

Two-Loop Corrections to Bhabha Scattering

Alexander Penin

TTP Karlsruhe & INR Moscow

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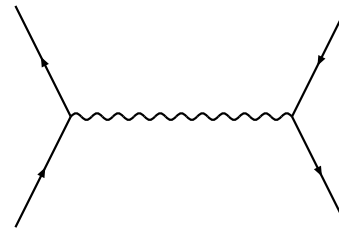
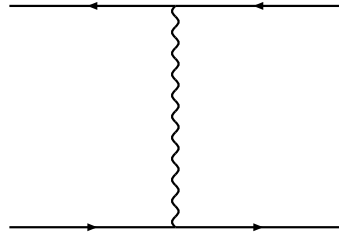
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- Test ground for new methods of multiloop calculations
- Classical problem of perturbative QED

Born approximation



$$\frac{d\sigma^{(0)}}{d\Omega} = \frac{\alpha^2}{s} \left(\frac{1-x+x^2}{x} \right)^2 + O(m_e^2/s), \quad x = \frac{1-\cos\theta}{2}$$

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Phenomenologically interesting:

- High energy region $s, t, u \gg m_e^2$
- Small angle Bhabha scattering $t \ll s, x \sim 0$
- Large angle Bhabha scattering $t \sim s, x \sim 1$

Radiative corrections

Only inclusive processes are IR finite and observable

$$\sigma = \sum_{n=0}^{\infty} \left(\frac{\alpha}{\pi} \right)^n \sigma^{(n)}, \quad \sigma^{(1)} = \sigma_v^{(1)} + \sigma_r^{(1)}, \quad \sigma^{(2)} = \sigma_{vv}^{(2)} + \sigma_{rv}^{(2)} + \sigma_{rr}^{(2)}, \dots$$

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Two types of IR divergences:

- Soft divergences, regulated by λ or ε . Disappear in soft-photon-inclusive cross section with the energy cutoff $\mathcal{E}_{cut}$ on the emitted photons
- Collinear divergences, regulated by m_e or ε . Disappear in collinear-photon-inclusive cross section with the angular cutoff θ_{cut} on the emitted photons

Inclusive cross section

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- Put $m_e = 0$
 - Define QED “structure function” for initial states
 - Define QED ‘jets’ with angular resolution $\theta_{cut} \gg \sqrt{m_e^2/s}$ for final states

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- Put $m_e = 0$
 - Define QED “structure function” for initial states
 - Define QED ‘jets’ with angular resolution $\theta_{cut} \gg \sqrt{m_e^2/s}$ for final states
- Keep $m_e \neq 0$
 - Split real radiation into “soft” and “hard” by $\mathcal{E}_{cut} \ll m_e$
 - Compute the virtual+soft real part analytically
 - Compute the hard real part with actual experimental cuts by means of Monte Carlo

Structure of the corrections

$$\begin{aligned}\frac{d\sigma^{(1)}}{d\sigma^{(0)}} &= \delta_1^{(1)} \ln\left(\frac{s}{m_e^2}\right) + \delta_0^{(1)} + O(m_e^2/s) \\ \delta_1^{(1)} &= 4 \ln\left(\frac{\mathcal{E}_{cut}}{\mathcal{E}}\right) + \dots, \quad \mathcal{E} = \sqrt{s}/2\end{aligned}$$

$$\begin{aligned}\frac{d\sigma^{(2)}}{d\sigma^{(0)}} &= \delta_2^{(2)} \ln^2\left(\frac{s}{m_e^2}\right) + \delta_1^{(2)} \ln\left(\frac{s}{m_e^2}\right) + \delta_0^{(2)} + O(m_e^2/s) \\ \delta_2^{(2)} &= 8 \ln^2\left(\frac{\mathcal{E}_{cut}}{\mathcal{E}}\right) + 12 \ln\left(\frac{\mathcal{E}_{cut}}{\mathcal{E}}\right) + \dots \\ \delta_1^{(2)} &= -16 \left[1 + \ln\left(\frac{1-x}{x}\right)\right] \ln^2\left(\frac{\mathcal{E}_{cut}}{\mathcal{E}}\right) + \dots\end{aligned}$$

History and current status of two-loop calculations

- Logarithmic corrections to SA scattering

A.B. Arbuzov, V.S. Fadin, E.A. Kuraev, L.N. Lipatov, N.P. Merenkov, L. Trentadue

- Logarithmic corrections to LA scattering

E.W. Glover, J.B. Tausk, J.J. van der Bij

- Full massless result for virtual correction

Z. Bern, L. Dixon, A. Ghinculov

- $m_e \neq 0$, fermion loop insertions

R. Bonciani, A. Ferroglia, P. Mastrolia, E. Remiddi, J.J. van der Bij

- Photonic corrections, leading order in m_e^2/s

A. Penin

Framework of calculation

- Purely photonic corrections
- Nonzero photon mass $\lambda \ll m_e$
- Leading order in m_e^2/s

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Systematic expansion of Feynman integrals in m_e^2/s :

Expansion by regions

(V.Smirnov)

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Systematic expansion of Feynman integrals in m_e^2/s :

Expansion by regions (V.Smirnov)

In the leading order in m_e^2/s the massless and massive results are related by change of IR regularization scheme:

Infrared subtractions

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$$\delta\mathcal{A} = \left[\mathcal{A}(\varepsilon) - \bar{\mathcal{A}}(\varepsilon) \right]_{\varepsilon \rightarrow 0}$$

- Compute the auxiliary amplitude $\bar{\mathcal{A}}$ for $\lambda, m_e \rightarrow 0$
- The amplitude $\mathcal{A}$ in the limit $\lambda, m_e \rightarrow 0$ is given by

$$\mathcal{A}(\lambda, m_e) \Big|_{\lambda, m_e \rightarrow 0} = \bar{\mathcal{A}}(\lambda, m_e) \Big|_{\lambda, m_e \rightarrow 0} + \delta\mathcal{A}$$

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IR singularities factorize and exponentiate

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Example: vector form factor $\mathcal{F} = \sum_{n=0}^{\infty} \left(\frac{\alpha}{\pi}\right)^n f^{(n)}, (Q^2 = -s)$

How to construct $\bar{\mathcal{A}}$?

IR singularities factorize and exponentiate

Example: vector form factor $\mathcal{F} = \sum_{n=0}^{\infty} \left(\frac{\alpha}{\pi}\right)^n f^{(n)}, (Q^2 = -s)$

$\lambda, m_e = 0:$

$$f^{(1)} = \left[-\frac{1}{2\varepsilon^2} - \frac{3}{4\varepsilon} - 2 + \frac{\pi^2}{24} + \left(-4 + \frac{\pi^2}{16} + \frac{7}{6}\zeta(3) \right) \varepsilon + \left(-8 + \frac{\pi^2}{6} + \frac{7}{4}\zeta(3) + \frac{47}{2880}\pi^4 \right) \varepsilon^2 \right] \frac{1}{Q^{2\varepsilon}}$$

$\lambda \ll m_e \ll Q:$

$$f^{(1)} = -\frac{1}{4} \ln^2 \left(\frac{Q^2}{m_e^2} \right) + \left[\frac{1}{2} \ln \left(\frac{\lambda^2}{m_e^2} \right) + \frac{3}{4} \right] \ln \left(\frac{Q^2}{m_e^2} \right) - \frac{1}{2} \ln \left(\frac{\lambda^2}{m_e^2} \right) - 1 + \frac{\pi^2}{12}$$

$m_e \ll \lambda \ll Q:$

$$f^{(1)} = -\frac{1}{4} \ln^2 \left(\frac{Q^2}{\lambda^2} \right) + \frac{3}{4} \ln \left(\frac{Q^2}{\lambda^2} \right) - \frac{7}{8} - \frac{\pi^2}{6}$$

Exponentiation

$$\mathcal{F} \sim \exp \left\{ \frac{\alpha}{2\pi} \left[\ln \left(\frac{Q^2}{m_e^2} \right) - 1 \right] \ln \left(\frac{\lambda^2}{m_e^2} \right) \right\}$$

(D.R.Yennie, S.C.Frautschi, H.Suura)

$$\frac{\partial}{\partial \ln(Q^2)} \mathcal{F} = \left[-\frac{\alpha}{2\pi} \ln(Q^2) + \phi(m_e, \lambda, \varepsilon, \alpha) \right] \mathcal{F}$$

(A.Mueller, J.Collins)

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$\lambda, m_e = 0:$

$$\mathcal{F} = \left(1 + O(\alpha) \right) \exp \left\{ -\frac{\alpha}{2\pi} \left(\frac{1}{\varepsilon^2} + \left(\frac{3}{2} + O(\alpha) \right) \frac{1}{\varepsilon} \right) \left(\frac{\mu^2}{Q^2} \right)^\varepsilon \right\}$$

$\lambda \ll m_e \ll Q:$

$$\mathcal{F} = \left(1 + O(\alpha) \right) \exp \left\{ \frac{\alpha}{4\pi} \left[-\ln^2 \left(\frac{Q^2}{m_e^2} \right) + 2 \left[\ln \left(\frac{Q^2}{m_e^2} \right) - 1 \right] \ln \left(\frac{\lambda^2}{m_e^2} \right) + \left(3 + O(\alpha) \right) \ln \left(\frac{Q^2}{m_e^2} \right) \right] \right\}$$

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Two-loop form factor

$\lambda, m_e = 0$ (T. Matsuura, S.C. van der Marck, W.L. van Neerven):

$$f^{(2)} = \frac{1}{2} \left(f^{(1)} \right)^2 - \left(\frac{3}{32} - \frac{\pi^2}{8} + \frac{3}{2} \zeta(3) \right) \frac{1}{2\varepsilon} \left(\frac{\mu^2}{Q^2} \right)^{2\varepsilon} - \frac{1}{128} + \frac{29}{96} \pi^2 - \frac{15}{8} \zeta(3) - \frac{2}{45} \pi^4$$

$\lambda \ll m_e \ll Q$ (P. Mastrolia, E. Remiddi):

$$f^{(2)} = \frac{1}{2} \left(f^{(1)} \right)^2 + \left(\frac{3}{32} - \frac{\pi^2}{8} + \frac{3}{2} \zeta(3) \right) \ln \left(\frac{Q^2}{m_e^2} \right) + \frac{11}{8} + \frac{17}{32} \pi^2 - \frac{9}{4} \zeta(3) - \frac{7}{240} \pi^4 - \frac{\pi^2 \ln(2)}{2}$$

$m_e \ll \lambda \ll Q$ (B. Feucht, J.H. Kühn, A.A. Penin, V.A. Smirnov):

$$f^{(2)} = \frac{1}{2} \left(f^{(1)} \right)^2 + \left(\frac{3}{32} - \frac{\pi^2}{8} + \frac{3}{2} \zeta(3) \right) \ln \left(\frac{Q^2}{\lambda^2} \right) + \frac{51}{128} + \frac{15}{16} \pi^2 + 5 \zeta(3) - \frac{83}{360} \pi^4 - \frac{2}{3} \pi^2 \ln^2(2) \\ + \frac{2}{3} \ln^4(2) + 16 \text{Li}_4 \left(\frac{1}{2} \right)$$

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● $A^{(n)}$ are two-component vectors in the chiral basis

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$$\mathcal{A} = \mathcal{F}^2 \tilde{\mathcal{A}}$$

● Reduced amplitude $\tilde{\mathcal{A}}$ is free of collinear logs

Exponentiation

$$\frac{\partial}{\partial \ln(Q^2)} \tilde{\mathcal{A}} = \frac{\alpha}{\pi} \ln\left(\frac{1-x}{x}\right) \tilde{\mathcal{A}}$$

(A.Sen; G.Sterman)

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$\lambda = 0$:

$$\tilde{\mathcal{A}} = \left(1 + O(\alpha) \right) \exp \left[-\frac{\alpha}{\pi} \ln \left(\frac{1-x}{x} \right) \frac{1}{\varepsilon} \left(\frac{\mu^2}{Q^2} \right)^\varepsilon \right]$$

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The auxiliary amplitude

$$\bar{A}^{(2)} = \frac{1}{2} \left(A^{(1)} \right)^2 + 2 \left[f^{(2)} - \frac{1}{2} \left(f^{(1)} \right)^2 \right]$$

Two-loop infrared divergences

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Our prediction

$$A^{(2)} = \frac{1}{2} \left(A^{(1)} \right)^2 + 2f^{(2)} - \left(f^{(1)} \right)^2 + \delta A^{(2)}$$

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Catani's formula

$$\begin{aligned} A^{(1)} &= \mathbf{I}^{(1)} + A_{\text{fin}}^{(1)} \\ A^{(2)} &= \left[-\frac{1}{2} \left(\mathbf{I}^{(1)} \right)^2 + \mathbf{H}^{(2)} \right] + \mathbf{I}^{(1)} A^{(1)} + A_{\text{fin}}^{(2)} \end{aligned}$$

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Scheme invariance

$$\begin{aligned} \mathbf{I}'^{(1)} &= \mathbf{I}^{(1)} + G, & A_{\text{fin}}'^{(1)} &= A_{\text{fin}}^{(1)} - G \\ \mathbf{H}'^{(2)} &= \mathbf{H}^{(2)} + F, & A_{\text{fin}}'^{(2)} &= A_{\text{fin}}^{(2)} - \left(\frac{1}{2} G^2 + F \right) \end{aligned}$$

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$$G = A_{\text{fin}}^{(1)}, \quad F = 2f^{(2)} - (f^{(1)})^2 - \mathbf{H}^{(2)}$$

Infrared matching

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- Change of the scheme

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$\mathbf{I}'^{(1)}(\varepsilon), \mathbf{H}'^{(1)}(\varepsilon)$

$\mathbf{I}'^{(1)}(\lambda, m_e), \mathbf{H}'^{(1)}(\lambda, m_e)$



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$\mathbf{I}'^{(1)}(\lambda, m_e), \mathbf{H}'^{(1)}(\lambda, m_e)$

- Inverse change of the scheme

$\mathbf{I}^{(1)}(\lambda, m_e), \mathbf{H}^{(1)}(\lambda, m_e)$

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Catani's operators in massless case

$$\begin{aligned}\mathbf{I}^{(1)} &= \frac{e^{-\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)} \left(\frac{1}{\varepsilon^2} + \frac{3}{2\varepsilon} \right) \left[- \left(\frac{\mu^2}{-s} \right)^\varepsilon - \left(\frac{\mu^2}{-t} \right)^\varepsilon + \left(\frac{\mu^2}{-u} \right)^\varepsilon \right] \\ \mathbf{H}^{(2)} &= \frac{e^{-\varepsilon\gamma_E}}{\Gamma(1-\varepsilon)} \frac{1}{\varepsilon} \left(\frac{3}{32} - \frac{\pi^2}{8} + \frac{3}{2}\zeta(3) \right) \left[- \left(\frac{\mu^2}{-s} \right)^\varepsilon - \left(\frac{\mu^2}{-t} \right)^\varepsilon + \left(\frac{\mu^2}{-u} \right)^\varepsilon \right]\end{aligned}$$

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$$\mathbf{H}^{(2)} = \frac{e^{-\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \frac{1}{\epsilon} \left(\frac{3}{32} - \frac{\pi^2}{8} + \frac{3}{2}\zeta(3) \right) \left[-\left(\frac{\mu^2}{-s} \right)^\epsilon - \left(\frac{\mu^2}{-t} \right)^\epsilon + \left(\frac{\mu^2}{-u} \right)^\epsilon \right]$$

Catani's operators in massive case

$$\mathbf{I}^{(1)} = -\frac{1}{2} \ln^2 \left(\frac{s}{m_e^2} \right) + \left[\ln \left(\frac{\lambda^2}{m_e^2} \right) + \frac{3}{2} - \ln \left(\frac{x}{1-x} \right) + i\pi \right] \ln \left(\frac{s}{m_e^2} \right) + \left[-1 + \ln \left(\frac{x}{1-x} \right) - i\pi \right]$$

$$\ln \left(\frac{\lambda^2}{m_e^2} \right) + 2 - \frac{2}{3} \pi^2 + \frac{3}{2} \ln \left(\frac{x}{1-x} \right) - \frac{1}{2} \ln^2(x) + \frac{1}{2} \ln^2(1-x) - \frac{3}{2} i\pi$$

$$\mathbf{H}^{(2)} = \left(\frac{3}{16} - \frac{\pi^2}{4} + 3\zeta(3) \right) \left[\ln \left(\frac{s}{m_e^2} \right) + \ln \left(\frac{x}{1-x} \right) - i\pi \right] + \frac{177}{64} + \frac{11}{24} \pi^2 - \frac{3}{4} \zeta(3) - \frac{7}{120} \pi^4 - \pi^2 \ln(2)$$

Result *(page 1 of 2)*

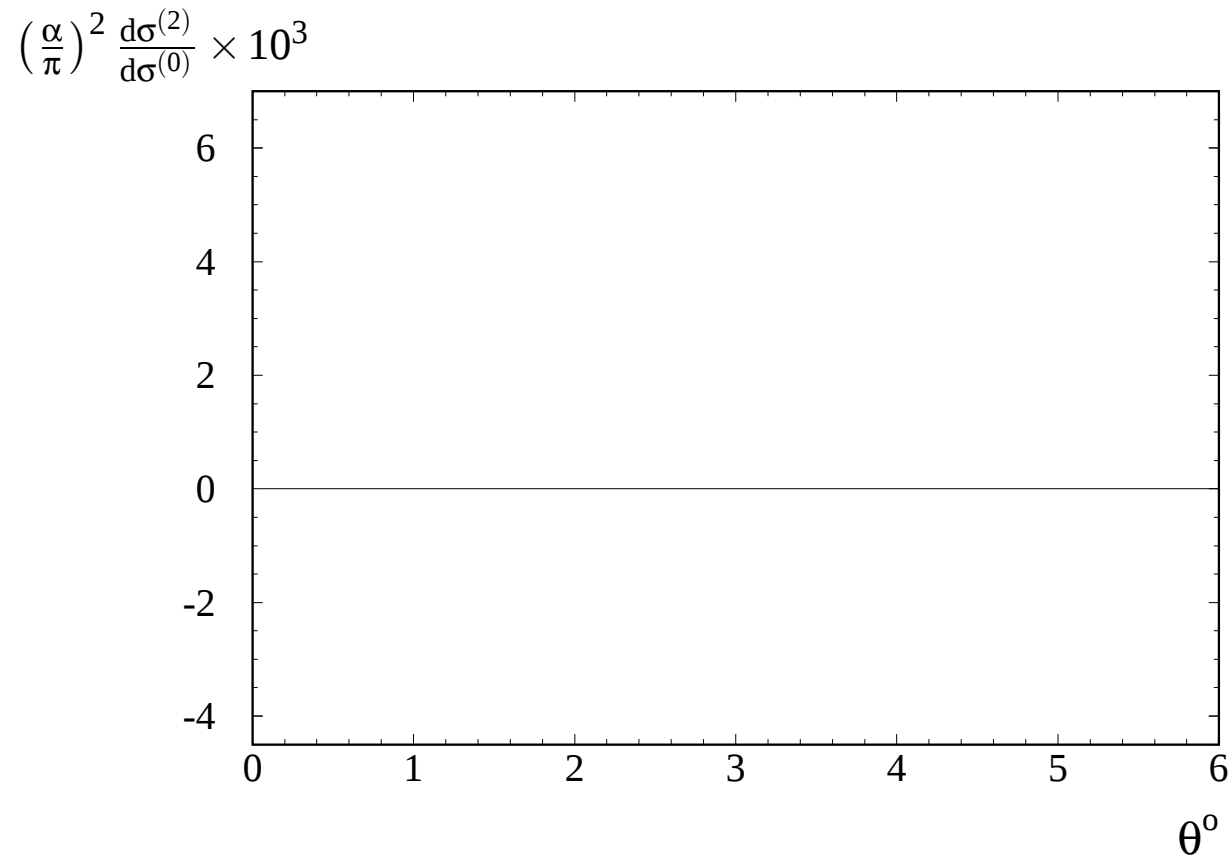
$$\begin{aligned}\delta_0^{(2)} = & 8\mathcal{L}_\varepsilon^2 + (1-x+x^2)^{-2} \left[\left(\frac{4}{3} - \frac{8}{3}x - x^2 + \frac{10}{3}x^3 - \frac{8}{3}x^4 \right) \pi^2 + (-12 + 16x - 18x^2 + 6x^3) \ln(x) \right. \\ & + (2x + 2x^3) \ln(1-x) + (-3x + x^2 + 3x^3 - 4x^4) \ln^2(x) + (-8 + 16x - 14x^2 + 4x^3) \ln(x) \\ & \times \ln(1-x) + (4 - 10x + 14x^2 - 10x^3 + 4x^4) \ln^2(1-x) + (1-x+x^2)^2 (16 + 8\text{Li}_2(x) \\ & \left. - 8\text{Li}_2(1-x)) \right] \mathcal{L}_\varepsilon + \frac{27}{2} - 2\pi^2 \ln(2) + (1-x+x^2)^{-2} \left(\left(\frac{83}{24} - \frac{125}{24}x + \frac{13}{4}x^2 + \frac{19}{24}x^3 - \frac{25}{24}x^4 \right) \right. \\ & \times \pi^2 + \left(-9 + \frac{43}{2}x - 34x^2 + 22x^3 - 9x^4 \right) \zeta(3) + \left(-\frac{11}{90} - \frac{5}{24}x + \frac{29}{180}x^2 + \frac{23}{180}x^3 - \frac{49}{480}x^4 \right) \pi^4 \\ & + \left[-\frac{93}{8} + \frac{231}{16}x - \frac{279}{16}x^2 + \frac{93}{16}x^3 + \left(-\frac{3}{2} + \frac{13}{4}x - \frac{7}{12}x^2 - \frac{11}{8}x^3 \right) \pi^2 + (12 - 12x + 8x^2 \right. \\ & \left. - x^3) \zeta(3) \right] \ln(x) + \left[\frac{9}{2} - \frac{43}{8}x + \frac{17}{8}x^2 + \frac{29}{8}x^3 - \frac{9}{2}x^4 + \left(\frac{x}{4} + \frac{x^2}{2} + \frac{5}{24}x^3 + \frac{19}{48}x^4 \right) \pi^2 \right] \ln^2(x) \\ & + \left(\frac{67}{24}x - \frac{5}{4}x^2 - \frac{2}{3}x^3 \right) \ln^3(x) + \left(\frac{7}{48}x + \frac{5}{96}x^2 - \frac{x^3}{12} + \frac{43}{96}x^4 \right) \ln^4(x) + \left\{ 3x + 3x^3 + \left(\frac{7}{6}x \right. \right. \\ & \left. \left. - \frac{73}{24}x^2 + \frac{15}{8}x^3 \right) \pi^2 + (-6 + 6x - x^2 - 4x^3) \zeta(3) + \left[-8 + \frac{21}{2}x - \frac{45}{4}x^2 + x^4 + \left(1 - \frac{x}{6} + \frac{x^2}{12} \right. \right. \right. \\ & \left. \left. \left. - \frac{x^3}{3} - \frac{x^4}{8} \right) \pi^2 \right] \ln(x) + \left(6 - 11x + \frac{35}{4}x^2 - \frac{15}{8}x^3 \right) \ln^2(x) + \left(\frac{2}{3} + \frac{x}{12} - \frac{x^3}{3} + \frac{5}{24}x^4 \right) \ln^3(x) \right\} \\ & \times \ln(1-x) + \left[\frac{7}{2} - 6x + \frac{45}{4}x^2 - 6x^3 + \frac{7}{2}x^4 + \left(-\frac{17}{24} + \frac{7}{6}x - \frac{25}{24}x^2 - \frac{13}{48}x^4 \right) \pi^2 + \left(-3 + \frac{23}{4}x \right. \right. \\ & \left. \left. - \frac{23}{4}x^2 + \frac{9}{8}x^3 \right) \ln(x) + \left(\frac{7}{2} - \frac{41}{8}x + \frac{31}{8}x^2 + \frac{3}{8}x^3 - \frac{13}{16}x^4 \right) \ln^2(x) \right] \ln^2(1-x) + \left[\frac{3}{8}x + \frac{1}{6}x^2 \right. \\ & \left. + \frac{3}{8}x^3 + \left(-4 + \frac{29}{6}x - \frac{49}{12}x^2 + \frac{5}{6}x^3 + \frac{7}{8}x^4 \right) \ln(x) \right] \ln^3(1-x) + \left(\frac{1}{32} - \frac{3}{4}x + \frac{71}{48}x^2 - \frac{29}{24}x^3 \right. \\ & \left. + \frac{9}{32}x^4 \right) \ln^4(1-x) + \left\{ 8 - 16x + 24x^2 - 16x^3 + 8x^4 + \left(\frac{7}{3} - 3x + \frac{3}{4}x^2 + \frac{5}{6}x^3 - \frac{2}{3}x^4 \right) \pi^2 \right\}\end{aligned}$$

Result *(page 2 of 2)*

$$\begin{aligned}
& + \left[-6 + \frac{11}{2}x - 4x^2 + x^3 + \left(2 - \frac{11}{4}x + \frac{7}{4}x^2 + \frac{x^3}{4} - x^4 \right) \ln(x) \right] \ln(x) + \left[\frac{3}{2}x - \frac{x^2}{4} + x^3 \right. \\
& + \left(-4 + 9x - \frac{15}{2}x^2 + 2x^3 \right) \ln(x) + \left(-1 - \frac{7}{2}x + \frac{25}{4}x^2 - 5x^3 + 2x^4 \right) \ln(1-x) \left. \right] \ln(1-x) + \left(2 \right. \\
& - 4x + 6x^2 - 4x^3 + 2x^4 \left. \right) \text{Li}_2(x) \left. \right\} \text{Li}_2(x) + \left\{ -8 + 16x - 24x^2 + 16x^3 - 8x^4 + \left[-\frac{2}{3} + \frac{4}{3}x \right. \right. \\
& + \frac{x^2}{2} - \frac{5}{3}x^3 + \frac{2}{3}x^4 \left. \right] \pi^2 + \left[6 - 8x + 9x^2 - 3x^3 + \left(\frac{3}{2}x - \frac{x^2}{2} - \frac{3}{2}x^3 + 2x^4 \right) \ln(x) \right] \ln(x) + \left[-x \right. \\
& - \frac{x^2}{4} - \frac{x^3}{2} + (10 - 14x + 9x^2) \ln(x) + \left(-8 + 11x - \frac{31}{4}x^2 + \frac{x^3}{2} + x^4 \right) \ln(1-x) \left. \right] \ln(1-x) \\
& + \left(-4 + 8x - 12x^2 + 8x^3 - 4x^4 \right) \text{Li}_2(x) + \left(2 - 4x + 6x^2 - 4x^3 + 2x^4 \right) \text{Li}_2(1-x) \left. \right\} \text{Li}_2(1-x) \\
& + \left[\frac{5}{2}x - 5x^2 + 2x^3 + (-4 - x + x^2 + 2x^3 - 2x^4) \ln(x) + (6 - 6x + x^2 + 4x^3) \ln(1-x) \right] \text{Li}_3(x) \\
& + \left[\frac{x}{2} - \frac{x^3}{2} + (-6 + 5x + 3x^2 - 5x^3) \ln(x) + (6 - 10x + 10x^3 - 6x^4) \ln(1-x) \right] \text{Li}_3(1-x) \\
& + \left(-2 + \frac{17}{2}x - \frac{17}{2}x^3 + 2x^4 \right) \text{Li}_4(x) + \left(7x - \frac{9}{2}x^2 - 4x^3 + 6x^4 \right) \text{Li}_4(1-x) + \left(-6 + 4x \right. \\
& + \frac{9}{2}x^2 - 7x^3 \left. \right) \text{Li}_4\left(-\frac{x}{1-x}\right),
\end{aligned}$$

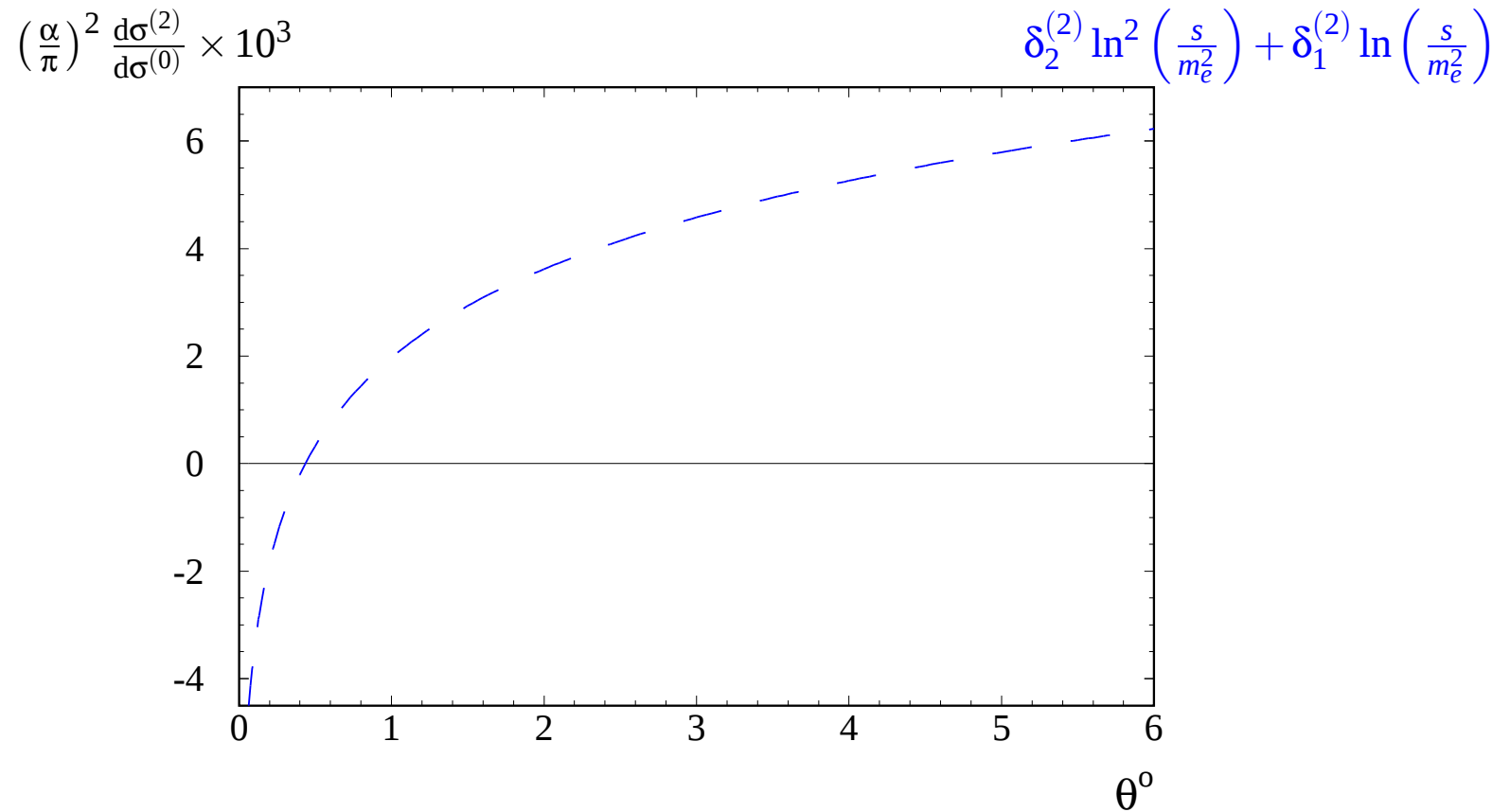
$$\mathcal{L}_\varepsilon = [1 - \ln(x/(1-x))] \ln(\mathcal{E}_{cut}/\mathcal{E}).$$

Two-loop photonic corrections to SA Bhabha scattering



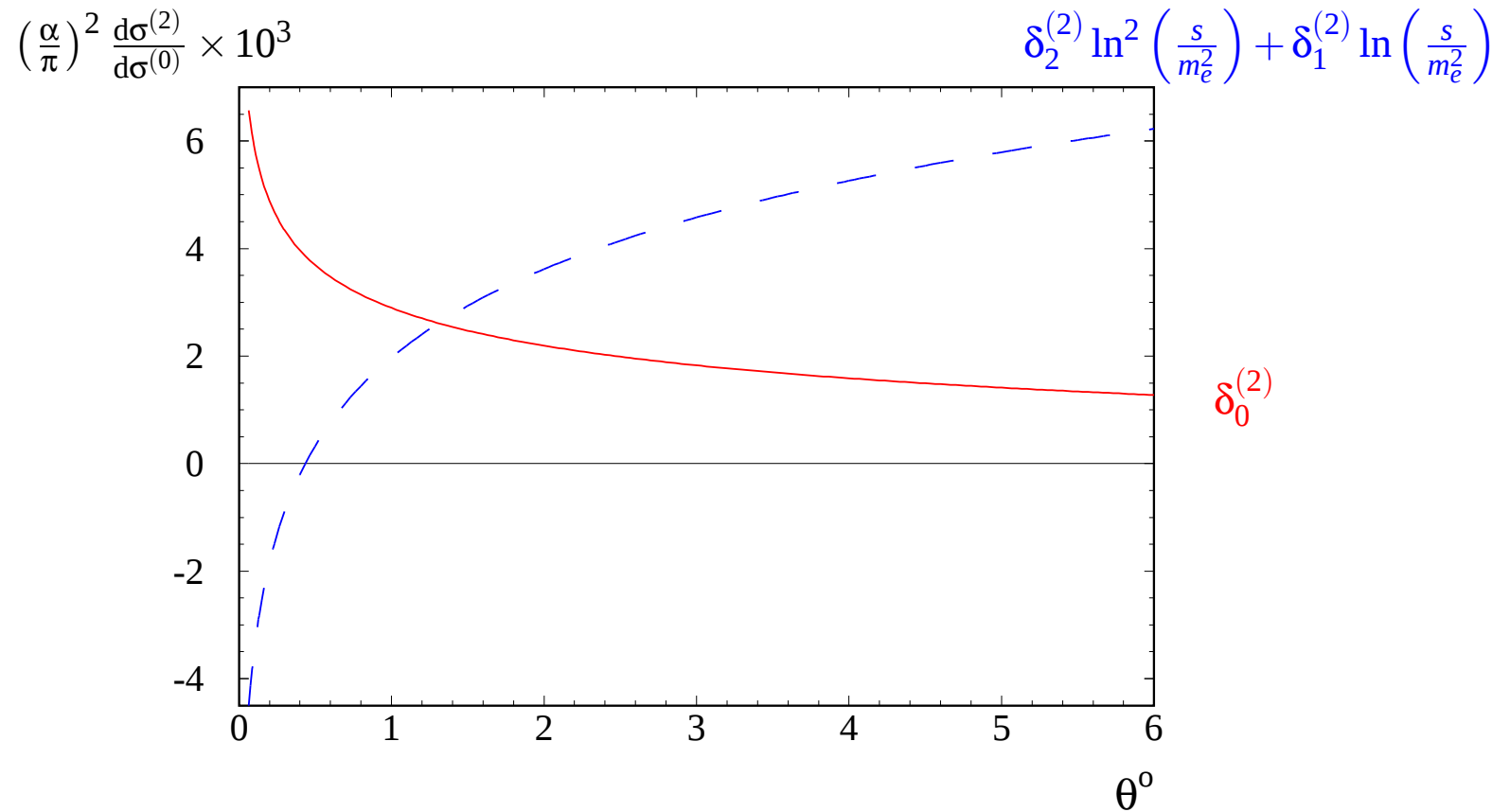
$$\sqrt{s} = 100 \text{ GeV}, \quad \ln\left(\frac{E_{cut}}{E}\right) = 0$$

Two-loop photonic corrections to SA Bhabha scattering



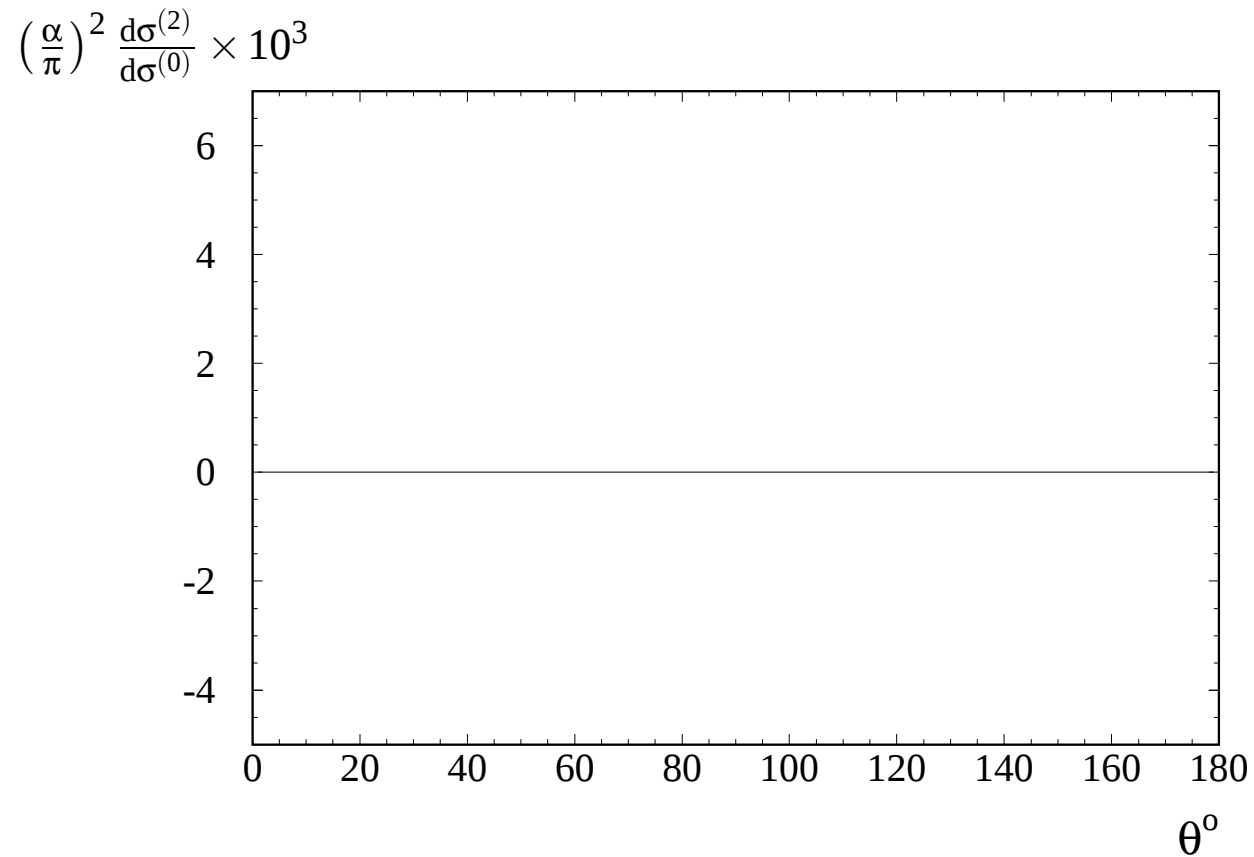
$$\sqrt{s} = 100 \text{ GeV}, \quad \ln\left(\frac{E_{cut}}{E}\right) = 0$$

Two-loop photonic corrections to SA Bhabha scattering



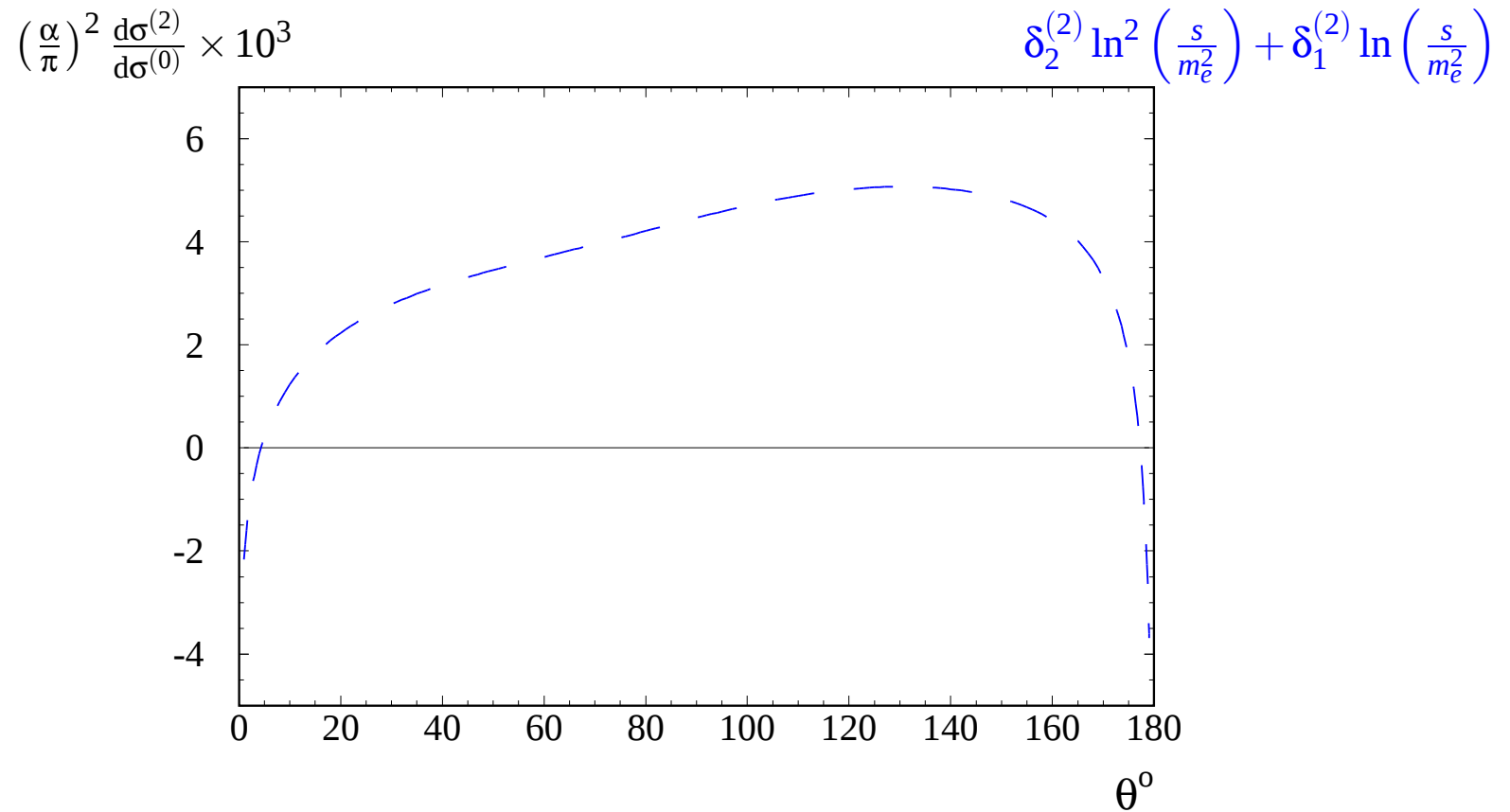
$$\sqrt{s} = 100 \text{ GeV}, \quad \ln\left(\frac{\mathcal{E}_{cut}}{\mathcal{E}}\right) = 0$$

Two-loop photonic corrections to LA Bhabha scattering



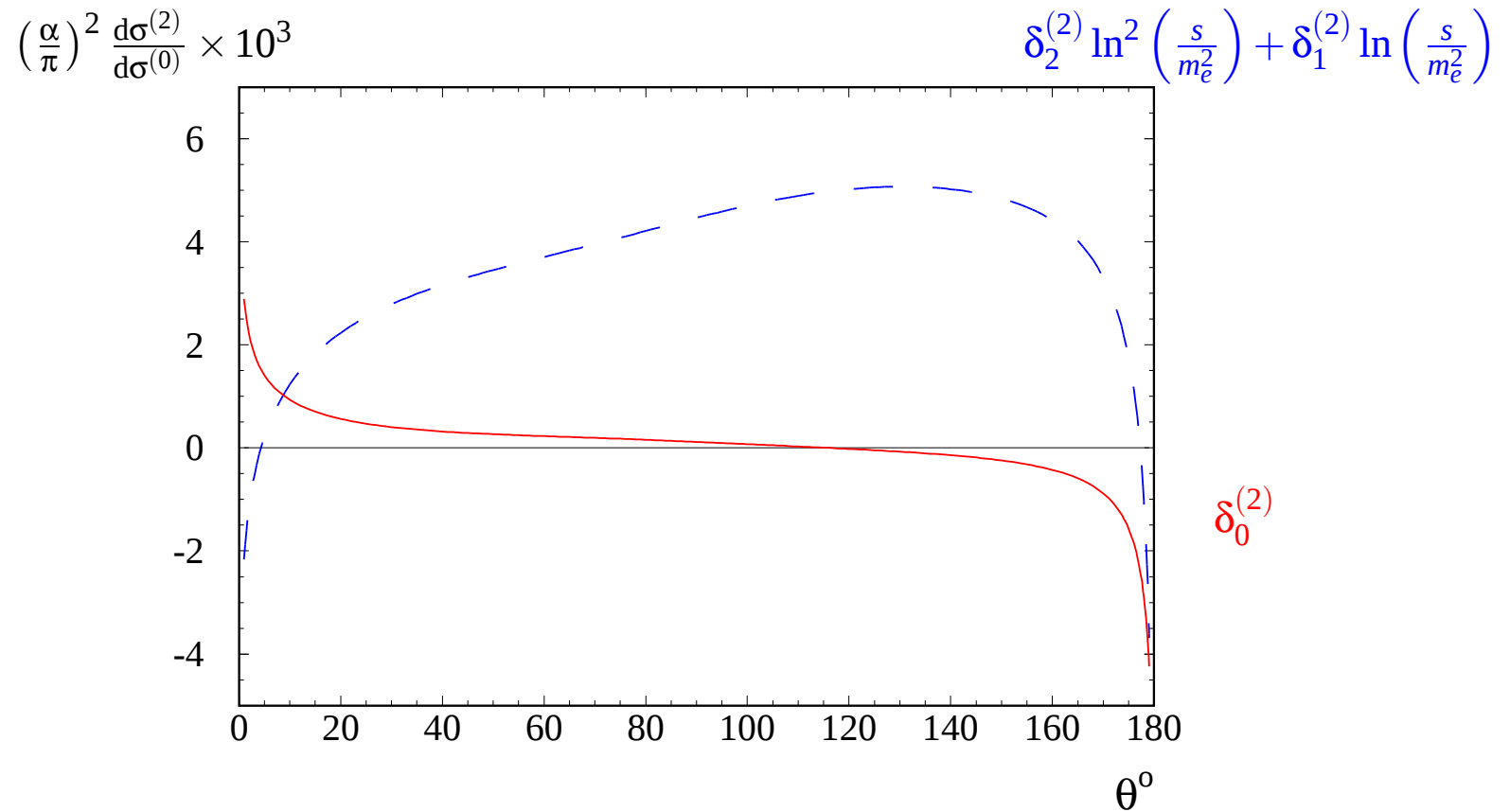
$$\sqrt{s} = 1 \text{ GeV}, \quad \ln\left(\frac{\mathcal{E}_{cut}}{\mathcal{E}}\right) = 0$$

Two-loop photonic corrections to LA Bhabha scattering



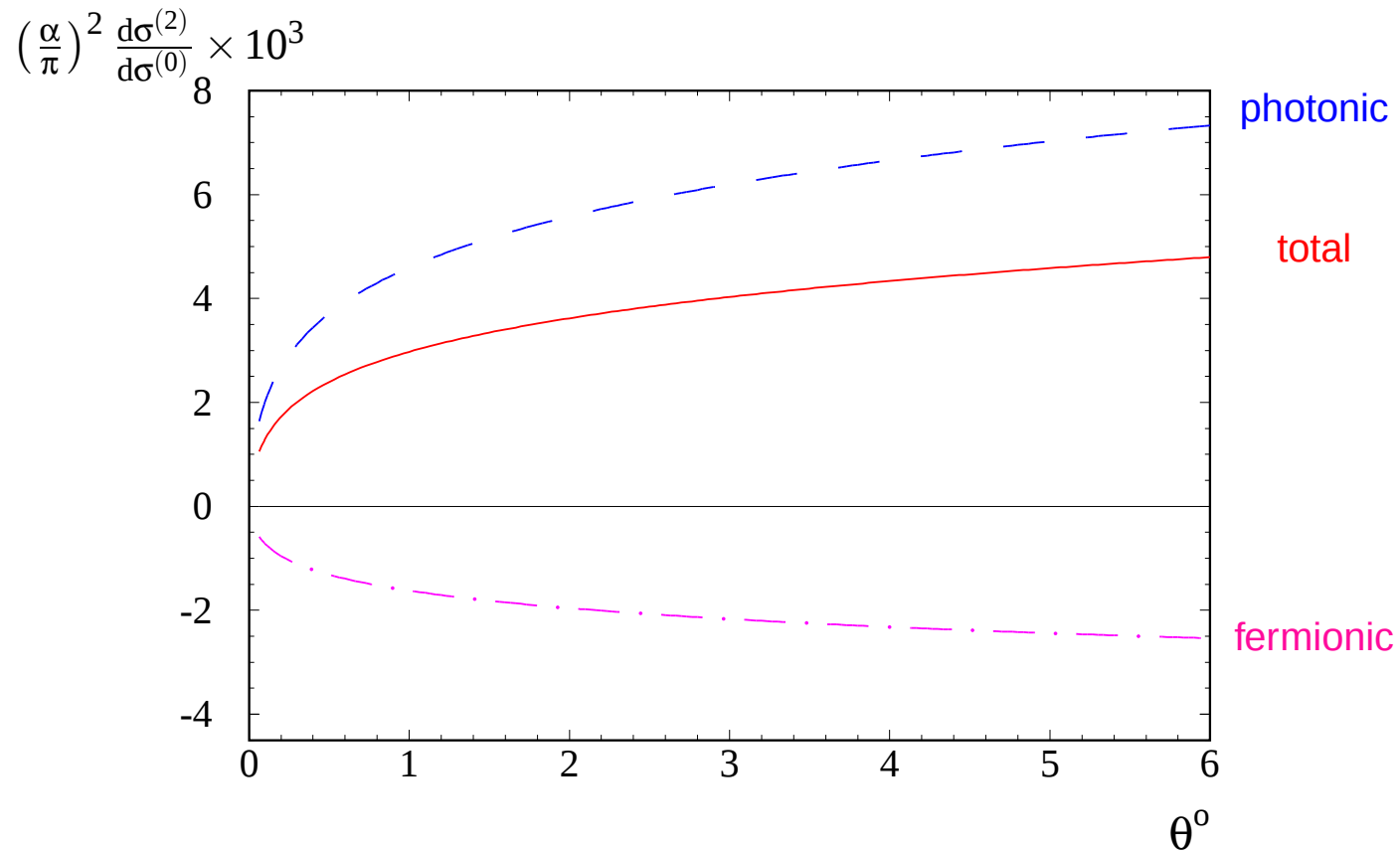
$$\sqrt{s} = 1 \text{ GeV}, \quad \ln\left(\frac{\mathcal{E}_{cut}}{\mathcal{E}}\right) = 0$$

Two-loop photonic corrections to LA Bhabha scattering



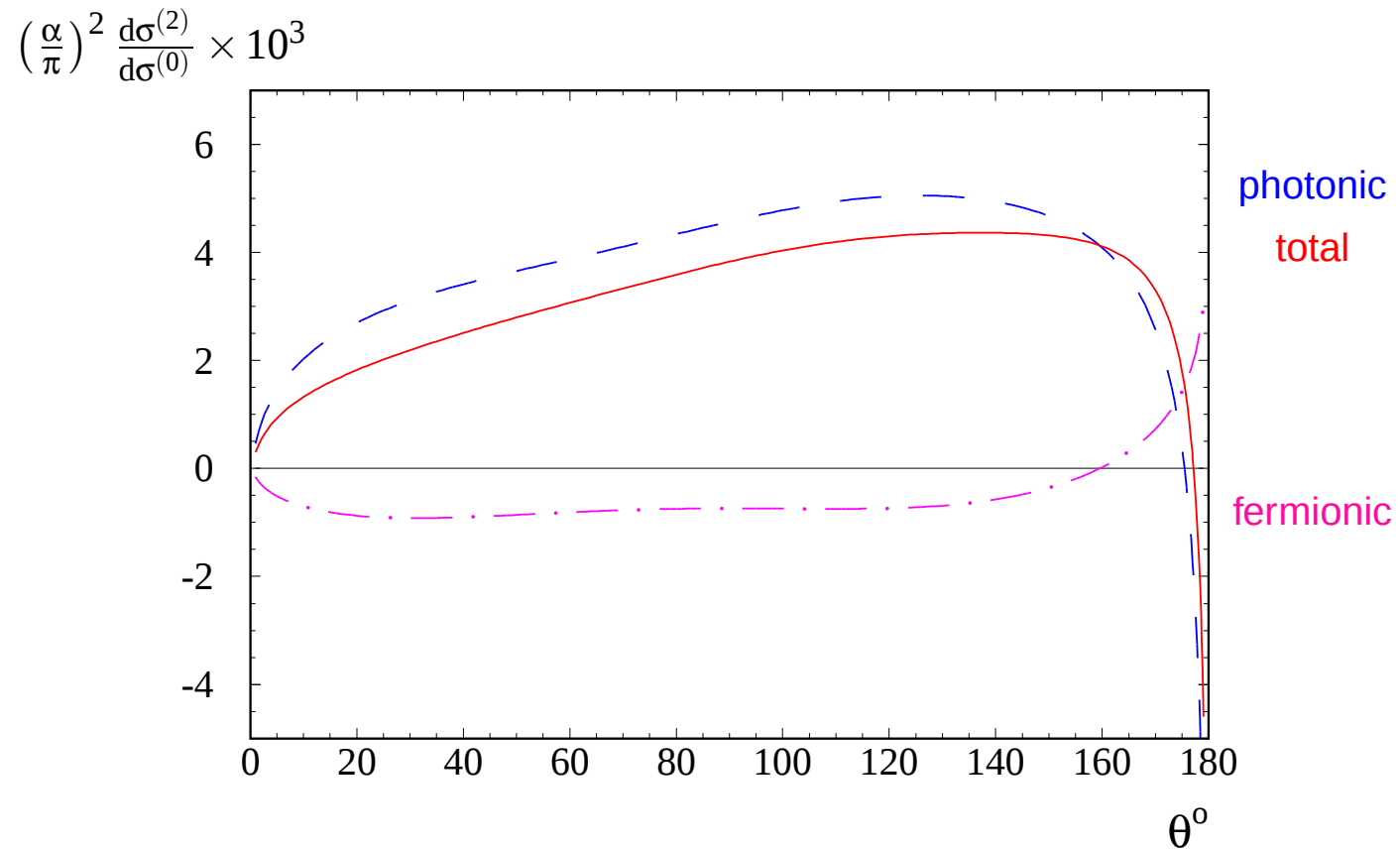
$$\sqrt{s} = 1 \text{ GeV}, \quad \ln\left(\frac{E_{cut}}{E}\right) = 0$$

Two-loop corrections to SA Bhabha scattering



$$\sqrt{s} = 100 \text{ GeV}, \quad \ln\left(\frac{\mathcal{E}_{cut}}{\mathcal{E}}\right) = \ln\left(\frac{\mathcal{E}_{cut}^{e^+e^-}}{\mathcal{E}}\right) = 0$$

Two-loop corrections to LA Bhabha scattering



$$\sqrt{s} = 1 \text{ GeV}, \quad \ln\left(\frac{\mathcal{E}_{cut}}{\mathcal{E}}\right) = \ln\left(\frac{\mathcal{E}_{cut}^{e^+e^-}}{\mathcal{E}}\right) = 0$$

Summary

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- After long way the two-loop QED corrections to Bhabha scattering are here.

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- They should be incorporated into the Monte Carlo event generators.