Proceedings of the Seventeenth Symposium on Energy Engineering Sciences
Argonne, IL, May 13-14, 1999, pp 106-113

ORDER OF MAGNITUDE SCALING OF COMPLEX ENGINEERING PROBLEMS

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ABSTRACT

This paper presents a methodology for obtaining order of magnitude estimations of complex engineering problems which are described by differential equations. It is often found that measurements and numerical treatment can be difficult in some of these problems. This might be due to the lack of reliability of idealizations, or the inability of dimensional analysis to reduce the number of arguments significantly. The methodology presented here overcomes these difficulties by bridging the fields of dimensional analysis and asymptotic considerations. The differential equations are transformed into a set of algebraic equations, which are much simpler to solve. The results obtained are estimations of the characteristic values of the unknown functions. It is not necessary to solve the differential equations in order to obtain the estimations; however, previous physical insight is necessary in order to perform the proper normalization and asymptotic considerations. The classical boundary layer problem is studied as a representative example, and it is shown that the estimations obtained are within a factor of 2 of the exact solution.

INTRODUCTION

Dimensional analysis and asymptotic considerations have been linked by some authors before. Barenblatt [1] focused on the application of dimensional analysis to obtain exact asymptotic solutions. Denn [2] introduces a scaling for pressure that depends on whether inertial or viscous forces dominate. He also uses the concept of dominant balance described by Bender *et al.*[4]. Chen [3] is the first to describe some of the properties of the dimensionless functions and their implications; however, he assumes that if the function and its arguments are normalized with their scale, all the dimensionless derivatives are of the order of one. This last statement is not generally valid and there are important cases for which it does not hold true. The order of magnitude methodology presented here applies some of Chen's concepts such as the emphasis on the normalization of the functions and its derivatives. The normalization is based on the scale of the unknown functions (velocity, temperature, etc.). The governing equations are normalized with the dominant terms of the equations. In this work the dominant terms are determined using a variation of the technique of dominant balance. One of the new concepts presented in this research is the requisite of an upper bound for the second derivative (the dimensionless second derivative must be of the order of one, and this implies the partition of the domain in some cases). Another novel concept is the transformation of a system of differential equations into an algebraic system. This useful simplification is possible by assuring that the unknown functions and derivatives are of the order of one, and their unknown scale is contained in a set of estimations. Also, a number of sets is introduced so matrix algebra can be used to simplify the calculation of the estimations.

Instead of having an abstract discussion, the order of magnitude scaling methodology is presented through an example. This example is the viscous boundary layer, for which the solution is known and can be used as a benchmark of the quality of the estimations obtained.

EXAMPLE: ORDER OF MAGNITUDE SCALING OF THE VISCOUS BOUNDARY LAYER

The objective is to obtain an estimation of the thickness and velocities in the viscous boundary layer. It is assumed that the fluid is incompressible and isothermal, and no external pressure gradients are applied.

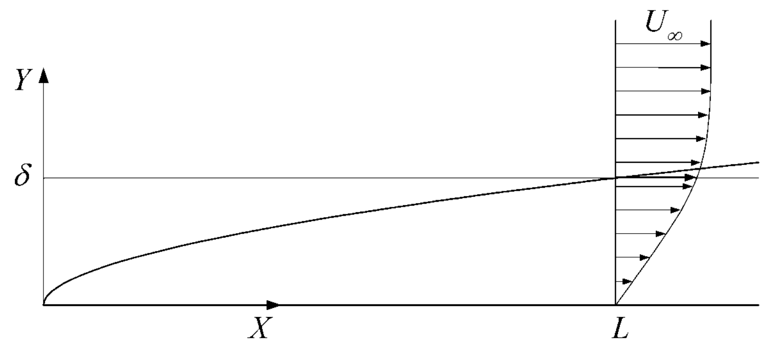

Figure 1: Schematic of the viscous boundary layer and the domain for scaling

Governing Equations, Boundary Conditions, and Domain for Scaling

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \tag{1}$$

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = -\frac{1}{\rho}\frac{\partial P}{\partial X} + \nu\left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2}\right) \tag{2}$$

$$U\frac{\partial V}{\partial X} + V\frac{\partial V}{\partial Y} = -\frac{1}{\rho}\frac{\partial P}{\partial Y} + \nu\left(\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2}\right) \tag{3}$$

The boundary conditions are: $U(X > 0, 0) = 0$, $V(X > 0, 0) = 0$, $U(-\infty, Y) = U_\infty$, $V(-\infty, Y) = 0$, $P(0, \infty) = 0$.

There are two independent arguments: X and Y. The equations are defined for an infinite domain over both independent arguments; however, a finite domain must be defined for the scaling. An arbitrary finite length L is defined as the domain limit in the X-direction. The domain in the Y-direction will be limited by the characteristic thickness of the boundary layer (δ). Because L is arbitrary, it is a known characteristic value, and belongs to the set $\{P\}$ of parameters that completely determine the problem. The thickness of the boundary layer is unknown, and its estimation ($\widehat{\delta}$) belongs to the set $\{S\}$ of estimations.

Sets of Parameters and Units

The set of parameters that completely determine the problem is obtained by inspection of the system of equations, its boundary conditions, and its domain, this way the complete set of parameters for this problem is:

$$\{P\}^T = \{\rho, \nu, U_\infty, L\} \tag{4}$$

A reasonable choice for the system of units is the SI. The set of reference units $\{R\}$ is obtained by expressing the units of each element of $\{P\}$ in the SI and inspecting the reference units involved:

$$\{R\}^T = \{\text{m, kg, s}\} \tag{5}$$

The matrix of dimensions $[U]^T$ for this problem is shown in Figure 2. Its rank is 3; therefore, three dimensionally independent parameters constitute a set of reference parameters:

$$\{P_k\}^T = \{\rho, \nu, U_\infty\} \tag{6}$$

	ρ	ν	U_∞	L
m	-3	2	1	1
kg	1			
s		-1	-1	

Figure 2: Matrix of dimensions $[U]^T$ for the viscous boundary layer. The submatrix on the left has rank 3, indicating that there are three dimensionally independent parameters

Scaling Relationships, Characteristic Values and Order of Magnitude Estimations

Scaling Relationships for the Independent Arguments

The domain for scaling is the rectangle $0 \leq X \leq L$, $0 \leq Y \leq \delta$, where L is the characteristic value for X, and δ the characteristic value for Y. Because δ is unknown, its estimation $\widehat{\delta}$ is used for the scaling relationships for the independent arguments:

$$X = Lx \tag{7}$$

$$Y = \widehat{\delta} y \tag{8}$$

Scaling Relationship for $U(X,Y)$

For laminar flow, the minimum value of U is 0, and the maximum is U_∞, therefore U_∞ is the characteristic value for U. Previous physical insight in this case comes from the well known solutions by Blasius [5]. There are no sharp changes in the slope of U inside the domain, therefore $u(x, y)$ and its first two derivatives are of the order of one. The scaling relationship for U then is:

$$U(X, Y) = U_\infty u(x, y) \tag{9}$$

Scaling Relationship for $V(X, Y)$

The transverse velocity V is 0 at the plate and upstream. Previous physical insight indicates that there are no sharp changes in the slope of V inside the domain, therefore $v(x, y)$ and its first two derivatives are of the order of one. The characteristic value of V is V_C, which is unknown; therefore, an estimation will be used in the scaling relationship.

$$V(X, Y) = \widehat{V}_C v(x, y) \tag{10}$$

Scaling Relationship for $P(X, Y)$

The minimum value of P is 0 far from the plate, but its characteristic value P_C is unknown. It is estimated by $\widehat{P}_C$, and the scaling relationship is:

$$P(X, Y) = \widehat{P}_C p(x, y) \tag{11}$$

Based on the scaling relationships defined above, the set of estimations $\{S\}$ is:

$$\{S\}^T = \{\widehat{\delta}, \widehat{V}_C, \widehat{P}_C\} \tag{12}$$

Dimensionless Governing Equations and Boundary Conditions

The original set of equations is normalized by using the scaling relationships and the dominant terms. The dominant terms are determined using a variation of the technique of dominant balance. In this work, a guess for a dominant term is verified by checking that all of the dimensionless coefficients in the equations are lesser or equal to one. The normal formulation for a dominant balance requires that the equations are simplified and solved. Equation 14 (conservation of momentum in the x-direction) was normalized with the viscous forces. This viscous forces create pressures, which are the forces used to normalize equation 15 (conservation of momentum in the y-direction). The expression of the coefficients N_i appears in matrix $[A]$ (Figure 3).

$$\frac{\partial u}{\partial x} + N_1 \frac{\partial v}{\partial y} = 0 \tag{13}$$

$$N_2 u \frac{\partial u}{\partial x} + N_3 v \frac{\partial u}{\partial y} = -N_4 \frac{\partial p}{\partial x} + \left(N_5 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \tag{14}$$

$$N_6 u \frac{\partial u}{\partial x} + N_7 v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial y} + N_8 \left(N_5 \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \tag{15}$$

The boundary conditions are: $u(x > 0, 0) = 0$, $v(x > 0, 0) = 0$, $u(-\infty, y) = 1$, $v(-\infty, y) = 0$, $p(0, \infty) = 0$.

Dimensionless Groups of Known Order of Magnitude

The boundary layer is the region where the flow transitions from stagnant (at the wall) to free flow (far from the wall). Close to the wall the viscous forces are dominant, and far from the wall the inertial forces dominate. A practical and physically meaningful way to define the boundary layer (used by Rivas and Ostrach [6]) is as the "region where the viscous forces are of the same order of magnitude as the inertial forces". From this consideration, the group N_3 can be estimated as equal to one. The group N_1 is also estimated as equal to one because it relates two terms that ar of the order of magnitude of one. The group N_4 will also be estimated as one, indicating that the pressure scales with the viscous and inertial forces.

Complete Set of Dimensionless Groups

From Buckingham's theorem it is known that this problem can be described completely by just one non-dimensional group. The Reynolds number ($\text{Re} = U_\infty L/\nu$) is arbitrarily chosen to describe the problem.

Expression of the Estimations

Matrix algebra will be used to illustrate its application, although this problem is simple enough as to be solved by simple inspection. The matrix $[A]$ of dimensionless groups is shown in Figure 3.

	ρ	ν	U_∞	L	$\widehat{\delta}$	$\widehat{V}_C$	$\widehat{P}_C$
N_1			-1	1	-1	1	
N_3		-1			1	1	
N_4	-1	-1	-1	-1	2		1
Re		-1	1	1			
N_2		-1	1	-1	2		
N_5				-2	2		
N_6	1		1	-1	1	1	-1
N_7	1					2	-1
N_8	1	1			-1	1	-1

Figure 3: Matrix $[A]$ for the viscous boundary layer. The internal lines divide the submatrices $[A_{ij}]$. The elements of the matrix are the exponents of the parameters in each dimensionless group

The matrix of estimations $[A_S]$ is shown in Figure 4. It is obtained by using the following equation:

$$[A_S] = -[A_{12}]^{-1}[A_{11}] \tag{16}$$

The expression for the estimation of the boundary layer thickness is obtained from matrix $[A_S]$:

$$\widehat{\delta} = \sqrt{\frac{\nu L}{U_\infty}} \tag{17}$$

Finally, the exact boundary layer thickness is $\delta = \widehat{\delta} g(\text{Re})$, where $g(\text{Re})$ is approximately equal to one.

	ρ	ν	U_∞	L
$\widehat{\delta}$		1/2	-1/2	1/2
$\widehat{V}_C$		1/2	-1/2	-1/2
$\widehat{P}_C$	1		2	

Figure 4: Matrix $[A_S]$ for the viscous boundary layer

Dimensionless Governing Equations and Boundary Conditions (In Terms of the Reference Dimensionless Groups)

Even though an estimation of the characteristic value of the functions in the problem was already obtained, rewriting the equations in terms of the reference dimensionless groups (in this case, the Reynolds number) is useful as a check of consistency and for added physical insight.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{18}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \left(\frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{19}$$

$$\frac{1}{\mathrm{Re}}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial y} + \frac{1}{\mathrm{Re}}\left(\frac{1}{\mathrm{Re}}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{20}$$

The equations above are equivalent to their dimensional counterparts, no physical aspects have been neglected or modified. The boundary layer is commonly studied at large Reynolds numbers, because that is the case when it can be considered thin and independent of Re. Inspecting the equations, it can be seen that at values of the Reynolds number larger than one, all term are of the order of one. The term $\partial p/\partial y$ in equation 20 is very small ($\mathrm{O}(\mathrm{Re}^{-1})$), indicating that the pressure gradient is approximately parallel to the x axis.

Comparison with Known Results

The expression of $g(\mathrm{Re})$ can be obtained from the literature [5]. This expression depends on the definition of boundary layer thickness used. The displacement thickness of the boundary layer is an integral definition that can be compared to the definition used here, and at high Reynolds numbers it is where the parallel velocity is 99% of that of the free flow. In this case the dimensionless function would be $g(\mathrm{Re}) = 5$. It is not difficult to try to improve the order of magnitude estimations according to this definition. The thickness of the boundary layer could be estimated as the region where the inertial forces are 100 times larger than the viscous; this statement is translated into $N_2 = 100$, with an estimated boundary layer thickness is $\widehat{\delta} = 10\sqrt{\nu L/U_\infty}$. In this last case, the dimensionless function is $g(\mathrm{Re}) = 0.5$.

DISCUSSION

This technique expands the capabilities of dimensional analysis by incorporating information from previous physical insight and the governing equations. For both standard dimensional

Table 1: Comparison of estimated and exact thickness of the viscous boundary layer for different definitions. It can be observed that the dimensionless function $g(\mathrm{Re})$ is approximately equal to one for all cases.

Definition	Estimation	N_2	δ	$\widehat{\delta}$	$g(\mathrm{Re})$
displacement thickness	inertial forces≈ viscous forces	1	$1.72\sqrt{\nu L/U_\infty}$	$\sqrt{\nu L/U_\infty}$	1.72
$99\% U_\infty$	inertial forces≈ viscous forces	1	$5\sqrt{\nu L/U_\infty}$	$\sqrt{\nu L/U_\infty}$	5
$99\% U_\infty$	inertial forces≈ 100×viscous forces	100	$5\sqrt{\nu L/U_\infty}$	$10\sqrt{\nu L/U_\infty}$	0.5

analysis and order of magnitude scaling, the unknowns are expressed in the form of a power law multiplied by an unknown function of the governing dimensionless parameters; however, in order of magnitude scaling that function is known to be approximately equal to one, while in dimensional analysis that function is unknown, and can have any behavior or order of magnitude.

This methodology can be applied to non-linear equations such as Navier-Stokes. Its present formulation, however, is limited to differential equations of second order or lower; the reason is that it is difficult to assure that lower order derivatives are of the order of one when dealing with higher order equations. Another limitation in the current formulation is that the equations must be written in scalar form, excluding vectorial notation. The reason is that the same vector might need to be assigned more than one scale (for example different scales in the X and Y directions). The circular logic of the dominant balance technique limits the generality of this methodology because not all of the dangerous cases can be identified beforehand. Special precautions should be taken when a dimensionless function can be of an order of magnitude smaller than one. In this case, differential equations might be transformed into into algebraic inequalities, which are difficult to analyze.

Problems for which many domain subdivisions are necessary (in order to reduce the magnitude of the second derivative) are beyond the scope of this methodology because they cannot be simplified significantly. Unstable systems, such as those presenting capillary instability may be of this type.

When using matrix algebra to implement this methodology the matrices involved are generally small, and the matrix operations relatively simple. The calculation process can be performed with commercial software tools.

CONCLUSIONS

Order of magnitude scaling is helpful to the engineer who needs to gain insight into a complex problem but cannot afford to tackle the full solution of the governing equations. The estimations can be obtained without solving the differential equations because the original system of differential equations is transformed into a linear algebraic system. These estimations can be used to determine the relative importance of the different driving forces in the problem, thus gaining deeper physical insight into it.

The estimations are related to the governing parameters through power laws, and the exact value of the characteristic values is related to the estimations through an unknown function of the governing dimensionless parameters. This function is approximately equal to one, and can be considered as exactly equal to one for order of magnitude approximations.

The approximations obtained can be refined by further calculations or experiments. The knowledge gained regarding what dimensionless groups can be neglected reduces significantly the necessary number of experiments or calculations. The simple expression of the solutions makes them suitable to be implemented in real-time control algorithms. When using matrix algebra to implement this methodology the matrices involved are generally small, and the matrix operations relatively simple. The calculation process can be performed with commercial software tools.

The boundary layer example illustrates the practical implementation of this methodology, and shows that the results obtained match satisfactorily the exact solution to the equations.

This work was supported by the United States Department of Energy, Office of Basic Energy Sciences.

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