The Matrix of Coefficients in Order of Magnitude Scaling

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1 Abstract

This paper introduces the matrix of coefficients, which summarizes the physical insight into a problem. This matrix is part of a larger methodology, Order of Magnitude Scaling, which provides closed form estimations of the unknowns, their range of validity, and a set of dimensionless groups that indicate the true ratio of driving forces. These results are obtained even when the problems are described by non-linear partial differential equations. Order of Magnitude Scaling focuses on problems with many driving forces and relatively simple geometries. The matrix of coefficients is the starting point for obtaining these results in a systematic way through matrix operations. This methodology can be computationally much faster than methods that numerically integrate the governing equations, and does not present convergence problems.

2 Introduction

Engineering problems are often described by a system of partial differential equations. These equations present three main challenges:

- there is uncertainty about their validity,
- they are very difficult to solve,

- even when the proper equations are solved, the generalization of results is very difficult.

Figure 1 shows how the introduction of digital computers allowed researchers to increase the complexity of problems analyzed, thus the difficulty in generalizing the results. The traditional tools for analysis are not completely satisfactory for generalizing the newer, more complex problems.

Order of Magnitude Scaling (OMS) is a framework that addresses these challenges by providing an approximate solution to the set of governing equations. This solution has important characteristics:

- it is achieved with little computational time
- it is expressed in closed-form
- its range of validity is clearly determined
- it discriminates the dominant factors in the problem from the less relevant ones

This paper provides engineers with a tool that can help them systematically find an approximate answer to a complex problem.

OMS is described in detail in a publication in preparation and in references [1–4]. This paper discusses the matrix of coefficients, which is one of the key elements of OMS.

2.1 Related Work

Research on the simplification of governing differential equations has come almost exclusively from three sources: mathematics, engineering, and artificial intelligence.

Mathematical approaches include theory of approximations [5], perturbations [6, 7] and asymptotic analysis [8]. They usually focus on an exact solution to one simplified equation. OMS borrows from the mathematical approaches the dominant balance technique [7]. OMS differs from these approaches by solving only for the characteristic values of the unknown functions, instead of solving for all values over the domain.

Engineering approaches vary greatly in their reliance upon intuition and formalism [9–11]. They usually involve dimensional analysis and inspection of the governing equations to determine the relevant dimensionless groups (inspectional analysis [12]). The normalization of the governing equations is considered proper when the coefficients of all terms in the equations are one or less. OMS borrows from these approaches the use of dimensionless groups and the emphasis on characteristic values. OMS differs from these approaches in the use of matrix algebra and the rules for normalization of functions.

Artificial Intelligence approaches include qualitative reasoning [13, 14], order of magnitude reasoning [15–21], and case-based reasoning [22]. These approaches focus on obtaining estimations of the order of magnitude of the solutions through a series of formal relations among quantities. OMS is similar to these approaches in that it can be implemented in the form of a computer algorithm. OMS and these approaches also share some common concepts, such as a description of the dominant physics of the problem in a space of parameters [23–25]. OMS differs from these approaches in that it can deal with partial differential equations and that it uses matrix algebra instead of sets of rules or constraints. Yip [26] presents an approach (AOM) aimed at partial differential equations. OMS has a larger scope than AOM because it provides an expression of the range of validity of the approximations, an estimation of the errors involved, and a reduced set of dimensionless groups that can be used to correct the approximations.

2.2 Scope of Order of Magnitude Scaling

Order of Magnitude Scaling is aimed at systems with simple geometry but many driving forces acting simultaneously, indicated by the shaded area in Figure 1. In this figure, the vertical axis represents geometrical complexity, measured as the number of dimensionless groups related to the geometry of the problem (m_g). The horizontal axis represents physical complexity, measured as the number of groups related to the physics of the system (m_p). Figure 1 divides modeling problems into four different categories relative to the difficulty of generalizing the results.

The region on the lower left of Figure 1 contains problems with simple geometries and simple physics. These problems are the easiest to generalize because they involve at most two parameters; therefore the problems can be completely represented in a simple two-dimensional map. Generalizations in this case can be based on induction from particular numerical models or experiments. Deduction from closed-form solutions can also be used to generalize these problems. The region on the upper left contains problems with relatively simple physics, but complex geometries. If geometric similarity is preserved, the physics of the problem can be generalized by induction from the analysis of particular cases through numerical models or experiments. The region on the bottom right represents systems with complex physics and simple geometry, and the region on the top right represents the most difficult cases to generalize: those with both complex physics and geometries. Problems belonging in these last two regions are very difficult to generalize with the current tools; OMS provides a systematic framework to generalize those with complex physics and simple geometry, indicated by the shaded area in Figure 1.

The dashed line in the figure shows the evolution in the modeling of the welding arc, which is representative of modeling of many other processes. The trend indicates that the increase in sophistication in the models focuses initially on adding more physical considerations, and only later, when the physics are well understood, is geometrical complexity increased. The dashed line lies mostly over the shaded area indicating the usefulness of OMS as models become more complex.

2.3 Methodology

OMS has two main stages, the first one is representing the governing equations in matrix form: the matrix of coefficients. The second stage involves iterations and matrix operations that solve for the unknowns, provide the range of validity of the approximations, and determine the quality of the approximations.

This paper focuses on the first stage, the obtention of the matrix of coefficients. This stage is the one that requires most judgment from the engineer. If the matrix of coefficients is correct, the second stage is automatic and can be carried out by a computer, with a minimum of human interaction for assessing the results.

3 Governing Equations in Engineering Problems

Governing equations in engineering problems often have the following characteristics listed below. OMS uses these features to obtain its results.

- Each equation is a sum of terms, with each term representing a driving force.
- The geometry of the problem can be divided into a few different regions where the second derivatives are not large

3.1 Basic Concepts

A physical system is modeled by describing its relevant attributes such as temperature at each point, velocities, or pressure. These attributes depend on the location of a given point within a system (space), and on the moment the attribute is considered (time). Since any time and location relevant to the system can be chosen, these magnitudes will be called "independent variables." The functions that relate the relevant attributes to the independent variables will be called "dependent variables."

As an illustration, we consider one of the Navier-Stokes equations in two-dimensions, steady-state, and rectangular coordinates:

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = -\frac{1}{\rho}\frac{\partial P}{\partial X} + \frac{\mu}{\rho}\left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}\right) \tag{1}$$

where U and V are the flow velocity components on the X and Y directions, P is the fluid pressure, μ is the viscosity and ρ the density of the fluid.

In equation 1, there are two independent variables, which describe position; and three dependent variables, which describe velocity and pressure at each point.

The solution of the governing equations will also depend on the values of ρ and μ, among other magnitudes that would appear in other governing equations and boundary conditions which are not shown here. If the mathematical formulation of the problem has a unique solution, that solution is completely determined by these magnitudes, which will be called "parameters." It is customary to call them "variables" in the dimensional analysis literature. Calling the set of parameters $\{P\}$, for a problem involving equation 1 we have:

$$\{P\}^T = \{\rho, \mu, \ldots\} \tag{2}$$

3.2 General Form of Governing Equations

Most governing equations in engineering have the form of a sum of terms, where each term is the product of a combination of parameters and a combination of dependent variables. For example, equation 1 can be expressed as indicated below, where the dependence of functions on X and Y has been omitted for simplicity.

$$
\begin{array}{cccccccccccccccc}
 & U\frac{\partial U}{\partial X} & + & & V\frac{\partial U}{\partial Y} & + & \frac{1}{\rho} & \frac{\partial P}{\partial X} & - & \frac{\mu}{\rho} & \frac{\partial^2 U}{\partial X^2} & - & \frac{\mu}{\rho} & \frac{\partial^2 U}{\partial Y^2} & = & 0 \\
 & \downarrow & & & \downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow & & \\
A_1 & F_1 & + & A_2 & F_2 & + & A_3 & F_3 & + & A_4 & F_4 & + & A_5 & F_5 & = & 0 \\
\downarrow & & & \downarrow & & & \downarrow & & & \downarrow & & & \downarrow & & & \\
\rho^0\mu^0 & & & \rho^0\mu^0 & & & \rho^{-1}\mu^0 & & & \rho^{-1}\mu^1 & & & \rho^{-1}\mu^1 & & &
\end{array}
$$

where A_i are the coefficients of the original governing equations, and F_i are the functions of the independent variables for each term of the original governing equations.

In general, a system of governing equations can be represented as group of equations of the following form:

$$\sum_{i=1}^{t} A_i F_i(\mathbf{X}) = 0 \tag{3}$$

where $\mathbf{X}$ represents the independent variables $X_1, X_2, X_3, \ldots$ The coefficients have the form of the product of the parameters raised to a power:

$$A_i = \prod_{j=1}^{n} P_j^{a_{ij}} \tag{4}$$

where a_{ij} is the exponent of parameter j in coefficient i. The exponents involved can be any real number, although they are frequently small natural numbers or ratios of small natural numbers. For practical reasons, the parameters are defined as always positive, adjusting the sign of the corresponding function of the independent variable if necessary.

This representation of the governing equations is valid for a broad range of governing equations used in engineering, including Fourier's law of heat transmission, Navier-Stokes equations for fluids, Maxwell's equations for electromagnetism, and equations of equilibrium in the theory of elasticity.

3.3 Normalized Expression of the Governing Equations

The normalization of the governing equations is a two-step process. First, the dependent and independent variables need to be normalized; second, the normalized variables are replaced into the original governing equations, which can then be normalized by any coefficient.

3.3.1 Normalization of the Independent Variables

A convenient way of normalizing the independent variables is by using their range. In general, the normalized independent variables will have the following form

$$x = \frac{X - X_{min}}{X_{max} - X_{min}} \tag{5}$$

where X is an independent variable that varies between X_{min} and X_{max}, and x is its normalized counterpart. One of the features of this normalization is that the normalized independent variables vary between 0 and 1.

3.3.2 Normalization of Functions of the Independent Variables

The proper normalization of the functions of the independent variables is key for the proper determination of the matrix of coefficients. These functions will be normalized by their characteristic values as indicated below:

$$f_i(\mathbf{x}) = \frac{F_i(\mathbf{X})}{F_{iC}} \tag{6}$$

where $F_i(\mathbf{X})$ is a function of the independent variables, F_{iC} is its characteristic value, and $f_i(\mathbf{x})$ is its normalized counterpart.

In this paper, we define the characteristic value of a function as its maximum absolute value over a given domain; therefore, the normalized functions will have a maximum absolute value of one. As with the independent variables, the characteristic values are always positive by definition.

The characteristic values of the functions can be estimated as a function of the characteristic values of simpler functions; for example, under certain conditions, the characteristic value of a product of functions can be estimated as the product of the characteristic value of each function. Similarly, the characteristic value of a differential expression can be estimated as a function of the characteristic values of the original function and the independent variables.

The simple aspect of equation 6 can be misleading, because this normalization has many important properties and consequences. One of its properties is that it enables us to compare the magnitude of each term in a normalized equation using only the coefficients. Another interesting property is that even when the characteristic value is unknown, the normalized function still has a maximum absolute value of one. This allows us to estimate the unknown characteristic values by extracting simple algebraic equations from the differential equations, which are usually very difficult to solve.

One of the important consequences of this normalization is that for governing equations involving differential expressions, the domain usually needs to be divided into different regions, each of them with a clear physical meaning. Also, the choice of the coordinate system in which the equations are expressed has an influence on the ability to estimate the characteristic values.

The expression of equation 1 using normalized functions is the following:

$$F_{1C} f_1 + F_{2C} f_2 = -\frac{F_{3C}}{\rho} f_3 + \frac{\mu F_{4C}}{\rho} f_4 + \frac{\mu F_{5C}}{\rho} f_5 \tag{7}$$

where F_{iC} are the characteristic values of the functions of each term of equation 1, and f_i is its normalized counterpart. For simplicity the functional dependence on x, and y, has been omitted in the notation. The coefficients of the governing equation with normalized functions are contained in set $\{C\}$. In our example:

$$\{C\}^T = \left\{F_{1C}, F_{2C}, \frac{F_{3C}}{\rho}, \frac{\mu F_{4C}}{\rho}, \frac{\mu F_{5C}}{\rho}\right\} \tag{8}$$

3.3.3 Normalization Using Unknown Characteristic Values

When the characteristic values are unknown, the functions are normalized by the estimated characteristic values:

$$f_i(\mathbf{x}) = \frac{F_i(\mathbf{X})}{\widehat{F}_{iC}} \tag{9}$$

where $\widehat{F}_{iC}$ is the estimation of characteristic value F_{iC}.

3.3.4 Order of Magnitude of the Normalized Functions

When a function of the independent variables is normalized by its characteristic value, its maximum absolute value will be 1, by definition. In other parts of the domain, the function can have smaller absolute values, including zero. If the function is normalized by an estimation of the characteristic value that has the correct order of magnitude, the maximum absolute value of the normalized function will also be of the order of magnitude of one. We will use the notation OM(1) for these type of normalized functions. In mathematical terms, the maximum absolute value of a function of OM(1) has upper and lower bounds of the order of magnitude of 1.

If the characteristic can only be estimated with an upper bound, the maximum absolute value of the normalized function could be of the the order of magnitude of one or less. We will use the notation O(1) for these type of normalized functions. In mathematical terms, the maximum absolute value of a function of O(1) has upper bound of the order of magnitude of 1, and a lower bound of zero.

The distinction between normalized functions of OM(1) and of O(1) is relevant when solving for the unknown characteristic values.

3.3.5 Normalization of Governing Equations

After replacing normalized functions in the governing equations, the equations themselves can be normalized by any coefficient. As an example, if we normalize equation 7 using the coefficient of the first term, we obtain

$$f_1 + \frac{F_{2C}}{F_{1C}} f_2 = -\frac{F_{3C}}{\rho F_{1C}} f_3 + \frac{\mu F_{4C}}{\rho F_{1C}} f_4 + \frac{\mu F_{5C}}{\rho F_{1C}} f_5 \tag{10}$$

The coefficients of the governing equation with normalized functions are contained in set $\{N\}$. In our example:

$$\{N\}^T = \left\{1, \frac{F_{2C}}{F_{1C}}, \frac{F_{3C}}{\rho F_{1C}}, \frac{\mu F_{4C}}{\rho F_{1C}}, \frac{\mu F_{5C}}{\rho F_{1C}}\right\} \tag{11}$$

There are several different sets $\{N\}$, depending on the coefficient used to normalize the equation.

3.3.6 Characteristic Value of Combinations of Functions

The multiplication of functions of OM(1) will yield another function of OM(1) when their absolute value is ≈ 1 at the same point in the domain, usually at a corner points of the regions that subdivide the domain.

The multiplication of a function of OM(1) by one of O(1) will yield a function of O(1).

The result of the addition of two functions of OM(1) is more difficult to determine. If they have their maximum absolute value at the same point in the domain, and the sign of the function is the same, it is reasonable to expect that their addition will be OM(1). This argument is not valid, however for successive additions, because the sum of many functions OM(1) can be larger than OM(1).

3.3.7 Characteristic Value of Differential Expressions

The normalization scheme shown below provides estimations of the order of magnitude of the first and second derivatives of a function.

$$\left.\widehat{\frac{\partial G}{\partial X}}\right|_C = \frac{G_u - G_l}{X_u - X_l} \tag{12}$$

$$\left.\widehat{\frac{\partial^2 G}{\partial X_i \partial X_j}}\right|_C = \frac{G_u - G_l}{\left(X_{iu} - X_{il}\right)\left(X_{ju} - X_{jl}\right)} \tag{13}$$

When all the normalized second derivatives are O(1), the first derivatives will be O(1). The order of magnitude of the normalized second derivatives can be estimated intuitively by inspecting numerical or experimental results.

Lower bounds are necessary to determine whether a function is OM(1). Lower bounds can be determined for the first and second derivatives by inspecting their value at the corner points of the regions that subdivide the domain. For example, if in a problem with two independent variables the maximum and the minimum of a normalized function are over the same line parallel to the X-axis, the partial derivative with respect to X is OM(1).

An important aspect of this normalization is that it requires dividing the domain of the independent variables into different regions. Each of these regions can experience different balances between its driving forces. This division helps to build an intuitive and insightful description of the mechanics of the system.

4 The Matrix of Coefficients

Equation 7 shows that the coefficients have the form of a power-law, which can be expressed in matrix form by using logarithms:

$$\left\{ \begin{array}{c} \ln C_1 \\ \ln C_2 \\ \ln C_3 \\ \ln C_4 \\ \ln C_5 \end{array} \right\} = \begin{bmatrix} & & 1 & & & & \\ & & & 1 & & & \\ -1 & & & & 1 & & \\ -1 & 1 & & & & 1 & \\ -1 & 1 & & & & & 1 \end{bmatrix} \left\{ \begin{array}{c} \ln \rho \\ \ln \mu \\ \ln F_{1C} \\ \ln F_{2C} \\ \ln F_{3C} \\ \ln F_{4C} \\ \ln F_{5C} \end{array} \right\} \tag{14}$$

where the elements with a value of zero have been omitted for simplicity. In this expression, the matrix that relates the coefficients to the parameters and characteristic values is the matrix of coefficients. Equation 14 can be written in general as

$$(C) = [C] \left\{ \begin{array}{c} (P) \\ (S) \end{array} \right\} \tag{15}$$

where the parenthesis indicate a vector of logarithms; thus, (C) is the vector of logarithms of coefficients, (P) is the vector of logarithms of the parameters, (S) is the vector of logarithms of the characteristic values, and $[C]$ is the matrix of coefficients. The general form of this matrix is:

$$
\begin{array}{c|ccc|ccc|}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{P_1} & \multicolumn{1}{c}{\cdots} & \multicolumn{1}{c}{P_n} & \multicolumn{1}{c}{S_1} & \multicolumn{1}{c}{\cdots} & \multicolumn{1}{c}{S_q} \\
\cline{2-7}
C_{1,1} & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
C_{1,t_1} & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
\cline{2-7}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cline{2-7}
C_{p,1} & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
C_{p,t_p} & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
\cline{2-7}
\end{array}
$$

where the horizontal lines divide the lines corresponding to terms of the same equation. On the left of the vertical line are the n columns associated with the parameters, and on the right, the q columns associated with the unknown characteristic values. The number of equations is p, with equation i having t_i terms. The total number of rows of $[C]$ is t.

5 Estimation of Unknowns in OMS

The matrix of coefficients is the starting point for the systematic estimation of the unknown characteristic values, their range of validity and the relevance of secondary driving forces.

These results are obtained through an iterative process derived from the dominant-balance technique. The first step in this process consists of selecting a number of equations equivalent to the number of unknown characteristic values.

The second step consists of assuming that one term in each selected equation is a dominant one, and another one is balancing. Each selected equation is normalized by the coefficient of the dominant term. If the normalization of functions for the construction of the matrix of coefficients is correct, the dominant term in the normalized equations is OM(1).

In the third step matrix operations are performed, with two possible outcomes: no estimation of the unknowns is possible (incompatible case), or an estimation of the unknowns is obtained. If an incompatible case is obtained, a new combination of equations and terms is selected and a new iteration starts.

If an estimation is obtained, the fourth step is a test of self-consistency of the results obtained. This test consists of looking at the estimated value of the coefficients of the secondary terms in the normalized equations. If any one of the secondary coefficients is larger than one, this means that terms assumed to be secondary are larger than the dominant, and the combination tested is inconsistent. If all the secondary terms in the normalized equations are less than one, then the combination of terms assumed dominant and balancing is self-consistent.

5.1 Normalization using the Matrix of Coefficients

The normalization of a governing equation can now be performed by subtracting a given row from all the rows corresponding to that equation. Equation 10, which is equation 7 normalized by its first term, can be expressed as

$$(N) = [N] \left\{ \begin{array}{c} (P) \\ (S) \end{array} \right\} \tag{16}$$

where (N) is the set of logarithms of coefficients, and

$$[N] = \left[\begin{array}{ccccccc} & & & & & & \\ & & -1 & 1 & & & \\ -1 & & -1 & & 1 & & \\ -1 & 1 & -1 & & & 1 & \\ -1 & 1 & -1 & & & & 1 \end{array} \right] \tag{17}$$

In this matrix the first row has only zeros for elements, because it corresponds to the term used to normalize the equation. Assuming that all equations are normalized by their first coefficient, the general form of the matrix of normalized coefficients $[N]$ is:

$$\begin{array}{c|ccc|ccc|} & P_1 & \cdots & P_n & S_1 & \cdots & S_q \\ \cline{2-7} N_{1,1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ N_{1,2} & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ N_{1,t_1} & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \hline \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline N_{p,1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ N_{p,2} & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ N_{p,t_p} & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cline{2-7} \end{array}$$

The vertical line divides matrix $[N]$ in two submatrices: $[N]_P$ (left), that relates to the problem parameters; and $[N]_S$ (right), that relates to the unknown characteristic values. The total number of rows of $[N]$ is t.

The matrix expression of the governing equations permits the normalization of the governing equations in a systematic way, suitable for computer implementation.

6 Other Results Obtained from the Matrix of Coefficients

The matrix of coefficients is the starting point for estimating the unknowns. Other important information can be obtained through matrix operations.

Range of Validity of Estimations The validity of the estimations can be determined by the range in which the balances are self-consistent. This can be determined in closed form expression as the range in the space of parameters or space of arbitrary dimensionless groups where all the secondary terms are less than one.

Estimation of Errors An upper bound for the errors in the estimations can be obtained from the secondary coefficients.

Number of Independent Dimensionless Groups If we call $[N^*]$ the matrix of coefficients when the unknowns are replaced by their estimations, the number of independent dimensionless groups that describe the problem n is

$$n = \text{rank}[N^*] \tag{18}$$

This result is obtained without using units, and is more accurate than what Buckingham's theorem predicts, because occasionally Buckingham's theorem predicts a larger number of dimensionless groups than are present in the governing equations.

Arbitrary Dimensionless Groups The arbitrary dimensionless groups are a set of n independent dimensionless groups chosen arbitrarily. Once this set is chosen, the governing equations can be expressed in dimensionless form automatically.

Natural Dimensionless Groups There are infinite possible sets of n dimensionless groups. OMS provides a criterion for selecting a unique set that contains the correct order of magnitude of ratios of driving forces. These dimensionless groups are formed from the secondary coefficients and have different expressions for different regimes. Using the dimensionless groups it is possible to rank the relevance of the secondary driving forces. Important natural dimensionless groups can be used as the argument in correction functions that relate the estimations to the real values.

7 Discussion

The complex problems addressed by OMS can often be solved by numerical methods. OMS provides quick, approximate answers, while numerical methods are more accurate, though slower, both in time of development and in computational time. Because of OMS simplicity and speed, it is useful to test different governing equations when there is uncertainty about them. Numerical methods are not flexible because they are very specific to the governing equations used.

Since OMS is based on the dominant-balance technique, it provides self-consistent answers. There is no proof that these answers are unique or correct. This is seldom a problem, since these are very rare cases. The utility of such proof would also be limited, since there are no proofs of existence or unicity of solution for most governing equations, except the simplest ones. Segel [6] discusses the relation between self-consistency and correctness of an approximation.

The matrix of coefficients summarizes the physical insight into a problem separating the mathematical manipulations from the generation of simplifying hypotheses. In order to obtain this matrix an engineer needs to have a general idea of the form of the expected solution. With this knowledge the domain of the problem can be divided into regions, and characteristic values, known or unknown, can be assigned. This need of previous knowledge is not a significant limitation, since this information is almost always available from experience, observation or previous calculations.

Even when the solution to the problem is known, OMS provides value by expressing it in a closed form, by determining the existence and range of different regimes with particular balances of forces, and by generating sets of dimensionless groups that represent the ratio of secondary to driving forces. Currently, no computational method of solution can provide this type of information for systems described by non-linear, partial differential equations.

8 Conclusions

Order of Magnitude Scaling (OMS) is a framework aimed at problems with many driving forces and relatively simple geometries. OMS can deal with problems described by non-linear partial differential equations. It provides an estimation of the characteristic values of the unknowns, the range of validity of these estimations, and a set of natural dimensionless groups that can be used to estimate and improve the accuracy of the estimations. The dominant forces in each regime are clearly established, contributing to an intuitive understanding of the physics of the problem.

These results are obtained in much less computational time than methods that solve the differential equations numerically, and the expressions obtained have closed-form.

The matrix of coefficients is the starting point for systematic matrix operations that provide the results of OMS. This matrix summarizes the physical insight of engineers into a problem, enabling them to concentrate on the choice of equations and simplifications instead of mathematical manipulations.

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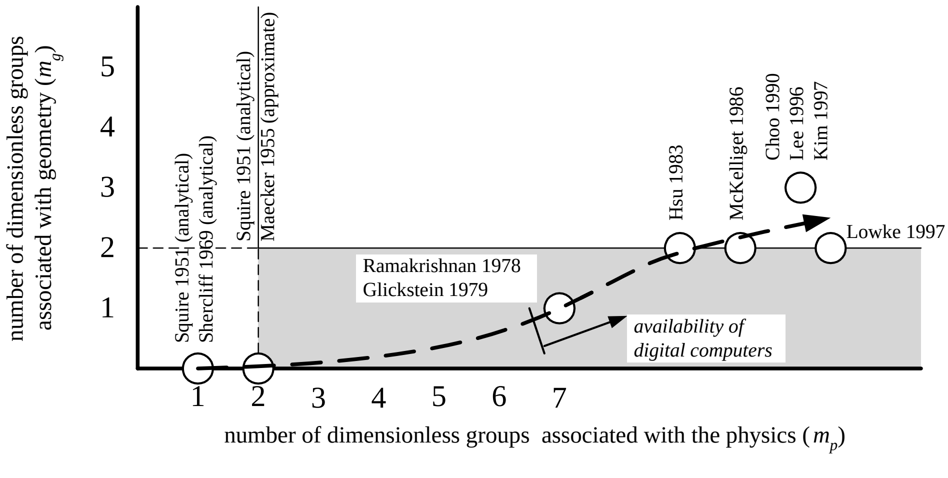

Figure 1: Evolution of the arc modeling. The horizontal and vertical axes indicate the degree of difficulty of generalizing the physics and geometry of the system respectively.