of dimensionless groups from the governing equations. This approach was briefly presented by Bridgman [12], made explicit by Ruark [13], and is included in classic textbooks on Transport Phenomena such as Geankopolis[14], Bird, Stewart, and Lightfoot [15], and Szekely and Themelis [16]. More recently, authors devoted entire chapters or whole books to exploring deeper aspects of scaling. Among them Denn [9] and Deen [6] devoted a whole chapter to scaling, Kline [17] devoted a whole book, and Dantzig and Tucker [7] put emphasis on scaling throughout their whole book on modeling of materials processing. Bender and Orszag focused on simplification methods based on self-consistency and explain the "dominant balance" method for simplification of differential equations [18]

Although little in comparison to other modeling techniques, peer reviewed journals have published articles specific to scaling. Sides [10], Chen [19], and Astarita [11] focused on heuristics useful for scaling. All this articles aim at the manual obtention of unknown characteristic values, thus put emphasis on rough estimations of the differential expressions, and avoid considering unwieldy coupling between the equations. Yip [20] proposed an artificial intelligence approach based on an automated search for self-consistent balances, yet still using rough estimates for the differential expressions. The approach presented in this paper was first introduced in [21], followed by [22, 23, 24, 25].

Traditional Scaling of the Boundary Layer

To put OMS within a context, it is useful to analyze a well known problem using traditional scaling techniques. The viscous boundary layer is chosen because it is a classic example of scaling that appears in many texts. The governing equations for the boundary layer are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{2}$$

$$u\frac{\partial v}{\partial x} + V\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{3}$$

with the following boundary conditions: $u(x > 0, 0) = 0$, $v(x > 0, 0) = 0$, $u(-\infty, y) = U$, $v(-\infty, y) = 0$, $p(x, \infty) = 0$.

In these governing equations x and y are the independent variables, represented in Figure 1. The dependent variables are the velocity in the x direction $u(x, y)$, the velocity in the y direction $v(x, y)$, and the pressure $p(x, y)$. The parameters for this example are the fluid density ρ =1000 kg/m^3, the kinematic viscosity $\nu = 10^{-6}$m^2/s, the free stream velocity U=0.1 m/s, and the length of the boundary layer L=0.1 m.

The goal of scaling in this case is to obtain expressions for the unknown characteristic values δ for the boundary layer thickness, v_c for the transverse velocity, and p_c for the pressure variations within the boundary layer. In typical scaling, the differential expressions are esti-

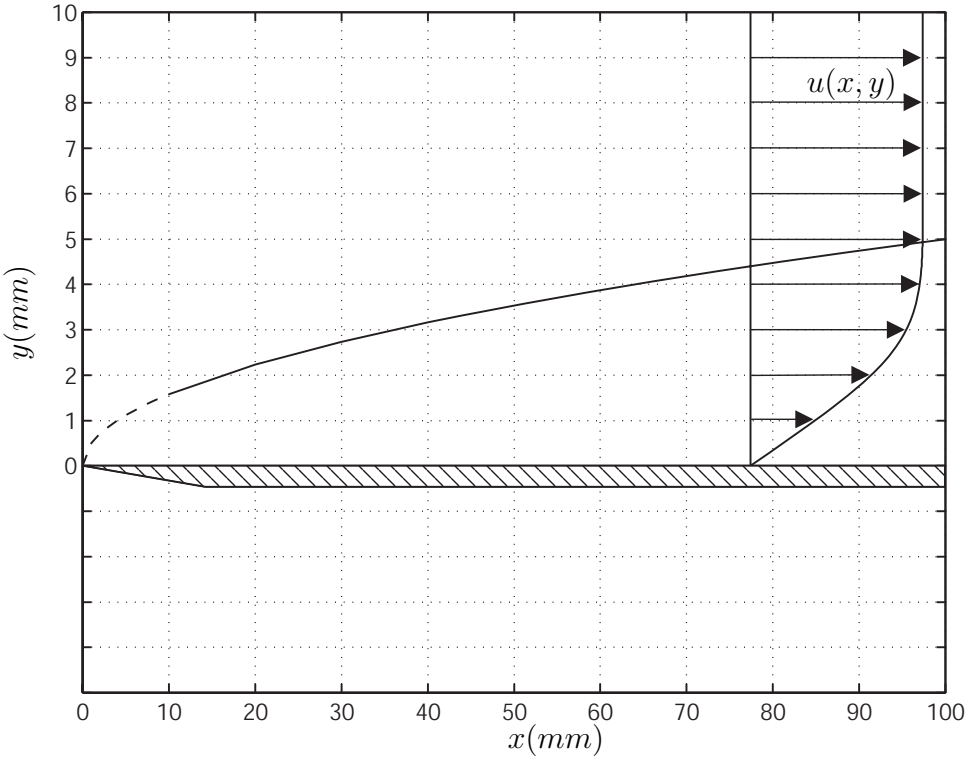

Figure 1: Behavior of a boundary layer of a 100 mm/s stream of water over a length of 100 mm.

mated as the variation of the function divided the variation of the independent variable, for example $\partial u/\partial x = (U/L)\partial u^*/\partial x^*$, and $\partial^2 u/\partial x^2 = (U/L^2)\partial^2 u^*/\partial x^{*2}$. The asterisk (*) indicates normalized variables. With this normalization approach, the estimated characteristic values do not involve a numeric coefficient, and Equations 1 through 3 become

$$\frac{U}{L}\frac{\partial u^*}{\partial x^*} + \frac{\widehat{v}_c}{\widehat{\delta}}\frac{\partial v^*}{\partial y^*} = 0 \tag{4}$$

$$\frac{U^2}{L}u^*\frac{\partial u^*}{\partial x^*} + \frac{\widehat{v}_c U}{\widehat{\delta}}v^*\frac{\partial u^*}{\partial y^*} = -\frac{\widehat{p}_c}{\rho L}\frac{\partial p^*}{\partial x^*} + \frac{\nu U}{L^2}\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\nu U}{\widehat{\delta}^2}\frac{\partial^2 u^*}{\partial y^{*2}} \tag{5}$$

$$\frac{U\widehat{v}_c}{L}u^*\frac{\partial v^*}{\partial x^*} + \frac{\widehat{v}_c^{\,2}}{\widehat{\delta}}v^*\frac{\partial v^*}{\partial y^*} = -\frac{\widehat{p}_c}{\rho\widehat{\delta}}\frac{\partial p^*}{\partial y^*} + \frac{\nu\widehat{v}_c}{L^2}\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\nu\widehat{v}_c}{\widehat{\delta}^2}\frac{\partial^2 v^*}{\partial y^{*2}} \tag{6}$$

with the following boundary conditions: $u^*(x^* > 0, 0) = 0$, $v^*(x^* > 0, 0) = 0$, $u^*(-\infty, y^*) = 1$, $v^*(-\infty, y^*) = 0$, $p^*(x^*, \infty) = 0$.

Typical scaling results obtained are:

$$\widehat{\delta} = \sqrt{\nu L/U} = 1 \text{ mm} \tag{7}$$

$$\widehat{v}_c = \sqrt{\nu U/L} = 1 \text{ mm/s} \tag{8}$$

Traditional analysis of the boundary layer typically scales pressure with ρU^2, eventually coming to the conclusion that the any meaningful characteristic value must be significantly smaller than that; thus, it does not provide an estimation of the characteristic pressure.

Addressing the need to perform iterations manually

This section introduces the formalism that enables to generate estimations of the characteristic values and test them for self consistency in an automatic way.

Normalization of the differential expressions and governing equations typically results in power-law expressions. Operations involving power-laws can be represented using matrix algebra, which can be easily implemented in computer algorithms.

The Matrix of Coefficients

The power-law coefficients present in Equations 4 through 6 can be captured in matrix form with the following expression:

$$\left\{\begin{array}{l}\ln C_1\\ \ln C_2\\ \hline \ln C_3\\ \ln C_4\\ \ln C_5\\ \ln C_6\\ \ln C_7\\ \hline \ln C_8\\ \ln C_9\\ \ln C_{10}\\ \ln C_{11}\\ \ln C_{12}\end{array}\right\} = \left[\begin{array}{cccc|ccc} & & 1 & -1 & & & \\ & & & & -1 & 1 & \\ \hline & & 2 & -1 & & & \\ & & 1 & & -1 & 1 & \\ -1 & & & -1 & & & 1 \\ & 1 & 1 & -2 & & & \\ & 1 & 1 & & -2 & & \\ \hline & & 1 & -1 & & 1 & \\ & & & & -1 & 2 & \\ -1 & & & & -1 & & 1 \\ & 1 & & -2 & & 1 & \\ & 1 & & & -2 & 1 & \end{array}\right] \left\{\begin{array}{l}\ln \rho\\ \ln \nu\\ \ln U\\ \ln L\\ \hline \ln \widehat{\delta}\\ \ln \widehat{v_c}\\ \ln t_c\widehat{p_c}\end{array}\right\} \tag{9}$$

where C_1, C_2, C_3, etc. represent power law coefficients. In this work the resulting matrix is called "Matrix of Coefficients" $[C]$, which was introduced in [21] and refined in [24]. The elements with a zero value have been omitted for simplicity. The vertical line in the matrix separates the parameters from the unknown characteristic values. The horizontal lines in the matrix divide the terms corresponding to different equations.

Notation

This section presents a consistent notation system for the several vectors, matrices and submatrices involved in the computer implementation of scaling iterations.

$\{\ldots\}$ Indicates a column vector. For example, $\{P\}$ indicates the column vector of parameters.

$(\ldots)$ Indicates a column vector of logarithms. This way, (P) indicates a column vector in which each element is a logarithm of the corresponding element $\{P\}$.

$[\ldots]$ Indicates a matrix, for example the Matrix of Coefficients defined in Section has the symbol $[C]$

$[\ldots]_{\text{cols}}$ The subscript indicates a submatrix based on the set of columns indicated by the subscript "cols." For example, the Matrix of Normalized Coefficients $[N]$ can be divided into three submatrices $[N]_K$, $[N]_P$, and $[N]_S$ in which each submatrix formed with the columns of $[N]$ corresponding to the numerical constants K, the parameters P or the unknowns S. Using this notation facilitates matrix operations such as

$$[N]\left\{\begin{array}{c}(K)\\(P)\\(S)\end{array}\right\} = [N]_K(K) + [N]_P(P) + [N]_S(S) \tag{10}$$

$[\ldots_{\text{rows}}]$, $\{\ldots_{\text{rows}}\}$, or $(\ldots_{\text{rows}})$ The subscript indicates a submatrix based on the set of rows indicated by the subscript. The subscript i indicates a dominant input row, the subscript o indicates a dominant output row, and the subscript s indicates a row corresponding to a secondary term.

$[\widehat{\ldots}]$ Indicates that in this matrix the columns corresponding to the unknowns have been replaced by the estimations through the following operation

$$[\widehat{\ldots}] = [\ldots]_{P'} + [\ldots]_S[S] \tag{11}$$

where set $\{S\}$ is the set of parameters, set $\{P'\}$ is defined in Equation 13, and matrix $[S]$ will be defined later in Equation 17. Using the Matrix of Normalized Coefficients as an example, with this notation we obtain $[\widehat{N}] = [N]_{P'} + [N]_S[S]$

Using this notation convention, Equation 9 can be written in a simpler and more general form as

$$(C) = [C]\left\{\begin{array}{c}(P')\\(S)\end{array}\right\} \tag{12}$$

where (C) is the vector of logarithms of coefficients, (P') is the vector of logarithms of the extended set of parameters, (S) is the vector of logarithms of the unknown characteristic values, and $[C]$ is the Matrix of Coefficients. The extended set of parameters $\{P'\}$ includes the numerical constants when they are present:

$$\{P'\} = \left\{\begin{array}{c}\{K\}\\\{P\}\end{array}\right\} \tag{13}$$

where $\{K\}$ is the vector of numerical constants.

Normalization of governing equations

The normalization of the governing equations is traditionally performed manually. The definition of the Matrix of Coefficients enables one to perform the normalization of the equations (Step 4) very easily in a computer. Normalization consists of subtracting the line of the term expected to be dominant from all the other lines corresponding to the same equation. Normalizing all equations this way, the Matrix of Coefficients $[C]$ is transformed into the Matrix of Normalized Coefficients $[N]$. By definition, one line for each equation in Matrix $[N]$ will be a row of zeros, corresponding to the term assumed to be dominant.

Solving for the characteristic values

Step 5 consists of choosing some terms in the normalized equations and make them equal to one. Current literature provides few guidelines for the choice of terms. Sides [10] suggests to choose coefficients that contain only one unknown characteristic value. This approach needs improvement for coupled problems, where unknown characteristic values typically appear simultaneously. In the methodology proposed here, the coefficients are represented by rows of the Matrix of Normalized Coefficients $[N]$. Thus, the rows selected for solving for the unknown characteristic values are chosen according to the following rules:

- Rows selected are non-zero
- No more than one row per equation is selected
- The total number of rows selected equals the number of unknown characteristic values

The rows selected constitute a subset of the Matrix of Normalized Coefficients: $[N_o]$. Step 5 states that the selected coefficients must be equal to 1. Taking logarithms on both sides, it can be expressed by the following operation:

$$[N_o] \left\{ \begin{array}{c} (P') \\ (\widehat{S}) \end{array} \right\} = \{0\} \tag{14}$$

or

$$[N_o]_{P'}(P') + [N_o]_S(\widehat{S}) = \{0\} \tag{15}$$

where $\{0\}$ is a vector of zeros and $\widehat{S}$ are estimations of the unknown characteristic values. Solving for $\widehat{S}$, a set of estimations is obtained simultaneously:

$$(\widehat{S}) = [S](P') \tag{16}$$

where the matrix of estimations $[S]$ contains the exponents of the parameters in the estimations of the unknowns.

$$[S] = -[N_o]_S^{-1}[N_o]_{P'} \tag{17}$$

If $[N_o]_S$ is singular, the chosen terms yield an incompatible system.

Consistency check

Step 6 consists of verifying that no term in the equation is larger than the terms considered dominant and balancing. This check is typically accomplished by replacing the estimated values of the unknown characteristic values into the coefficients of the normalized equations. If any coefficient is larger than 1, then the corresponding choice of terms is inconsistent, and a new combination should be chosen. This is a very tedious process for coupled systems, which often involve many equations with many coefficients each, easily reaching into the several tens of coefficients.

The formulation proposed here is also instrumental for the computer implementation of the consistency check. The numerical values of the coefficients are calculated as

$$(N) = [N]_{P'}(P') + [N]_S(S) \tag{18}$$

The unknown scaling factors $\{S\}$ can be estimated using Equation 16, then:

$$(\widehat{N_s}) = [\widehat{N_s}](P') \tag{19}$$

where

$$[\widehat{N_s}] = [N_s]_{P'} + [N_s]_S[S] \tag{20}$$

Because these operations are based on the logarithms of the values of the parameters, the self-consistency condition can be stated as

$$[\widehat{N_s}](P') \leq \{0\} \tag{21}$$

If any coefficient is larger than 1, Step 7 indicates that a new combination of terms must be chosen (back to Step 4). Each balance is be identified by the matrix of balances $[B]$ with one row per equation chosen, and three columns: the first one indicating the equations considered, the second one indicating the terms used to normalize the equations, and the third indicating the terms chosen to be equal to 1.

Subtleties of self-consistency

All scaling approaches are based on self-consistency, which is a circular type of reasoning that reveals an incorrect choice of terms for the estimations but cannot assure that a self-consistent choice is indeed correct. This aspect of self-consistency has been analyzed in detail by Segel [8], Riley [26], and Denn [9]. There are some indications that using the matrix procedure proposed might be possible to identify this situation. The condition number of $[N_o]_S$ becomes very large while approaching this condition. Further work is necessary to confirm this observation.

A particular case of self consistency occurs when a normalized equation has more than two terms of the order of magnitude of 1. In this case, even if the secondary forces are considered zero, the balance between three terms indicates that Step 5 must be revisited, because equating the unknown terms to 1 is inconsistent with the fact that some neglected terms are of the same order than those considered. The matrix procedure presented here helps solve this minor inconsistency; the analysis of this case is beyond the scope of this paper.

Another consistency issue is the balance between two terms of the same dominant order of magnitude, but with both playing the same role as "inputs" or "outputs." Generalizing from conservation equations, terms can be classified as inputs or outputs (e.g. mass input). Clearly, a proper balance must match an input term with an output term. A balance between two inputs, neglecting all outputs has no physical sense. In the computer implementation proposed, a "vector of inputs and outputs" $\{\Phi\}$ is defined. Each element of this vector corresponds to a coefficient, and can be +1, -1, or 0 for an input role, output role or unknown role respectively. A term is defined as input if it has a positive sign when on the left hand of the equation.

Order of Magnitude Scaling of the Boundary Layer Problem

For the boundary layer problem stated above, the Matrix of Coefficients is presented in Equation 9. An analysis of each term based on available qualitative information yields the following vector of inputs and outputs:

$$\{\Phi\}^T = \{-1, 1, -1, 1, 0, -1, 1, -1, 1, 0, -1, 1\} \tag{22}$$

This type of qualitative information is necessary even for traditional scaling procedures. Fortunately, coarse, qualitative information is almost always available. In this case, the term

roles were determined by observation of Blasius solutions [27]. Since the pressure terms are not determined in those solutions, the role of the pressure terms is unknown.

Computer code in MATLAB was developed by Prof. N. Stier of Columbia University and by Mr. J. P. Díaz-Orbán at MIT to test all possible combinations of inputs and outputs of the Matrix of Coefficients. A total of 64 combinations were tested, of which 35 (55%) were incompatible, 22 (34%) were inconsistent, and 7 (11%) were self consistent.

The self-consistent solutions could be grouped in two classes yielding two different matrices of scaling factors $[S]$:

$$[S]_1 = \begin{array}{c} \begin{array}{cccc} \rho & \nu & U & L \end{array} \\ \left[\begin{array}{cccc} & & & 1 \\ & & 1 & \\ 1 & & 2 & \end{array}\right] \end{array} \tag{23}$$

$$[S]_2 = \begin{array}{c} \\ \left[\begin{array}{cccc} & 1/2 & \text{-}1/2 & 1/2 \\ & 1/2 & 1/2 & \text{-}1/2 \\ 1 & 1 & 1 & \text{-}1 \end{array}\right] \end{array} \tag{24}$$

Class 1 has includes four balances, and Class 2 three balances. The normalized systems of equations are different for each class of balances. Class 1 yields the following set of normalized equations:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{25}$$

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \mathrm{Re}^{-1} \frac{\partial^2 u^*}{\partial x^{*2}} + \mathrm{Re}^{-1} \frac{\partial^2 u^*}{\partial y^{*2}} \tag{26}$$

$$u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial y^*} + \mathrm{Re}^{-1} \frac{\partial^2 v^*}{\partial x^{*2}} + \mathrm{Re}^{-1} \frac{\partial^2 v^*}{\partial y^{*2}} \tag{27}$$

where Re is the Reynolds number UL/ν. Re is large in this example (Re=10^4). This class of self-consistent balances is discarded because it violates the common definition of boundary layer as the region where the viscous terms are of the same order of magnitude of the inertial terms. Class 2 yields the following set of normalized equations:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{28}$$

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\mathrm{Re}^{-1} \frac{\partial p^*}{\partial x^*} + \mathrm{Re}^{-1} \frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}} \tag{29}$$

$$u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial y^*} + \mathrm{Re}^{-1} \frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}} \tag{30}$$

Class 2 of balances does not violate the definition of boundary layer and is self-consistent. The estimations of boundary layer thickness and transverse velocity are the same as with traditional scaling performed manually (Equations 7 and 8). Additionally, Class 2 yields an estimation of pressure variation within the boundary layer. The normalization typically used

in the manual procedure ($p^* = p/(\rho U^2)$) does not provide an estimation of the characteristic value. The estimation obtained here can be expressed as.

$$\widehat{p_c} = \rho {v_c}^2 \tag{31}$$

This estimation is seldom seen in texts scaling the boundary layer equations. It has been previously obtained by Walz [28] as a further derivation from Prandtl's analysis.

Discussion

Similarly to traditional scaling, the approach presented is also based on self consistency. Thus, it is not possible to prove that a solution is right, it is only possible to realize when a proposed scaling is wrong. While a proof of having found the right answer would be desirable, it might no be feasible, given that there is no proof of existence and unicity of a solution for many of the large, coupled, non-linear systems of equations that appear in advanced transport problems.

This paper does not address subtleties in the estimation of characteristic values for differential expressions. For manual scaling, the prevailing wisdom is "the cruder the better" [11], meaning that ignoring factors of the order of 2 or 5 is tolerable. For large coupled systems, these factors might aggregate to errors beyond an order of magnitude, and better estimations are necessary. A better estimation of the characteristic values of differential expressions is currently work in progress.

Still within a frame of self-consistency, the matrix approach presented is useful to estimate the errors involved in neglecting the secondary driving forces. The approach proposed is also helpful to obtain directly from the computer a set of dimensionless groups ranked by order of relevance, which can be used to construct process maps. Also, a backwards application of Buckingham's theorem [29] to the set of dimensionless groups obtained permits to build an extended set of reference units that can simplify the application of dimensional analysis to a whole family of problems.

Conclusions

A formalism called here Order of Magnitude Scaling (OMS) for automating portions of the traditional scaling procedure was developed. OMS enables the computer implementation of traditional scaling techniques. This way much larger systems of coupled equations can be analyzed exhaustively to obtain all meaningful scaling laws representing a problem.

Within this formal framework, a Matrix of Coefficients is defined, which enables one to obtain estimations of the unknown characteristic values in a system using standard linear algebra tools. With this matrix approach it is also possible to automate normalization, dominant balance, and check for self-consistency. Representing the set of equations with the Matrix of Coefficients, an estimation of the unknown characteristic values can be made using Equation 17.

Application of the proposed formalism to the analysis of the viscous boundary layer without an imposed pressure gradient yields the traditional scaling factors for boundary layer thickness and transverse velocity, and in addition, it provides a meaningful scaling factor

for pressure variations within the boundary layer. This last scaling factor is typically not addressed in the manual analysis.

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